the transmission of information, characteristic of our profession, to higher forms of education and human enrichment. In my friend and colleague Ernst Snapper, to whom this Chauvenet paper was dedicated, the aspiring young mathematician could see the rich and varied academic career available for those who apply high standards of excellence in all their professional activities. He is a fountain of eternal mathematical youth. Finally, while in graduate school I had the privilege of a close personal relationship with a professor, Guido Weiss, well known as an outstanding mathematical expositor. It is a pleasure to follow him, years later, in receiving the Chauvenet Prize. To these people, and many others unnamed, my debt is great."

David P. Roselle, *Secretary*

THE GEOMETRY OF ROOT SYSTEMS AND SIGNED GRAPHS

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Dedicated to Professor Fred Supnick
of The City College and the City University of New York:
A small return for much given.

This essay tells of a newly discovered connection among root systems, graphs, and matroids.

Root systems are sets of vectors which satisfy certain requirements of symmetry and metric regularity. They arose in Lie theory, where they are important because they correspond one-to-one to Lie algebras and hence to Lie groups and because many properties of the algebra and group involve the root system. They have since found other applications, such as to line graphs and the search for finite simple groups.[1]

Graphs, or networks, which consist of nodes joined by arcs, arise in all kinds of combinatorial analysis. Yet signed graphs, in which each arc is labeled by + or −, are rarely discussed or applied. (So much so that some people at first think they are another form of directed graph. They are not.) They will find good use in this article, for with five exceptions every root system can be concisely and faithfully represented by a signed graph.

One of the problems encountered in Lie theory is that of counting the pieces into which space is cut by all the hyperplanes dual to elements of a root system. The usual method of solution, which is classical and well known, depends on translating the problem into one concerning an automorphism group of the root system. But it is not necessary to take that approach. Instead, by tackling the problem directly with combinatorial techniques, one can count the pieces derived not only from full root systems but from many subsystems; roughly speaking, the root systems correspond to complete graphs, while the additional systems solvable by combinatorics correspond to arbitrary subgraphs. The principal tool is the characteristic polynomial of an arrangement of hyperplanes (see Section 4), a polynomial borrowed from matroid theory. This theory, which I shall not need to mention again by name, is nonetheless the quiet ground of my discourse.

1. Root Systems. A *root system* is a finite set R of vectors in $\mathbb{R}^n$, called *roots*, with the properties:

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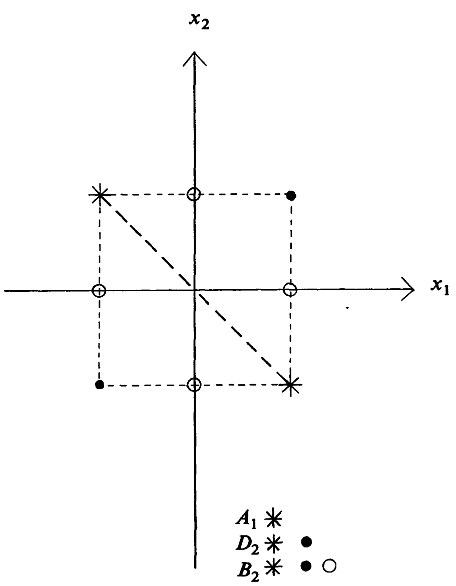
(1) if $\alpha \in R$, the only integral multiples $k\alpha$ which are in R are α and $-\alpha$ (thus 0 is not a root);

(2) if α, β are linearly independent roots and if $p, q \geqslant 0$ are the largest integers such that $\alpha + k\beta$ is a root for all integers k in the range $-p \leqslant k \leqslant q$, then $(\alpha, \beta)/(\alpha, \alpha) = -(p+q)$.

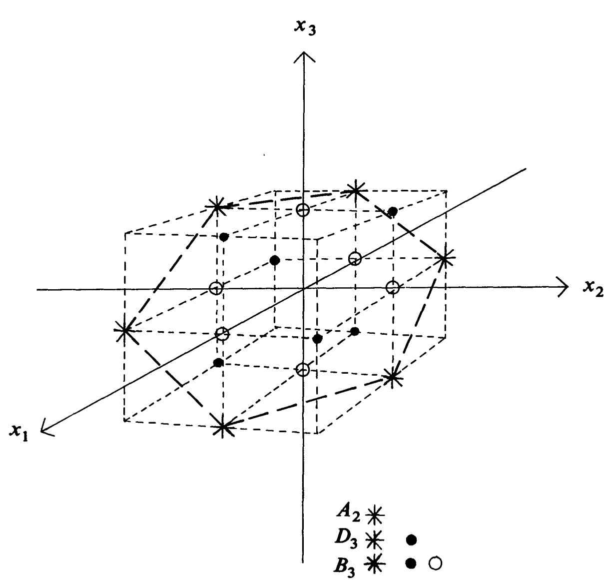
It turns out that there are very few root systems which cannot be built out of others in a simple way. Let us suppose $S \subseteq \mathbb{R}^l$ and $T \subseteq \mathbb{R}^m$ are root systems. Then their union $S \cup T$ is in a natural way a subset of $\mathbb{R}^{l+m}$, with S and T lying in orthogonal subspaces. This set $S \cup T$, the *direct sum* of S and T, is obviously also a root system. A root system which cannot be decomposed as a direct sum is *irreducible*. Now let's look at irreducible root systems. Two of them are called *similar*, and considered to be essentially identical, if there is a change of scale (a similarity transformation) which makes them isometrically isomorphic. The remarkable fact is that, aside from five "exceptional" root systems (known as G_2, F_4, E_6, E_7, and E_8), there are only four families of irreducible root systems. They are the *classical root systems* A_{n-1}, B_n, C_n, and D_n, which are traditionally represented in $\mathbb{R}^n$ as the vector sets

$$A_{n-1} = \{b_i - b_j\}_{i \neq j},$$
$$D_n = A_{n-1} \cup \{\pm(b_i + b_j)\}_{i \neq j},$$
$$B_n = D_n \cup \{\pm b_i\},$$
$$C_n = D_n \cup \{\pm 2b_i\},$$

where $b_1, b_2, \ldots, b_n$ are an orthonormal basis of $\mathbb{R}^n$. The proof of this classification theorem, which is long and complicated, appears in most books on Lie algebras. (The dimension of the system is indicated by its subscript; so all span $\mathbb{R}^n$ except A_{n-1}. In dimensions $\leqslant 3$ there are some similarity relations and reducibilities among the classical systems, which will not concern us.)



(a). The three root systems A_1, D_2, B_2 in $\mathbb{R}^2$.

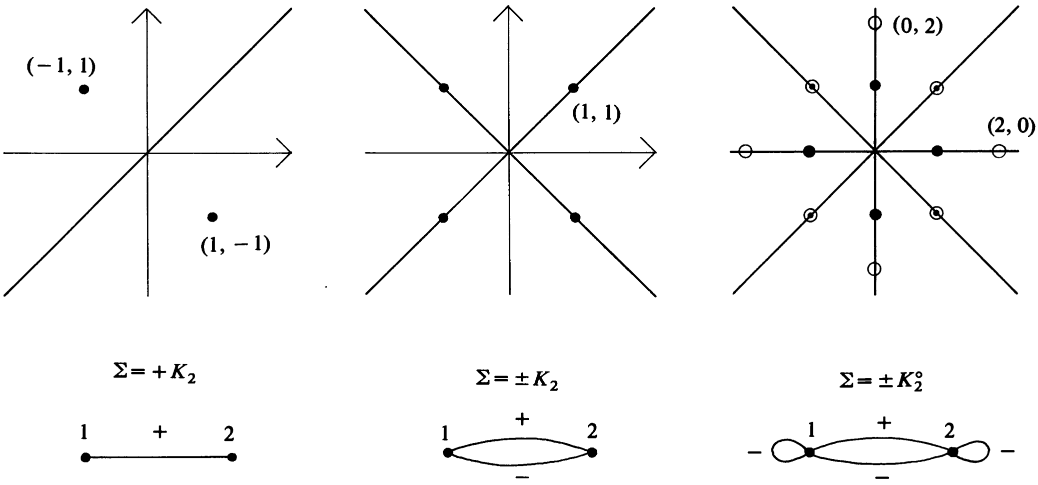


(b). The three root systems A_2, D_3, B_3 in $\mathbb{R}^3$.

FIG. 1

2. Arrangements, Regions, and Chambers. Let R be a root system in $\mathbb{R}^n$. If we take the hyperplane perpendicular to each root we get a finite set of hyperplanes, R^*, which dissects $\mathbb{R}^n$ into n-dimensional pieces called *Weyl chambers*. They are the components of $\mathbb{R}^n \setminus \cup \{h \in R^*\}$.

We could easily be more general. Let H be any finite set of hyperplanes. We call H an *arrangement of hyperplanes* and the components of $\mathbb{R}^n \setminus \cup \{h \in H\}$ the *regions* of the arrangement. (If $H = R^*$ the "regions" are also "chambers"; here we see the meeting of two independent traditions.)



