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HOW COLORFUL THE SIGNED GRAPH?

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The zero-free chromatic number χ^* of a signed graph Σ is the smallest positive number k for which the vertices can be colored using $\pm 1, \pm 2, \ldots, \pm k$ so the endpoints of a positive edge are not colored the same and those of a negative edge are not colored oppositely. We establish the value of χ^* for some special signed graphs and prove in general that χ^* equals the minimum size of a vertex partition inducing an antibalanced subgraph of Σ, and also the minimum chromatic number of the positive subgraph of any signed graph switching equivalent to Σ. We characterize those signed graphs with the largest and smallest possible χ^*, that is n, $n-1$, and 1, and the simple ones with the maximum and minimum χ^*, that is $\lceil n/2 \rceil$ and 1, where n is the number of vertices. We give tighter bounds on χ^* in terms of the underlying graphs, but they are not sharp. We conclude by observing that determining χ^* is an NP-complete problem.

We introduce the (zero-free) chromatic number of a signed graph, a graph with edges labelled by signs. We look for structural formulas and upper and lower bounds in terms of vertex partitions, the positive and negative edge sets, and the doubly signed adjacencies. We also study the signed graphs with the largest or the smallest chromatic number having given order, underlying graph, or doubly signed adjacencies, and we characterize the extremal examples among all signed graphs and among signed simple graphs.

Signed graphs and balance were first defined by Harary [4]; coloring[1] was introduced in [7]. A *signed graph* $\Sigma = (\Gamma, \sigma)$ consists of a graph $\Gamma = (V, E)$, which may have loops and multiple edges,[2] and a sign function $\sigma : E \to \{+, -\}$. We also denote the underlying graph Γ by $|\Sigma|$. The *order* of Γ, or Σ, is $n = n(\Gamma) = n(\Sigma) = |V|$; we assume $n \geq 1$. The complement of a vertex set $X \subseteq V$ is denoted by $X^c = V \backslash X$; and Γ^c denotes the complementary graph of a simple graph Γ. The positive edge set of Σ is $E_+ = \sigma^{-1}(+)$; the positive part of Σ is the all-positive subgraph $\Sigma_+ = (V, E_+, +)$. Similarly we define E_- and Σ_-. The negative of Σ is $-\Sigma = (\Gamma, -\sigma)$.

A subgraph Σ_1 is *balanced* if every circuit has positive sign product, *antibalanced* if $-\Sigma_1$ is balanced. If X and Y are disjoint subsets of V, the *balanced-induced subgraph* $\Sigma \cdot \langle X, Y \rangle$ is the signed graph whose vertex set is $X \cup Y$, whose

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edge set is $\{e \in E: \sigma(e) = +$ and e has both endpoints in X or both in Y, or else $\sigma(e) = -$ and e has one endpoint in X and the other in $Y\}$, and whose signs are as in Σ. An easy but fundamental criterion for balance is

Lemma 1 ([4]; also see [6, Theorem X.11]). *Σ is balanced if and only if there is a partition of V into $X \cup X^c$ so that every positive edge has both ends in X or both in X^c and every negative edge has one end in X and one in X^c.*

Switching Σ by $X \subseteq V$ means reversing the signs of all edges with one endpoint in X and the other in X^c. A signed graph Σ' obtained by switching Σ is said to be *switching equivalent* to it, written $\Sigma' \sim \Sigma$. A basic and easy theorem is that $\Sigma_1 \sim \Sigma_2$ if and only if Σ_1 and Σ_2 have the same underlying graph and the same balanced circuits. It follows that Σ can be switched to be all positive (all negative) if and only if it is balanced (antibalanced).

A *zero-free coloring* of *Σ in k (unsigned) colors*, where k is a non-negative integer, is a mapping $c: V \to \{\pm 1, \pm 2, \ldots, \pm k\}$. (A *coloring* allows 0 as another color. Here we treat only zero-free colorings, but there is a close relationship: see [8, Section 1].) It is *proper* if the endpoints of each positive edge have different signed colors and those of each negative edge have colors that are not negatives of each other. Thus unbalanced loops do not affect the properness of a zero-free coloring, but a signed graph with a positive loop has no proper colorings. The *zero-free chromatic number* of Σ, denoted $\chi^*(\Sigma)$, is the smallest number of (unsigned) colors for which Σ has a proper zero-free coloring. Since we treat only zero-free colorings in this article, we will omit the modifier, speaking only of 'colorings'.