(a) A_1, A_1^*, and $\Sigma = +K_2$. There are 1 hyperplane (a line) and 2 chambers.

(b) D_2, D_2^*, and $\Sigma = \pm K_2$. There are 2 lines and 4 chambers.

(c) $B_2(\bullet)$, $C_2(\circ)$, $B_2^* = C_2^*$, and $\Sigma = \pm K_2^\circ$. There are 4 lines and 8 chambers.

FIG. 2. The classical root systems in $\mathbb{R}^2$, their dual arrangements of hyperplanes, and their corresponding signed graphs.

Let $c(H)$ denote the number of regions. In Lie theory the number of chambers, $c(R^*)$, is important enough to be the subject of a chapter. It is calculated by means of a certain symmetry group of R, the Weyl group, whose order equals the number of chambers. But it seemed to me that it should not be necessary to rely on the symmetry of R to count its chambers. Such a combinatorial problem should have a combinatorial solution. As it happened, I knew a purely combinatorial technique for counting the regions of an arrangement of hyperplanes without any assumption of symmetry. In the rest of this paper (after a brief classical interlude) I will show how to use it, by way of the medium of signed graphs, to find the numbers of chambers of all the classical root system arrangements as well as the numbers of regions of many of their subarrangements (not all, because the calculations become too complicated).

3. The Classical Approach to Weyl Chambers. In Lie theory the number of chambers of R, $c(R^*)$, is calculated by showing that a certain group $\mathfrak{W}(R)$, the *Weyl group*, permutes the chambers and that for each two chambers, C_1 and C_2, exactly one $w \in \mathfrak{W}(R)$ carries C_1 to C_2. Thus $c(R^*) =$ the order of $\mathfrak{W}(R)$. The Weyl group is generated by the reflections S_α for $\alpha \in R$, where S_α means orthogonal reflection of $\mathbb{R}^n$ in the hyperplane h_α perpendicular to α. It is easy to compute that

$$\mathfrak{W}(A_{n-1}) \cong \mathfrak{S}_n,$$
$$\mathfrak{W}(B_n) \cong \mathfrak{W}(C_n) \cong \mathfrak{O}_n,$$
$$\mathfrak{W}(D_n) \cong \mathfrak{O}_n^+,$$

where $\mathfrak{S}_n =$ the symmetric group on n letters, or the group of $n \times n$ permutation matrices; $\mathfrak{O}_n$ is the hyperoctahedral group, or the group of $n\times n$ signed permutation matrices; and $\mathfrak{O}_n^+$ denotes the subgroup of $\mathfrak{O}_n$ consisting of the matrices with evenly many minus signs. From the one-to-one correspondence between elements of the Weyl group and the Weyl chambers, we deduce:

THEOREM 1. $c(A_{n-1}^*)=n!$, $c(D_n^*)=2^{n-1}n!$, *and* $c(B_n^*)=c(C_n^*)=2^n n!$.

Now let me show you how to calculate these same numbers without reference to the Weyl group.

4. The Combinatorial Approach to Regions. Here is the formula for the number of regions of an arrangement of hyperplanes in $\mathbb{R}^n$. Let H be the arrangement. Let us write $d(S)=\dim(\cap S)$ for any $S\subseteq H$. The *characteristic polynomial* of H is the polynomial defined by [2]

$$p_H(\lambda)=\sum_{S\subseteq H}(-1)^{\#(S)}\lambda^{d(S)-d(H)}.$$

THEOREM 2. $c(H)=(-1)^{n-d(H)}p_H(-1)$.

The proof of Theorem 2 is a good illustration of the powerful inductive method of deletion and contraction, which I will use again in the course of the paper.[3]

Pick a hyperplane $h\in H$. The other hyperplanes in H dissect h into $(n-1)$-dimensional pieces, which are the regions of the *induced arrangement* of hyperplanes in h,

$$H/h=\{h_1\cap h: h_1\in H, h_1\neq h\}.$$

H/h is the "contraction" of H to h. The "deletion" is $H\backslash h$. The method of deletion and contraction begins by proving two parallel equations:

$$c(H)=c(H\backslash h)+c(H/h) \tag{1}$$

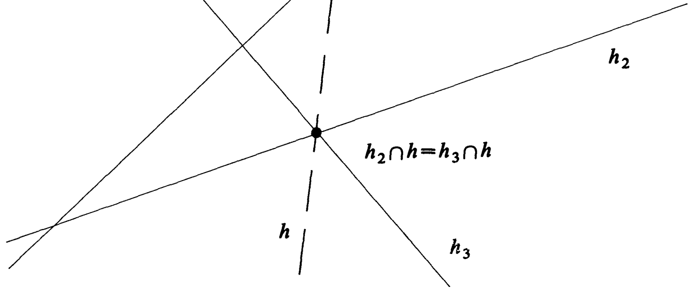


FIG. 3 (a). Removing a line h from an arrangement of lines $H=\{h,h_1,h_2,h_3\}$, yielding $H_1=\{h_1,h_2,h_3\}$. The induced arrangement H/h consists of two points on h. (In this example, because all hyperplanes do not pass through a common point, the definition of $p_H(\lambda)$ must be modified: see [14, pp. 3 and 13]. But the method of deletion and contraction remains valid.)

$$p_{H_1}(\lambda)=\lambda^2-3\lambda+3, \qquad c(H_1)=7=|p_{H_1}(-1)|,$$
$$p_{H/h}(\lambda)=\lambda-2, \qquad c(H/h)=3=|p_{H/h}(-1)|,$$
$$p_H(\lambda)=\lambda^2-4\lambda+5, \qquad c(H)=10=|p_H(-1)|.$$

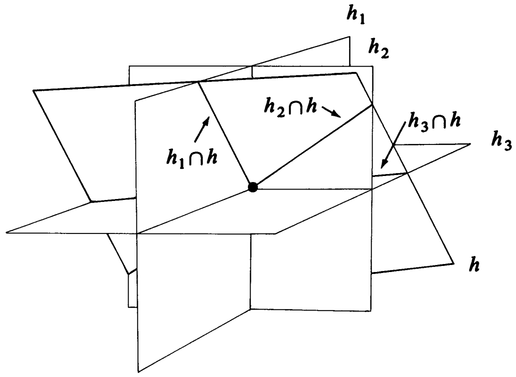


FIG. 3 (b). Removing a (hyper)plane h from an arrangement $H=\{h, h_1, h_2, h_3\}$ of planes, leaving $H_1=\{h_1, h_2, h_3\}$. The induced arrangement H/h consists of 3 lines in h, all passing through a common point.

$$p_{H_1}(\lambda)=\lambda^3-3\lambda^2+3\lambda-1, \qquad c(H_1)=8,$$
$$p_{H/h}(\lambda)=\lambda^2-3\lambda+2, \qquad c(H/h)=6,$$
$$p_H(\lambda)=\lambda^3-4\lambda^2+6\lambda-3, \qquad c(H)=14.$$

for any $h \in H$, and

$$p_H(\lambda)=p_{H\setminus h}(\lambda)-p_{H/h}(\lambda) \tag{2}$$

for any $h \in H$ such that $\cap H = \cap(H\setminus h)$, or $h \supseteq \cap(H\setminus h)$. Since the deletion and the contraction have fewer hyperplanes than H, we can carry out induction on the size of H, deducing Theorem 2 for H from its validity for $H\setminus h$ and H/h. The only hitch will be when (2) does not apply, which is the case when no $h \in H$ contains $\cap(H\setminus h)$. But if that is so, H must consist of $n-d(H)$ hyperplanes in general position; it is easy to see that $c(H)=2^{\#(H)}$ and it is an easy calculation that $p_H(\lambda)=(\lambda-1)^{\#(H)}$ (since $d(S)=n-\#(S)$). Now clearly Theorem 2 is valid.

The heart of the proof is thus the verification of Equations (1) and (2). You will have no trouble carrying out the rather long calculations necessary for (2) once I point out that there is a one-to-one correspondence between subarrangements $S \subseteq H$ which contain h and subarrangements $S' \subseteq H/h$, namely, $S \leftrightarrow S/h$, and that $\#(S)=\#(S')+1$ and $\cap S = \cap S'$. (It is only fair to admit there are some complications. There may be several hyperplanes which coincide; nevertheless they have to be treated as distinct objects. And one must allow the degenerate "hyperplane," which is the whole space—an arrangement containing it has no regions.)

By contrast Equation (1) is purely pictorial. Each region of H arises from a region C of $H\setminus h$. If h hits C, it cuts it into two parts, C_1 and C_2, which are regions of H, and one $(n-1)$-dimensional piece, $h \cap C$, which is a region of H/h. All the regions of H/h come about this way. If h misses C, however, then C is a region of H. Adding everything up, we get (1). That proves everything we needed for Theorem 2.

5. Equations and Signed Graphs. How does all this apply to the root system arrangements? We have to have some way of calculating their characteristic polynomials. With the help of signed graphs we'll be able to compute $p_{A^*_{n-1}}$, $p_{D^*_n}$, $p_{B^*_n}=p_{C^*_n}$, and more besides.[4]

Let's look at the hyperplanes of B^*_n. Each of them has one of the two forms

$$h^\varepsilon_{ij}\colon x_i=\varepsilon x_j \qquad \text{where } \varepsilon=\pm 1 \text{ and } i \neq j,$$
$$h_i\colon x_i=0.$$

(I think of a hyperplane as being the same as its equation for all practical purposes.) There is a nice, compact way of describing this. Let us construct a signed graph Σ on the n nodes

$\{1,2,\ldots,n\}$. Each node i corresponds to the coordinate x_i. A hyperplane (or equation) h_{ij}^{ε} of the first type corresponds to an arc of Σ which links i to j. To distinguish h_{ij}^{+} from h_{ij}^{-} we label the arc by the sign of h_{ij}^{ε}. Call this arc e_{ij}^{ε}. As for a hyperplane h_i of the second type, it is the same as $h_{ii}^{-}: x_i = -x_i$ so it corresponds to an arc e_{ii}^{-} (a negative loop at i). We have the following table:

$$h_{ij}^{\varepsilon}: x_i = \varepsilon x_j \leftrightarrow e_{ij}^{\varepsilon}: \text{linking } i \text{ and } j,$$

$$h_i: x_i = 0 \leftrightarrow e_{ii}^{-}: \text{loop at } i.$$

Note that according to our scheme a positive loop e_{ii}^{+} corresponds to the equation $x_i = x_i$, which is the whole space (the "degenerate hyperplane"). A negative loop, however, is a real hyperplane.

So if we take any subarrangement $H \subseteq B_n^*$ we can describe it by a signed graph. Conversely any signed graph Σ on the n nodes $\{1,2,\ldots,n\}$ describes a unique arrangement,

$$H[\Sigma] = \{h_{ij}^{\varepsilon}: \text{there is an arc } e_{ij}^{\varepsilon} \text{ in } \Sigma\}.$$

What we wanted from all this was a way to calculate $p_H(\lambda)$. It can be written down directly from the signed graph Σ corresponding to H, but as doing so for arbitrary H requires some relatively esoteric concepts, I will skip the general solution.[5] Instead I will discuss two kinds of subarrangement for which the value $p_H(\lambda)$ is particularly intriguing and easy to formulate.

6. "Special" Subarrangements. Let's call a subarrangement $H \subseteq B_n^*$ "special" if it contains all the coordinate hyperplanes and it has *sign symmetry*: whenever $h_{ij}^{\varepsilon} \in H$ (for $i \neq j$), also $h_{ij}^{-\varepsilon} \in H$.

The "special" arrangements correspond to the signed graphs I call "full signed expansions of ordinary graphs." Let Γ be an ordinary graph (for the sake of simplicity, without loops or multiple arcs). By $\pm\Gamma$ I mean the signed graph which has the same nodes as Γ and, for each arc e_{ij} of Γ, both the signed arcs e_{ij}^{+} and e_{ij}^{-}. This graph is *sign-symmetric* because it contains $e_{ij}^{-\varepsilon}$ whenever it contains e_{ij}^{ε} (for $i \neq j$). By $(\pm\Gamma)^{\circ}$, loosely written just $\pm\Gamma^{\circ}$, I mean the signed graph $\pm\Gamma$ with a negative loop added to every node. Having all these loops makes $\pm\Gamma^{\circ}$ a *full* sign-symmetric signed graph. The "special" arrangements are just those which equal $H[\pm\Gamma^{\circ}]$ for some ordinary graph Γ. For example, $B_n^* = H[\pm K_n^{\circ}]$. The point of singling them out is the simplicity of their characteristic polynomials. To state the theorem we need the *chromatic polynomial* $\chi_{\Gamma}(\lambda)$ of the graph Γ. If λ is a positive integer, $\chi_{\Gamma}(\lambda)$ is the number of proper colorings of Γ by λ colors: which means assigning an integer from the set $\{1,2,\ldots,\lambda\}$ to each node so that the two endpoints of an arc have different colors. The function χ_{Γ} turns out to be a polynomial; this lets us define it for other values of λ besides positive integers.

THEOREM 3. *Let Γ be a graph on the nodes $\{1,2,\ldots,n\}$ and $H = H[\pm\Gamma^{\circ}]$. Then*

$$p_H(\lambda) = 2^n \chi_{\Gamma}\left(\frac{\lambda-1}{2}\right). \tag{3}$$

In particular

$$p_{B_n^*}(\lambda) = p_{C_n^*}(\lambda) = 2^n \left(\frac{\lambda-1}{2}\right)_n,$$

where $(\lambda)_n$ denotes the falling factorial $\lambda(\lambda-1)\cdots(\lambda-n+1)$.

The proof of the particular case depends on the observation that $\chi_{K_n}(\lambda) = (\lambda)_n$.

I will prove the theorem by again applying deletion and contraction. The necessary recursion for the right-hand side is both well known and easy to prove. It is

$$\chi_{\Gamma}(\lambda) = \chi_{\Gamma \setminus e}(\lambda) - \chi_{\Gamma/e}(\lambda) \tag{4}$$

for $e = e_{ij}$ = any arc of Γ (except a loop, which we've ruled out anyway), where $\Gamma \setminus e$ is Γ with e removed and Γ/e means Γ "contracted" by e: the endpoints of e are merged and e itself is thrown away. We can prove (4) for every positive integer λ by counting colorings of Γ by λ

colors. If we ignore $e=e_{ij}$, letting i and j be colored the same, we have $\chi_{\Gamma\backslash e}(\lambda)$ colorings. Some of them are proper for Γ (i and j are colored differently); there are $\chi_\Gamma(\lambda)$ of these. The rest are the ways to color $\Gamma\backslash e$ so that i and j have the same color; their number equals $\chi_{\Gamma/e}(\lambda)$. Thus (4) holds for all positive integers λ. Since $\chi_\Gamma(\lambda)$ is a polynomial, it follows that (4) is a polynomial identity, valid for all λ.[6]

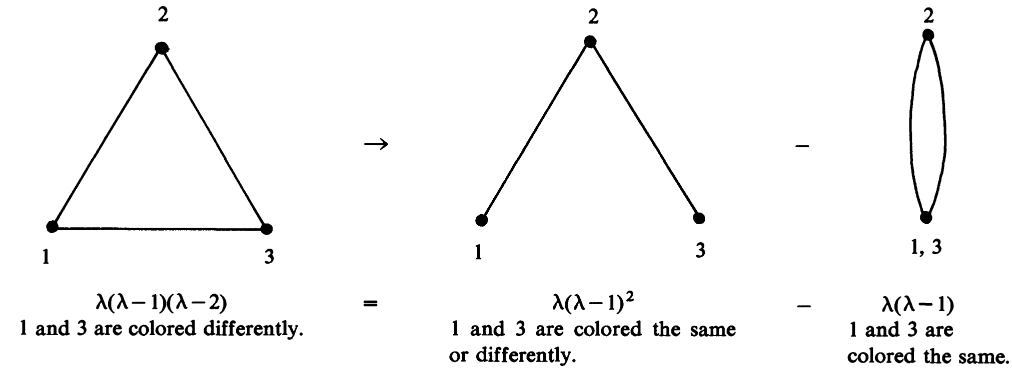


FIG. 4. Graph coloring, showing deletion and contraction. Under each graph is its chromatic polynomial.

Now we can prove Theorem 3 by induction on the number of arcs in Γ. We begin, of course, with the arcless graph on the node set $\{1,2,\ldots,n\}$; call it V_n. Its chromatic polynomial is $\chi_{V_n}(\lambda)=\lambda^n$. The corresponding "special" arrangement $H[\pm V_n^\circ]$ consists of the coordinate hyperplanes. Its characteristic polynomial is easily seen from the definition to be $(\lambda-1)^n$. This equals $2^n\chi_{V_n}(\frac{1}{2}(\lambda-1))$, as we wanted.