The chromatic number of an unsigned graph is denoted $\chi(\Gamma)$.

Some particular signed graphs are: The all-positive signed graph $+\Gamma$, denoting $(\Gamma, +)$ or the graph Γ with all edges positive. The all-negative signed graph $-\Gamma$. The signed expansion $\pm\Gamma = +\Gamma \cup -\Gamma$ (meaning the edge-disjoint union of two graphs on the same vertices). The signed complete graphs, (K_n, σ), also written K_Γ to signify the signed K_n on vertex set $V(\Gamma)$ with negative edge set $E(\Gamma)$. These should not be confused with complete signed graphs, where every vertex pair is adjacent at least once. Important examples of the latter that are not signed K_n's are $+\Gamma \cup -K_n$ and $-\Gamma \cup +K_n$, which have respectively all possible negative (or positive) adjacencies and the positive (or negative) adjacencies of Γ; here again Γ and K_n are assumed to have the same vertex sets.

For some of these examples the chromatic number is quite easy to determine. Let $\lceil x \rceil$ denote the least integer $\geqslant x$. We have

$$\chi^*(+\Gamma) = \lceil \tfrac{1}{2}\chi(\Gamma) \rceil, \tag{1}$$

since coloring $+\Gamma$ is like coloring Γ but we save in the count by using both signs of each unsigned color value;

$$\chi^*(-\Gamma) = 1, \tag{2}$$

since we can color every vertex the same; and

$$\chi^*(\pm\Gamma) = \chi(\Gamma), \tag{3}$$

for since every edge appears doubled with both signs, the color signs play no role.

Because adding edges only makes a graph harder to color properly, we have the useful fact that

$$\chi^*(\Sigma_1) \leqslant \chi^*(\Sigma_2) \quad \text{if } \Sigma_1 \text{ is a subgraph of } \Sigma_2. \tag{4}$$

To define *switching of a colored graph,* if we switch Σ by X, we must also negate all the colors of the vertices in X. Then it is clear that switching Σ does not affect its chromatic number.

The basic formulas for the chromatic number are given in Theorem 1. A vertex set is called an *(anti)balanced set* in Σ if it induces an (anti)balanced subgraph.

Theorem 1. *The zero-free chromatic number $\chi^*(\Sigma)$ is equal to*

(a) *the mininum number of antibalanced sets into which V can be partitioned, and*

(b) *the minimum of $\chi(|\Sigma'_+|)$ over all $\Sigma' \sim \Sigma$.*

Proof. (a) Let p denote the minimum partition size. Let c be a minimal proper coloring. By definition of properness $c^{-1}(\pm i)$ is an antibalanced set for each i. Thus $p \leqslant \chi^*$.

Conversely let V be partitioned into p antibalanced sets $V_1, V_2, \ldots, V_p$. By Lemma 1, each V_i splits into two subsets (possibly void) such that each edge in a subset is negative and each edge between subsets is positive. If we color the subsets respectively $+i$ and $-i$, we have a proper coloring. Thus $\chi^* \leqslant p$.

(b) Suppose we color $|\Sigma_+|$ by the positive colors $+1, +2, \ldots, +\chi(|\Sigma_+|)$. Then all positive edges of Σ are properly colored, and so are all negative edges, so $\chi^* \leqslant \chi(|\Sigma_+|)$.

Conversely let V be partitioned into χ^* antibalanced sets V_i. We can switch Σ to Σ' in which the edges lying in each antibalanced set V_i are all negative. Then clearly $\chi(|\Sigma'_+|) \leqslant \chi^*$. Part (b) follows. □

It would be desirable to have a 'bottom' analog of Theorem 1(b), say of the form $\chi^*(\Sigma) = \max_{\Sigma' \sim \Sigma} f(|\Sigma'_+|)$ for some function f.