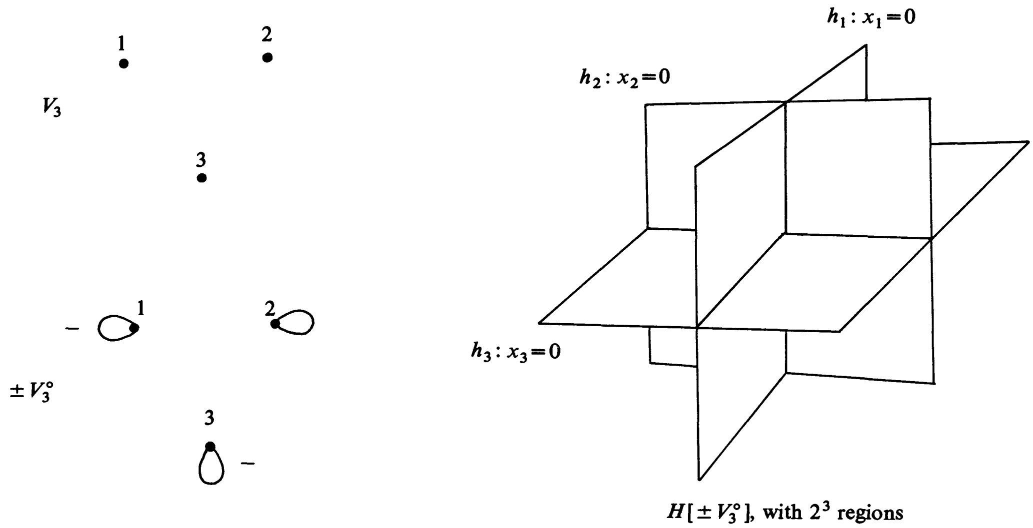
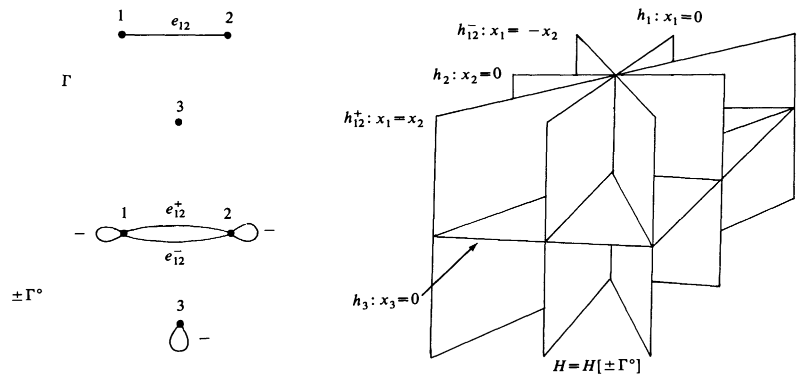


FIG. 5. The graph V_3, the signed graph $\pm V_3^\circ$, and the "special" arrangement $H[\pm V_3^\circ]$.

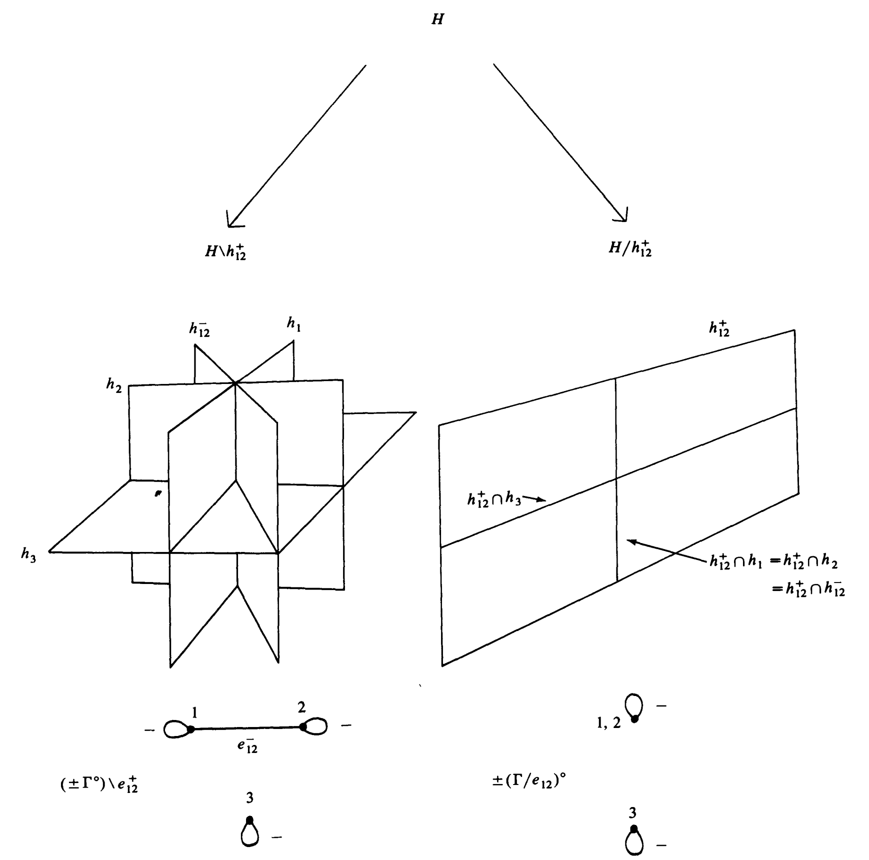
Next we carry out the induction. It is a relatively complicated application of deletion and contraction: it uses it twice. First we pick one arc e_{ij} of Γ and apply deletion and contraction by h_{ij}^+ to $H=H[\pm\Gamma^\circ]$, using Equation (2). The result is

$$p_H(\lambda)=p(H\backslash h_{ij}^+;\lambda)-p(H/h_{ij}^+;\lambda). \tag{5}$$

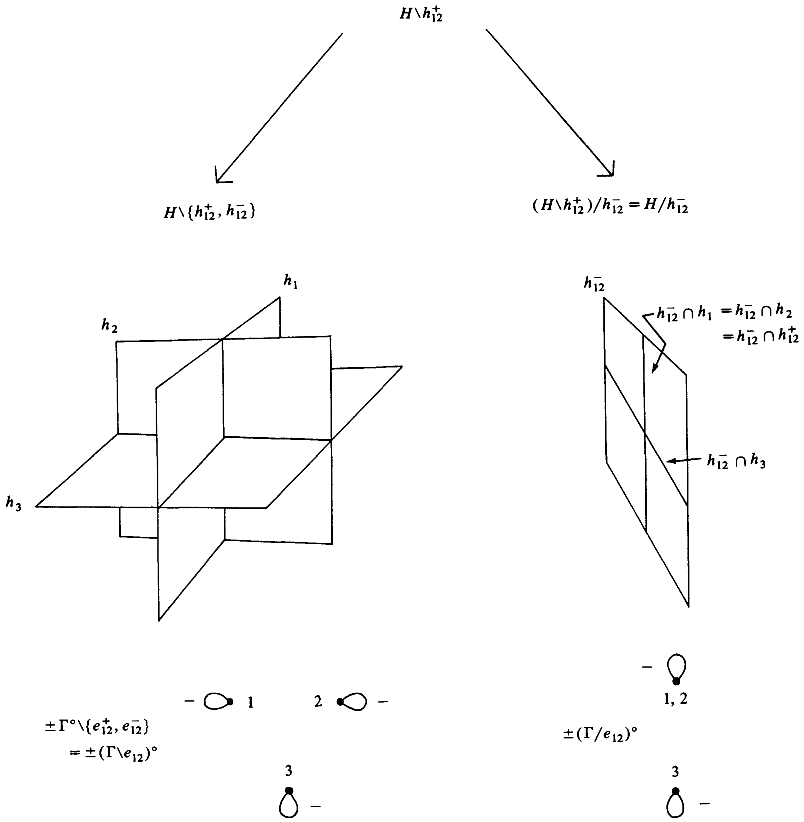
FIG. 6. An example of the inductive procedure used to prove Theorem 3.



(a) The initial graph Γ, full signed expansion graph $\pm\Gamma^{\circ}$, and "special" arrangement $H[\pm\Gamma^{\circ}]$.



(b) The arrangements resulting from deletion of and contraction by h_{12}^{+}, together with the signed graphs which describe them.



(c) The arrangements resulting from deletion and contraction of $H\backslash h_{12}^+$ by h_{12}^-, together with the signed graphs which describe them.

Then we do the same to $H\backslash h_{ij}^+$, deleting and contracting with respect to h_{ij}^-. We obtain

$$p(H\backslash h_{ij}^+;\lambda)=p(H\backslash\{h_{ij}^+,h_{ij}^-\};\lambda)-p((H\backslash h_{ij}^+)/h_{ij}^-;\lambda). \tag{6}$$

Obviously $H\backslash\{h_{ij}^+,h_{ij}^-\}=H[\pm(\Gamma\backslash e_{ij})^\circ]$, which is a smaller "special" arrangement than H. But what about the contractions H/h_{ij}^+ and $(H\backslash h_{ij}^+)/h_{ij}^-$?

They are also "special"; they both equal $H[\pm(\Gamma/e_{ij})^\circ]$. Let us see why. The arrangement H/h_{ij}^+ is the one induced by H on $h_{ij}^+: x_i=x_j$. That means we take every equation in H and set x_j identically equal to x_i. An equation $x_k=\varepsilon x_j$ becomes the same as $x_k=\varepsilon x_i$; h_j: $x_j=0$ and $h_{ij}^-: x_i=-x_j$ both become repetitions of h_i: $x_i=0$. The effect is as if we had merged i and j in Γ. (Notice that h_{ij}^- becomes the same as h_i, so that $(H\backslash h_{ij}^-)/h_{ij}^+=H/h_{ij}^+$. By the sign symmetry of the situation, $(H\backslash h_{ij}^+)/h_{ij}^-=H/h_{ij}^+$.) Thus we have proved

$$H/h_{ij}^+=(H\backslash h_{ij}^+)/h_{ij}^-=H[\pm(\Gamma/e)^\circ].$$

In the light of this fact, combining (5) and (6) we have

$$p_{H[\pm\Gamma^\circ]}(\lambda)=p_{H[\pm(\Gamma\backslash e)^\circ]}(\lambda)-2p_{H[\pm(\Gamma/e)^\circ]}(\lambda). \tag{7}$$

Equation (7) is what we need to do induction, since $\Gamma\backslash e$ and Γ/e have fewer arcs than Γ. We can complete the proof by substituting from (3) in the right-hand side of (7)—remembering that Γ/e has $n-1$ nodes.

COROLLARY 4. *$H[\pm\Gamma^\circ]$ has $2^n|\chi_\Gamma(-1)|$ regions. In particular $c(B_n^*)=2^n n!$.*

The proof is by Theorems 2 and 3.

Notice how we have gotten $c(B_n^*)$ without any group theory. As a bonus we see that the 2^n and the $n!$ enter into $c(B_n^*)$ for different reasons: the 2^n because B_n^* is "special," the $n!$ because it derives from the complete graph.

7. Graphic Subarrangements. A subarrangement of B_n^* is called *graphic* if it contains only hyperplanes with plus signs, $h_{ij}^+ : x_i = x_j$. If Γ is a graph with node set $\{1,2,\ldots,n\}$, let

$$H[\Gamma]=\{h_{ij}^+ : \text{there is an arc } e_{ij} \text{ in } \Gamma\}.$$

Obviously $H[\Gamma]$ is graphic; conversely, if H is a graphic arrangement, it is derived from the graph whose arcs are $\{e_{ij}: h_{ij}^+ \in H\}$. This is the reason for the name "graphic."

THEOREM 5. *Let Γ be a graph on the nodes $\{1,2,\ldots,n\}$, having c components, and let $\chi_\Gamma(\lambda)$ be its chromatic polynomial. Then*

$$p_{H[\Gamma]}(\lambda)=\chi_\Gamma(\lambda)/\lambda^c.$$

In particular

$$p_{A_{n-1}^*}(\lambda)=(\lambda-1)_{n-1}=(\lambda-1)(\lambda-2)\cdots(\lambda-n+1).$$

COROLLARY 6. *$H[\Gamma]$ has $|\chi_\Gamma(-1)|$ regions. In particular $c(A_{n-1}^*)=n!$.*

The theorem is a consequence of the fact that $H[\Gamma]$ represents the graphic geometry (or polygon matroid) of Γ.[7]

Corollary 6 was first noticed by Curtis Greene. He also saw that the corollary can be strengthened: there is a one-to-one correspondence between the regions of $H[\Gamma]$ and the "acyclic orientations" of Γ. (It was these discoveries of Greene's, dating from 1975, that interested me in graphic arrangements.) An *acyclic orientation* of Γ is a way of directing the arcs so there is no closed path which follows their directions. There is an analogous, although more complex, interpretation of the regions of $H[\pm\Gamma^\circ]$ in terms of Γ. There is also a close connection between the graphic and "special" arrangements associated with Γ. The fact that

$$c(H[\pm\Gamma^\circ])=2^n c(H[\Gamma])$$

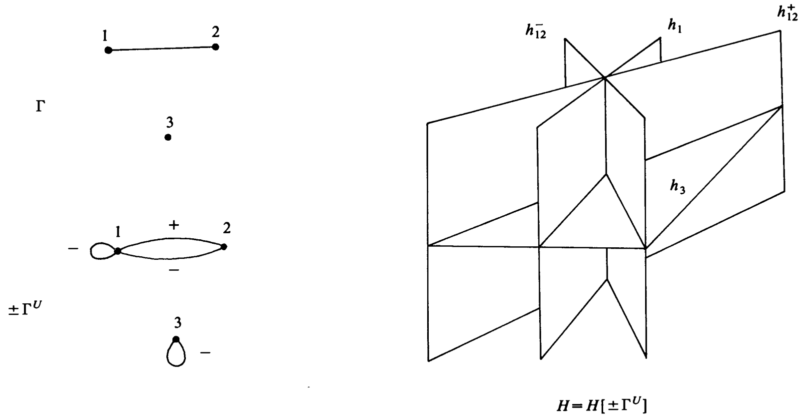
suggests that the regions of $H[\Gamma]$ may be individually sliced into 2^n parts each by the extra hyperplanes in $H[\pm\Gamma^\circ]$. And so they are—although not, as one might think, through successive halving of each old region by n of the new hyperplanes.

8. Subarrangements with Sign Symmetry. So far we've counted the chambers of B_n^* and A_{n-1}^* as special cases of the "special" and graphic arrangements, $H[\pm\Gamma^\circ]$ and $H[\Gamma]$. Although D_n^* belongs to neither of these types, it too is highly symmetrical: it has sign symmetry, for $D_n^*=H[\pm K_n]$. I will now show how to calculate the number of regions of a general sign-symmetric arrangement; in particular, that will take care of D_n^* and all $H[\pm\Gamma]$.

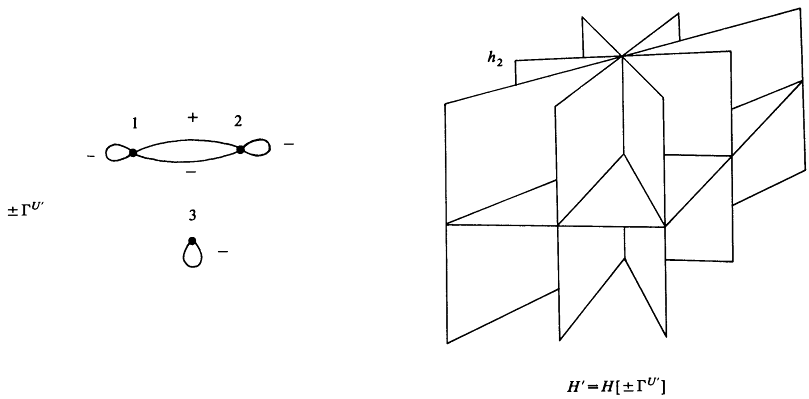
A sign-symmetric arrangement $H\subseteq B_n^*$ has associated to it an ordinary graph Γ, defined as having an arc e_{ij} linking distinct nodes i and j whenever H has hyperplanes h_{ij}^+ and h_{ij}^-. Suppose H corresponds to the signed graph Σ. Then Σ contains $\pm\Gamma$ and, since H consists of $H[\pm\Gamma]$ plus perhaps some coordinate hyperplanes, the remaining arcs of Σ must all be negative loops. We will need a way to describe Σ. So if U is a subset of the node set, let $\pm\Gamma^U$ be the signed graph which has all the arcs of $\pm\Gamma$ and in addition a negative loop at each node in U. Now we can say: Every sign-symmetric subarrangement of B_n^* is $H[\pm\Gamma^U]$ for some Γ and U. Conversely, every $H[\pm\Gamma^U]$ is obviously sign-symmetric.

What we want is a computational scheme for all arrangements of this form. That means we need to find the characteristic polynomials of all $H[\pm\Gamma^U]$. The problem can be reduced to the

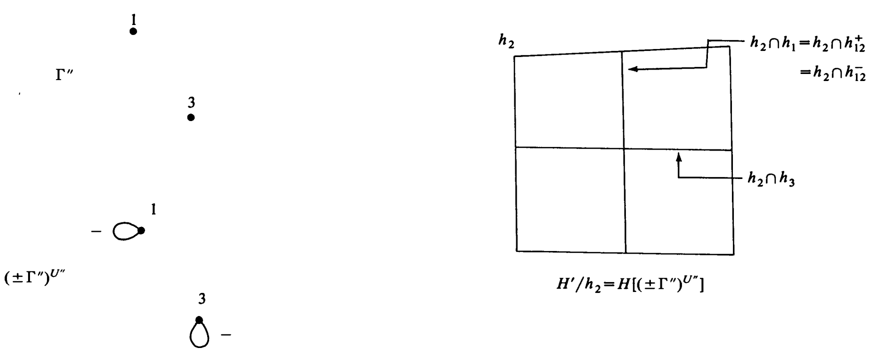
FIG. 7. An illustration of Theorem 7 in the case $q=0$ (all isolated nodes of Γ are in U).



(a) The original arrangement $H=H[\pm\Gamma^U]$, where $U=\{1,3\}$.