Theorem 1 implies (1)–(3) and also (5)–(7) now stated. Let $m(\Gamma)$ denote the size of a largest matching in Γ. We have

$$\chi^*(+\Gamma \cup -K_n) = \chi(\Gamma), \tag{5}$$

$$\pm\Gamma \subseteq \Sigma \subseteq +\Gamma \cup -K_n \ \text{ implies } \ \chi^*(\Sigma) = \chi(\Gamma). \tag{6}$$

These results follow from (3) and (4) since $+\Gamma \cup -K_n$ clearly has a proper coloring using $\chi(\Gamma)$ positive colors. And

$$\chi^*(-\Gamma \cup +K_n) = n - m(\Gamma^c). \tag{7}$$

The reason: the $+K_n$ prevents any antibalanced set from having more than two vertices. The $-\Gamma$ prevents an antibalanced pair that is adjacent in Γ; thus the antibalanced pairs of a proper coloring form a matching in Γ^c. We conclude that $\chi^* \geqslant n - m(\Gamma^c)$. Conversely a maximum matching in Γ^c yields a proper coloring in $n - m(\Gamma^c)$ colors, whence $\chi^* \leqslant n - m(\Gamma^c)$. Formula (7) follows.

Formulas (5) and (7) show that the chromatic number problem for signed graphs includes both that for ordinary graphs and the maximum matching problem. This can also be seen from [8, Section 6]; in fact, (5) and (7) follow from the results there.

Corollary 1. *Let Σ have no loops. Then $\chi^*(\Sigma) = n$ if $\Sigma = \pm K_n$; $\chi^*(\Sigma) = n-1$ if Σ consists of $\pm K_n$ with either a nonvoid set of edges at one vertex, or an unbalanced triangle, removed; and otherwise $\chi^*(\Sigma) \leqslant n-2$. And $\chi^*(\Sigma) = 1$ if and only if Σ is antibalanced.*

Proof. The values given are obvious, as are the characterizations for chromatic number 1 and n. Suppose now that $\chi^*(\Sigma) = n-1$, and let $E_0 = E(\pm K_n) \setminus E(\Sigma)$. There can be no two nonadjacent edges in E_0. Thus either E_0 consists of some edges at one vertex, or it is a triangle, with perhaps doubled edges, on vertices $u, v, w \in V$. If E_0 is anything but an unbalanced triangle, then $\{u, v, w\}$ is an antibalanced set and $\chi^*(\Sigma) \leqslant n-2$. □

Now we come to deduce bounds on the chromatic number. An obvious lower bound is

$$\chi^*(\Sigma) \geqslant \lceil \tfrac{1}{2}\chi(|\Sigma_+|) \rceil. \tag{8}$$

That the bound is not sharp even up to switching is shown by $\Sigma = \pm\Gamma$. So it will not give a 'bottom' formula for χ^*.

Let Δ_Σ be the graph of doubly signed adjacencies in Σ: that is, v and w are Δ_Σ-adjacent if they are both positively and negatively adjacent in Σ. Then we have

$$\chi(\Delta_\Sigma) \leqslant \chi^*(\Sigma) \leqslant n - m(\Delta_\Sigma^c). \tag{9}$$

The first inequality follows from (3) and (4). For the second, choose a maximum matching in Δ_Σ^c and use it to color Σ as in the proof of (7).

Although these are quite weak bounds, it is regrettably difficult to find all the graphs satisfying them exactly. But for signed complete graphs we can do so. The next result gives sharp absolute bounds (in terms of the order) for all signed simple graphs.

Theorem 2. *Let Σ be a signed simple graph. Then $\chi^*(\Sigma) \leq \lceil n/2 \rceil$, with equality precisely when Σ is complete and balanced, or n is even and Σ contains a balanced K_{n-1}, or $n=4$ and Σ is an unbalanced 4-circuit, or $n=6$ and Σ switches to K_Γ where Γ is a 5-circuit plus an isolated vertex. Also $\chi^*(\Sigma) \geq 1$, with equality precisely when Σ is antibalanced.*

Proof. The upper bound follows directly from Corollary 1. The lower bound is trivial; the case of equality follows from Theorem 1(a).

As for equality in the upper bound, it holds by (1) when Σ is complete and balanced. Suppose n is odd, $n = 2m+1$, and Σ is incomplete or unbalanced. Then it has three vertices not all adjacent, or it is complete and hence has an unbalanced triangle [2, Theorem 3.2]. Since the remaining $2m-2$ vertices can be divided into $m-1$ antibalanced pairs, we have $\chi^* \leq m < \lceil n/2 \rceil$.