(b) The enlarged arrangement $H'=H\cup\{h_2\}$, which equals $H[\pm\Gamma^{U'}]$ where $U'=\{1,2,3\}$. Here $i=2$ and the added hyperplane is h_2.



(c) The contracted arrangement H'/h_2, which equals $H[(\pm\Gamma'')^{U''}]$ where $\Gamma''=\Gamma:\{2\}^c$ and $U''=\{1, 3\}$.

"special" case. If W is a set of nodes of Γ, the *subgraph induced by* W, written $\Gamma: W$, is the graph whose node set is W and whose arcs are all those of Γ both of whose endpoints are in W. In case there are no such arcs, we call W a *stable* set of nodes.

A node is *isolated* in Γ if it lies on no arcs.

THEOREM 7. *Let* Γ *be a graph on the nodes* $\{1,2,\ldots,n\}$, U *be any subset of the node set,* $q=$ *the number of isolated nodes of* Γ *which lie outside* U, *and* $H=H[\pm\Gamma^U]$. *Then*

$$p_H(\lambda)=\lambda^{-q}\sum_W 2^{\#(W)}\chi_{\Gamma:W}\left(\frac{\lambda-1}{2}\right), \tag{8}$$

where the sum is taken over every node set $W\supseteq U$ *whose complement is stable in* Γ. *In particular*

$$p_{D_n^*}(\lambda)=2^n\left(\frac{\lambda-1}{2}\right)_n+n2^{n-1}\left(\frac{\lambda-1}{2}\right)_{n-1}.$$

Let me write, informally, $H: W$ for the arrangement $H[\pm(\Gamma: W)^\circ]$, which lies in $\mathbb{R}^W$. If you refer to Theorem 3 you will see that Theorem 7 is equivalent to the formula

$$p_H(\lambda)=\lambda^{-q}\sum_W p_{H:W}(\lambda). \tag{9}$$

I will prove (8) in two stages: via (9) for the case $q=0$ by deletion and contraction (again!) along with induction on the number of nodes not in U (which is the number of coordinate hyperplanes not in H); then (8) directly for $q>0$ by an *ad hoc* reduction to the first case.

To begin with, if all nodes are in U, the range of W in the summation is merely $W=\{1,2,\ldots,n\}$. So (9) is trivially true.

If $q=0$ but U is not everything, pick a fixed $i\notin U$; write $U'=U\cup\{i\}$ and $H'=H[\pm\Gamma^{U'}]$. By deletion and contraction,

$$p_H(\lambda)=p_{H'}(\lambda)+p_{H'/h_i}(\lambda). \tag{10}$$

(Our appeal to deletion and contraction depends on the fact that $\cap H'=\cap H$. This is true because, since $i\notin U$, i is not isolated; so there is an arc e_{ij}. Therefore h_{ij}^+ and h_{ij}^- are in H, so $h_i\supseteq h_{ij}^+\cap h_{ij}^-\supseteq\cap H$.)

Now we have to know what H'/h_i is. But it is merely the arrangement in h_i which results from setting $x_i=0$ in all the hyperplanes of H'. An equation h_j: $x_j=0$ or h_{jk}^ε: $x_j=\varepsilon x_k$ in which $j,k\neq i$ will remain the same, but h_{ij}^ε: $x_i=\varepsilon x_j$ becomes transformed to h_j: $x_j=0$. That is, for every node j which is adjacent to i in Γ, H'/h_i will contain the corresponding coordinate hyperplane. In the language of signed graphs: Set $\Gamma''=\Gamma:\{i\}^c$ (where $\{i\}^c$ means the complement of $\{i\}$) and $U''=U\cup\{j: j\neq i$ is adjacent to i in $\Gamma\}$. Then $H'/h_i=H[(\pm\Gamma'')^{U''}]$.

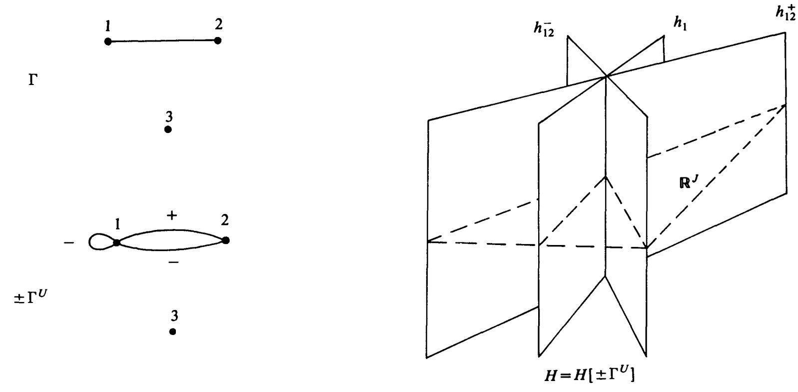
Notice that H' and H'/h_i have fewer missing coordinate hyperplanes than H. Also their signed graphs have no isolated nodes outside U' or U''. Hence we can use induction to assume the validity of (9) for H' and H'/h_i. Substituting in (10) leads to Equation (9) for H. Thus we have proved Theorem 7 for the case $q=0$.

What if $q>0$? Let Q be the set of isolated nodes of Γ which are not in U; and let J be its complement in the full node set, Q^c. I will reduce both sides of (8) to $\Gamma: J$ instead of Γ.

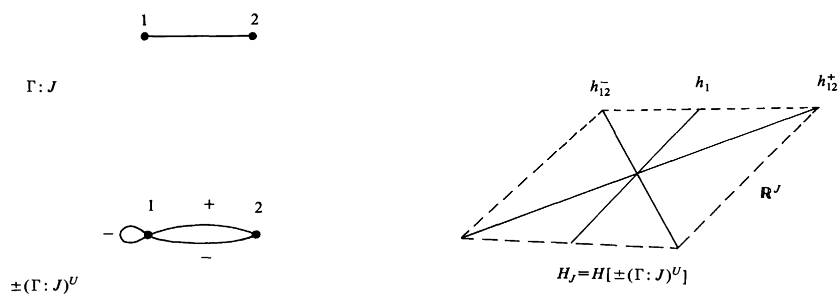
That means H will be replaced by $H_J=H[\pm(\Gamma: J)^U]$, the faithful cross section of H obtained by forgetting the Q coordinates. Everything in H_J is the same as in H except that all dimensions are lowered by q. Since that doesn't affect the characteristic polynomial, we have $p_H(\lambda)=p_{H_J}(\lambda)$.

On the other side of (8), let's split W into $Y=W\cap Q$ and $Z=W\cap J$. Since $\Gamma: W$ is the disjoint union of $\Gamma: Z$ and $\#(Y)$ isolated nodes, $\chi_{\Gamma:W}(\lambda)=\lambda^{\#(Y)}\chi_{\Gamma:Z}(\lambda)$. Moreover $W\supseteq U$ and W^c is stable in Γ, if and only if $Z\supseteq U$ and $J\backslash Z$ is stable in $\Gamma: J$. From this it is easy to check that

$$\sum_W 2^{\#(W)}\chi_{\Gamma:W}\left(\frac{\lambda-1}{2}\right)=\lambda^q\sum_{Z\subseteq J}2^{\#(Z)}\chi_{\Gamma:Z}\left(\frac{\lambda-1}{2}\right).$$



(a) The original arrangement $H=H[\pm\Gamma^U]$, where $U=\{1\}$. The set of isolated nodes not in U is $Q=\{3\}$; $q=1$; and $J=Q^c=\{1, 2\}$.



(b) The arrangement H_J, the cross section of H by $\mathbb{R}^J$. It is combinatorially equivalent to H.

FIG. 8. An illustration of Theorem 7 in the case $q>0$ (there are isolated nodes of Γ which are not in U).

That is the reduction we wanted to $\Gamma: J$. Thus the general part of Theorem 7 is proved.

Evaluating (8) for $D_n^* = H[\pm K_n]$, the stable node sets are $\varnothing$ and the singletons. So $W=\{1,2,\ldots,n\}$, giving $\Gamma: W=K_n$, and $W=\{i\}^c$ for $i=1,2,\ldots,n$, giving $\Gamma: W=K_{n-1}$. Since $q=0$, that gives us the particular formula.

COROLLARY 8. *Let Γ be a graph on the nodes $\{1,2,\ldots,n\}$, U be a subset of the node set, and $H=H[\pm\Gamma^U]$. Then H has*

$$c(H)=\sum_W (-1)^{n-\#(W)}2^{\#(W)}|\chi_{\Gamma:W}(-1)|$$

regions, where W ranges over every node set $W\supseteq U$ whose complement is stable in Γ. If $U=\varnothing$, W ranges over all complements of stable node sets. In particular $c(D_n^*)=2^n(n-1)!$.