Now let n be even. If Σ contains a balanced K_{n-1}, then (1) and (4) imply $\chi^* = n/2$. This takes care of the case $n=2$. When $n=4$, it leaves only antibalanced Σ's, for which $\chi^* = 1$, and the unbalanced 4-circuit, for which clearly $\chi^* = 2$. So let $n \geq 6$.

Suppose Σ has maximum degree less than $n-1$. If $|\Sigma|^c$ has two non-adjacent edges, then (since $n \geq 6$) Σ contains two disjoint antibalanced vertex triples; then $\chi^* < n/2$. Otherwise, Σ is a signed K_{n-1} plus one isolated vertex, and by the odd case $\chi^* = n/2$ if and only if the K_{n-1} is balanced.

We now assume that $\chi^* = n/2$ and Σ has a vertex v_n of degree $n-1$. Let us switch so v_n has only positive edges, and set $\Psi = \Sigma \backslash v_n$. If $(\Psi_+)^c$ contained a triangle uvw, then $uvwv_n$ would be an antibalanced vertex quadruple and $\chi^* < n/2$ by Theorem 1(a). So $(\Psi_+)^c$ is triangle-free. If it contains no two nonadjacent edges, then it is a *claw*: a graph whose edges all meet at a common vertex, say u. In that case $\Sigma \backslash u$ is a balanced K_{n-1}.

Assume $(\Psi_+)^c$ contains two nonadjacent edges, say uv and wx. If u and v were not adjacent in Σ, then uvz and wxv_n (where z is any sixth vertex) would be disjoint antibalanced triples in Σ, hence $\chi^* < n/2$. So Σ must be complete and $(\Psi_+)^c = \Psi_-$. If there were a vertex z nonadjacent in Ψ_- to u and v, then uvz and wxv_n would be disjoint antibalanced triples in Σ. Therefore every vertex other than u, v, w, x is a neighbor in Ψ_- of u or v (but not both) and w or x (but not both). So if Ψ_- contains a 5-circuit, it must be an induced one. Suppose Ψ_- does contain a 5-circuit. When $n=6$, this results in a Σ with $\chi^* = 3$, since Ψ cannot be colored properly in two unsigned colors without using all four signed colors. When $n > 6$, there is a seventh vertex z, which must be adjacent to exactly one end of each edge of the circuit; but that is impossible. So there is no induced 5-circuit in Ψ_- when $n > 6$. A longer induced circuit is impossible because it would contain edges uv and wx and a vertex z not adjacent to either u or v. Hence Ψ_- is bipartite. Moreover, it is connected. Finally, it is complete. For suppose its diameter were at least 3. Then there would be an induced path $uvwx$ and another vertex z. Suppose z is adjacent to v: then u is adjacent to x or w, hence to x by biparticity. Suppose z is adjacent to u: then x is adjacent to u or z, hence to u by biparticity. But one of those cases must apply. So Ψ_- has diameter 2, which makes it a complete bipartite graph on $V\backslash v_n$. Now switch Σ so Ψ is all positive. In the new signed graph, the negative edges form a claw. This proves the theorem. □

A stronger lower bound on $\chi^*(K_\Gamma)$ is that of formula (8), which becomes $\chi^*(K_\Gamma) \geq \lceil \frac{1}{2}\chi(\Gamma^c) \rceil$. There are many graphs Γ yielding equality; it would be interesting to see them classified, but I suspect this is difficult.

Finally, we observe that $\chi^*(\Sigma)$ is computationally about as difficult as $\chi(\Gamma)$. (For general background see for instance [3].)

Proposition 1. *The problem* (SQ_k) *"Is* $\chi^*(\Sigma) \leq k$?" *is NP-complete if* $k \geq 2$, *and polynomially bounded if* $k = 1$.

Proof. Obviously all (SQ_k) are in the class NP. The first case, (SQ_1), is equivalent to determining balance of $-\Sigma$, which is solvable in time quadratic in n [5]. For $k \geq 3$, (SQ_k) contains as a subproblem (the case $\Sigma = \pm\Gamma$, by Eq. (3)) the NP-complete problem (Q_k) "Is $\chi(\Gamma) \leq k$?" And (SQ_2) contains (Q_4) as the special case $\Sigma = +\Gamma$. □

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