The proof, of course, is by combining Theorems 2 and 7. As for D_n^*, we have

$$\begin{aligned} c(D_n^*) &= 2^n|(-1)_n|+n(-1)2^{n-1}|(-1)_{n-1}| \\ &= 2^n n! - n2^{n-1}(n-1)! \\ &= 2^{n-1}n!. \end{aligned}$$

This calculation is combinatorially interesting. It suggests that, rather than each chamber of B_n^* being the union of two from D_n^* as one might have thought from the formula $c(B_n^*)=2c(D_n^*)$, it is more likely that some chambers of D_n^* are not subdivided when one adds the coordinate hyperplanes while others are divided into three or more pieces. Analyzing the characteristic inequalities defining each chamber shows this to be true. One can also see that each successive coordinate hyperplane added to D_n^* halves $2^{n-1}(n-1)!$ regions. (*Proof.* Let $D_n^{*(k)}$ be D_n^* with k coordinate hyperplanes added. Then

$$c(D_n^{*(k)}) = 2^{n-1}n! + k2^{n-1}(n-1)!$$

by Corollary 8. So in $D_n^{*(k+1)}$, $2^{n-1}(n-1)!$ regions of $D_n^{*(k)}$ are halved.) Is there any significance attached to which chambers of D_n^* are cut by any given number of coordinate hyperplanes or are separated into a given number of chambers of B_n^*?

9. Faces and Flats. With all this lengthy discussion of regions and chambers, I have not yet mentioned the lower-dimensional faces of an arrangement of hyperplanes. Take a region of an arrangement H in $\mathbb{R}^n$. It is an n-dimensional convex polyhedron, with flat sides, not bounded since it is a cone radiating from the origin—rather like an infinite wedge or pyramid. Each flat side (of whatever dimension) is called a *face* of the region and of the arrangement.

I can define faces more precisely by applying the notion of a *flat* of H: a subspace which is the intersection of hyperplanes in H. I include as flats the whole space and each hyperplane. One way to define a face is as the relative interior of an intersection $\bar{C} \cap t$, where $\bar{C}$ is the topological closure of a region C and t is any flat. Another way (completely equivalent) is as any region of any arrangement H/t induced by H on a flat. The largest faces are the regions. The smallest face is the intersection of all the hyperplanes; in most cases this is the origin (but not for graphic arrangements, where it always contains the line $x_1 = \cdots = x_n$).

For instance D_2^* (see Fig. 2(b)) has 4 regions, 4 one-dimensional faces (they are rays from the origin), and 1 zero-dimensional face (the origin).

Remarkably enough, with little extra effort we can calculate the number of k-dimensional faces for any k, which is denoted f_k, and the number of k-dimensional flats, denoted a_k, for all sign-symmetric root system subarrangements. The secret weapon is a certain polynomial, the *Whitney polynomial* of H,[8]

$$w_H(x,\lambda) = \sum_{T \subseteq S \subseteq H}\sum x^{n-d(T)}(-1)^{\#(S)-\#(T)}\lambda^{d(S)-d(H)},$$

where $d(S)$, you may recall, $=\dim(\cap S)$. Now f_k is equal to the coefficient of x^{n-k} in $(-1)^{n-d(H)}w_H(-x,-1)$; while the coefficient of $x^{n-k}\lambda^{k-d(H)}$ in $w_H(x,\lambda)$ is a_k. To see why, use a second version of w_H,

$$w_H(x,\lambda) = \sum_t x^{n-\dim t}p_{H/t}(\lambda), \tag{11}$$

summed over all flats of H; the second definition of a face, from which it follows that

$$f_k = \sum_{t:\dim t = k} c(H/t);$$

and Theorem 2 applied to H/t. (*Proof of Equation* (11): Fix T; let $t = \cap T$ and $T_1 = \{h \in H \setminus T: h \supseteq t\}$. If $T_1 \neq \varnothing$ then the sum over all $S \supseteq T$ will equal 0. On the other hand, the T such that $T_1 = \varnothing$ are in one-to-one correspondence with the flats of H.)

Our problem, then, is to compute the Whitney polynomial. So long as we stick to the arrangements we've investigated—the graphic and sign-symmetric subarrangements of the root systems—and remember Theorems 3, 5, and 7, the calculations are quite straightforward. I will skip them and just give their results.[9] The *contraction* Γ/T of a graph Γ by an arc set T is the graph resulting from coalescing each group of nodes which are connected by T and then discarding T; the arcs of Γ/T are thus $E(\Gamma)\setminus T$. The *Whitney polynomial of the graph* Γ is

$$w_\Gamma(x,\lambda) = \sum_{T \subseteq E(\Gamma)} x^{c(T)}\chi_{\Gamma/T}(\lambda),$$

$c(T)$ being the number of components into which T connects the nodes of Γ (it is the number of

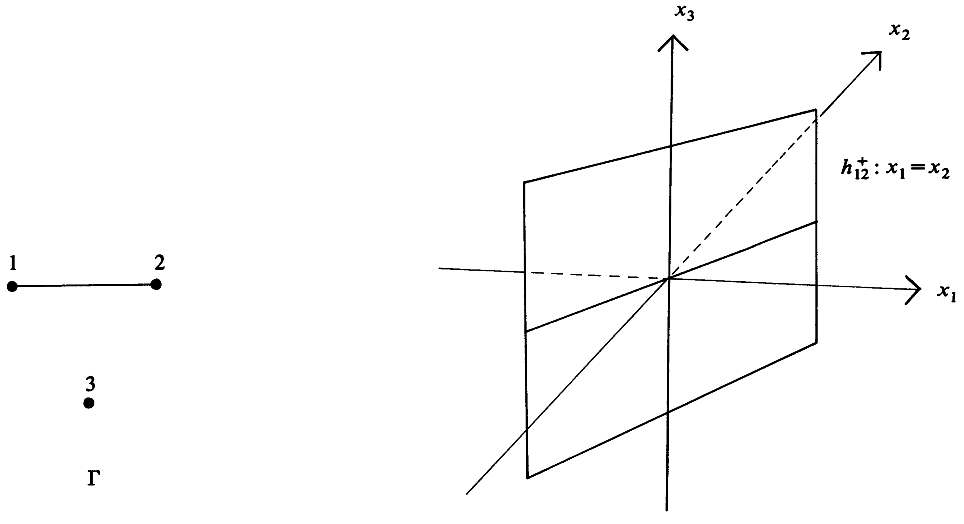


FIG. 9. A sample calculation of $\alpha_k(\Gamma)$ and $\phi_k(\Gamma)$ for use in Corollaries 10 and 11. You can see by inspecting Γ that

$\alpha_1(\Gamma)=0$ since Γ is not connected,
$\alpha_2(\Gamma)=1$ since N has one partition into 2 connected blocks,
$\alpha_3(\Gamma)=1$ since N has one partition into 3 connected blocks.

The ϕ_k can (says Corollary 11) be seen in the picture of $H[\Gamma]$:

$\phi_0(\Gamma)=0$ since $H[\Gamma]$ has no vertices,
$\phi_1(\Gamma)=0$ since it has no 1-faces (rays or edges),
$\phi_2(\Gamma)=1$ since it has one 2-face (the plane h_{12}^+),
$\phi_3(\Gamma)=2$ since h_{12}^+ divides the space into two regions.

nodes of Γ/T). I should explain that $\chi_{\Gamma/T}(\lambda)\equiv 0$ if there is an arc $e\notin T$ whose endpoints are connected by T; because Γ/T then has a loop, so no colorings. If there is no such arc, T is called *closed*. Then $\chi_{\Gamma/T}(\lambda)$ is monic of degree $c(T)$.

THEOREM 9. *Let Γ be a graph on the node set $N=\{1,2,\ldots,n\}$ and $c(\Gamma)=$the number of components of Γ. The Whitney polynomial of the graphic arrangement $H[\Gamma]$ is*

$$w_{H[\Gamma]}(x,\lambda)=\lambda^{-c(\Gamma)}w_\Gamma(x,\lambda).$$

That of the "special" arrangement $H=H[\pm\Gamma^\circ]$ is

$$w_H(x,\lambda)=\sum_{W\subseteq N} x^{n-\#(W)}2^{\#(W)}w_{\Gamma:W}\left(x,\frac{\lambda-1}{2}\right).$$

Let U be a subset of the node set. The Whitney polynomial of the sign-symmetric arrangement $H=H[\pm\Gamma^U]$ is

$$w_H(x,\lambda)=\lambda^{-i(N)}\sum_{W\subseteq N} x^{\#(W^c)-i(W^c)}2^{\#(W)}w_{\Gamma:W}\left(x,\frac{\lambda-1}{2}\right),$$

where $W^c=N\backslash W$ and $i(Y)=$the number of isolated nodes of $\Gamma:Y$ which lie outside U.

COROLLARY 10. *Let $\alpha_k(\Gamma)=$the number of partitions of the nodes of Γ into k connected blocks. Then*

$$a_k(H[\Gamma])=\alpha_k(\Gamma),$$

$$a_k(H[\pm\Gamma^\circ]) = \sum_{W\subseteq N} 2^{\#(W)-k}\alpha_k(\Gamma:W).$$

If $U\subseteq N$,

$$a_k(H[\pm\Gamma^U]) = \sum_W 2^{\#(W)-k}\alpha_k(\Gamma:W)$$

summed over those node sets W *such that all the isolated nodes of* $\Gamma: W^c$ *are in* U.

To prove the case $H=H[\pm\Gamma^U]$ we only have to look at the terms of highest degree in $\lambda^{i(N)}w_H(x,\lambda)$, for a_k is the coefficient of $x^{n-k}\lambda^k$. I should point out that $d(H)=i(N)$. You can satisfy yourself that there is a one-to-one correspondence between closed arc sets and the partitions of Γ enumerated by α, so the terms of highest degree in $w_{\Gamma:W}(x,\frac{1}{2}(\lambda-1))$ are $\alpha_k(\Gamma:W)x^{\#(W)-k}\lambda^k 2^{-k}$. The rest is easy.

The analog for f_k requires the numbers $\phi_l(\Gamma)$, defined by

$$\phi_l(\Gamma) = \sum_{T_l}(-1)^l\chi_{\Gamma/T_l}(-1)$$

summed over all the sets of arcs T_l which connect the nodes of Γ into l blocks.[10] Because a summand is 0 if T_l is not closed and is the number of regions in the arrangement $H[\Gamma/T_l]$ if T_l is closed, ϕ_l is positive. Given the definitions of ϕ_l and w_H and the fact that $f_k(H)$ is the coefficient of x^{n-k} in $(-1)^{n-i(N)}w_H(-x,-1)$, the deduction of Corollary 11 from Theorem 9 is merely a series of formal manipulations.

TABLE 1.

The numbers of flats and faces of some signed-graphic arrangements of planes, calculated by Corollaries 10 and 11 and the data in Fig. 9. The arrangements are $H[\pm\Gamma^U]$, where Γ is the graph in Fig. 9 and U is various.

(a) The arrangement $H[\pm\Gamma^\circ]$ (i.e., $U=N=\{1,2,3\}$). This arrangement is depicted in Fig. 6 (a).

k	0	1	2	3
Number of k-flats, a_k	1	5	5	1
Number of k-faces, f_k	1	10	24	16

(b) The arrangement $H[\pm\Gamma^{\{1,3\}}]$, depicted in Fig. 7 (a).

k	0	1	2	3
a_k	1	4	4	1
f_k	1	8	18	12

(c) The arrangement $H[\pm\Gamma^{\{1\}}]$, depicted in Fig. 8 (a).

k	0	1	2	3
a_k	0	1	3	1
f_k	0	1	6	6

(d) The arrangement $H[\pm\Gamma]$ (i.e., $U=\varnothing$), which consists of the two planes $x_1=\pm x_2$, meeting in the line $x_1=x_2=0$.

k	0	1	2	3
a_k	0	1	2	1
f_k	0	1	4	4

COROLLARY 11. *The face numbers of graphic and sign-symmetric arrangements are given by*

$$f_k(H[\Gamma]) = \phi_k(\Gamma),$$

$$f_k(H[\pm\Gamma^\circ]) = \sum_{W\subseteq N} 2^{\#(W)}\phi_k(\Gamma:W),$$

$$f_k(H[\pm\Gamma^U]) = \sum_{i=0}^{k}(-1)^i \sum_{W:\, i(W^c)=i} 2^{\#(W)}\phi_{k-i}(\Gamma:W).$$

When we are dealing with the root system arrangements A^*_{n-1}, B^*_n, and D^*_n, or with $D^{*(p)}_n = D^*_n$ plus p coordinate hyperplanes, Γ is the complete graph K_n. Every partition of the nodes is into connected blocks, so $\alpha_j(K_n) = S(n,j)$, the number of partitions of n objects into j blocks: the well-known Stirling number of the second kind. The contraction K_n/T_l, where T_l is closed, is a loop-free complete graph; its chromatic polynomial is therefore $(\lambda)_l$. Since there are $S(n,l)$ such sets T_l, $\phi_l(K_n) = S(n,l)l!$. Corollary 12 collects all the formulas we can now derive from Corollaries 10 and 11. The calculations are straightforward enough except that one needs the identity $S(n,k)/k = S(n-1,k) + S(n-1,k-1)/k$ in the evaluation of $f_k(D^{*(p)}_n)$.

COROLLARY 12. *The flat and face numbers of* A^*_{n-1}, B^*_n, *and* $D^{*(p)}_n$ *are:*

$$a_k(A^*_{n-1}) = S(n,k),$$

$$f_k(A^*_{n-1}) = k!\, S(n,k),$$

$$a_k(B^*_n) = 2^{-k}\left[\sum_{j=k}^{n} 2^j\binom{n}{j}S(j,k)\right],$$

$$f_k(B^*_n) = k!\left[\sum_{j=k}^{n} 2^j\binom{n}{j}S(j,k)\right],$$

$$a_k(D^{*(p)}_n) = 2^{-k}\left[\sum_{j=k}^{n} 2^j\binom{n}{j}S(j,k) - (n-p)2^{n-1}S(n-1,k)\right],$$

$$f_k(D^{*(p)}_n) = k!\left[\sum_{j=k}^{n} 2^j\binom{n}{j}S(j,k) - (n-p)2^n\frac{1}{k}S(n,k)\right].$$

10. The End . . . That wraps up my discussion of subarrangements of the classical root system arrangements of hyperplanes. I believe I have made a case for the claim that just about anything about graphic and sign-symmetric arrangements can be reduced to ordinary graph theory. Arrangements which are neither graphic nor sign-symmetric can also be handled, but it takes a theory of signed graphs.[11]

A kind of question I have only touched on concerns the connection between the geometry of the arrangements based on an ordinary graph Γ and the combinatorics of Γ. Curtis Greene's discovery that regions of $H[\Gamma]$ correspond to acyclic orientations of Γ (see Section 7) suggests what one might find. As a matter of fact, regions of $H[\Sigma]$ correspond to acyclic orientations of the signed graph Σ. From that one can derive a combinatorial description of the way a region of $H[\Gamma]$ is subdivided when one passes to $H[\pm\Gamma^\circ]$. I believe there are more good problems along this line; certainly looking at geometry will lead to new ideas about graphs, and vice versa, as the connection between them is made increasingly strong.

Acknowledged (with pleasure). This work is a sea child, born on the warm, slow cruise of the *Rachael and Ebenezer* out of Rockland, Maine. To her and her people, many thanks. Thanks also to my colleagues at MIT, Joe Kung, Richard Stanley, and Jay Sulzberger, for their interest in and encouragement of signed graphs. To Jeff Lagarias and Bob Proctor for reading and editing the manuscript. To the NSF and SGPNR for their financial support. And foremost to Fred Supnick, who initiated me into combinatorial geometry, whose enthusiastic support sustained me mathematically through several hard, lean years, and who deserves many dedications.

Notes

1. For root systems and their connection to Lie algebras, see any book on Lie algebras, such as those of Wan, Adams, and Serre, or the excellent brief account of Veldkamp in [6, Section 3]. There are several definitions, all equivalent either to ours or to the slightly broader one adopted by Serre and Veldkamp (who call our root systems "reduced").

2. The characteristic polynomial is usually defined in terms of the lattice of flats; for this see [14]. The proof of Theorem 2 was found first by Winder, later (independently and in more generality) by me.

3. The first analysis of deletion and contraction invariants (called by Brylawski "Tutte-Grothendieck invariants") was Tutte's study of graphs. Later Brylawski extended the analysis to matroids (of which arrangements of hyperplanes are typical examples).

4. Signed graphs and some basic notions were invented by Harary. The connection between signed graphs and characteristic polynomials is implicit in Dowling's article, although only the case corresponding to B_n^* was discussed there.

5. For the characteristic (or rather, chromatic) polynomial of an arbitrary signed graph, see [16].

6. This is a classical proof. For various properties of chromatic polynomials, including a proof that they are polynomials, see Read's survey or a book on graph theory.

7. For a proof of this vector representation of a graph, see Theorem 2 in Section 9.5 of Welsh's book.

8. The Whitney polynomial was introduced in [14], there called the "Möbius" polynomial. Specialists will realize that "Whitney" is a better name, because the coefficients of $x^k\lambda^{n-k-d(H)}$ and $x^0\lambda^{n-k-d(H)}$ are the Whitney numbers W_k and w_k of the lattice of flats.

9. A proof from a different viewpoint appears in [16].

10. The ϕ_l have a combinatorial meaning for Γ. A theorem of Stanley's implies that the number of acyclic orientations of all contraction graphs Γ/T_l is $\phi_l(\Gamma)$.

11. See [15] and [16].

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15. ______, Signed graphs (submitted).

16. ______, Signed graph coloring (submitted).