Line Graphs Eigenvalues and Root Systems

Thomas Zaslavsky

Binghamton University (State University of New York)

C R Rao
Advanced Institute of
Mathematics, Statistics and Computer Science

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1. What is a Line Graph?

Graph $\Gamma = (V, E)$: Simple (no loops or multiple edges).

$$V = \{v_1, v_2, \ldots, v_n\}, \qquad E = \{e_1, e_2, \ldots, e_m\}.$$

Line graph $L(\Gamma)$:

$$V_L := E(\Gamma), \text{ and } e_k e_l \in E_L \iff e_k, e_l \text{ are adjacent in } \Gamma.$$

Adjacency matrix:

$$A(\Gamma) := (a_{ij})_{i,j \leq n} \text{ with } a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent,} \\ 0 & \text{if not,} \\ 0 & \text{if } i = j. \end{cases}$$

Unoriented incidence matrix:

$$\mathrm{H}(-\Gamma) = (\eta_{ik}(-\Gamma))_{i \leq n, k \leq m} \text{ where } \eta_{ik}(-\Gamma) = \begin{cases} 1 & \text{if } v_i, e_k \text{ are incident,} \\ 0 & \text{if not.} \end{cases}$$

Oriented incidence matrix:

$$\mathrm{H}(+\Gamma) = (\eta_{ik}(+\Gamma))_{i \leq n, k \leq m} \text{ where } \eta_{ik}(+\Gamma) = \begin{cases} \pm 1 & \text{if } v_i, e_k \text{ are incident,} \\ 0 & \text{if not,} \end{cases}$$

in such a way that there are one $+1$ and one -1 in each column.

Kirchhoff ('Laplacian') matrix of Γ:

$$K(+\Gamma) := \mathrm{H}(+\Gamma)\mathrm{H}(+\Gamma)^{\mathrm{T}} = D(\Gamma) - A(\Gamma),$$
$$K(-\Gamma) := \mathrm{H}(-\Gamma)\mathrm{H}(-\Gamma)^{\mathrm{T}} = D(\Gamma) + A(\Gamma),$$

where $D(\Gamma) := \mathrm{diag}(d(v_i))_i$ is the degree matrix of Γ.

Theorem 1.1. $\mathrm{H}(-\Gamma)^{\mathrm{T}}\mathrm{H}(-\Gamma) = 2I + A(L(\Gamma))$.

2. What is an Eigenvalue?

Eigenvalue of Γ: An eigenvalue of the adjacency matrix, $A(\Gamma)$.

Stage 1 of the History of Eigenvalues ≥ -2.

Corollary 2.1. *All eigenvalues of* $L(\Sigma)$ *are* ≥ -2.

Thus began the hunt for graphs whose eigenvalues are ≥ -2.
Hope: They are line graphs and no others.
Hope is disappointed.

Stage 2 of the History of Eigenvalues ≥ -2.

Generalized line graph $L(\Gamma; r_1, \ldots, r_n)$:

- The vertex star $E(v_i) := \{e_k : e_k \text{ is incident with } v_i\} \to$ *vertex clique* in L.
- A *cocktail party graph* $\mathrm{CP}_r := K_{2r} \setminus M_r$ (a perfect matching).

Alan Hoffman:

$$L(\Gamma; r_1, \ldots, r_n) := L(\Gamma) \text{ with } \mathrm{CP}_{r_i} \text{ joined to the vertex clique of } v_i.$$

(*Joined* means use every possible edge.)

Theorem 2.2. *All eigenvalues of* $L(\Gamma; r_1, \ldots, r_n)$ *are* ≥ -2.

A mystery!

Stage 3 of the History of Eigenvalues ≥ -2.

Solution by Cameron, Goethals, Seidel, & Shult:

3. WHAT IS A ROOT SYSTEM?

Root system:
A finite set $R \subseteq \mathbb{R}^d$ such that

RS1. $x \in R \implies -x \in R$ (central symmetry),

RS2. $x, y \in R \implies 2\dfrac{x \cdot y}{x \cdot x} \in \mathbb{Z}$ (integrality),

RS3. $x, y \in R \implies y - 2\dfrac{x \cdot y}{x \cdot x} x \in R$ (reflection in $y^{\perp}$),

RS4. $0 \notin R$.

Observation: $\big(R_1 \times \{0\}\big) \cup \big(\{0\} \times R_2\big) \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is a root system, called 'reducible'.

Origin: Classification of simple, finite-dimensional Lie groups and algebras by classifying irreducible root systems. The classification:

$$\begin{aligned} A_{n-1} &\cong \{x \in \mathbb{R}^n : x = \pm(b_j - b_i) \text{ for } i < j\}, \\ D_n &\cong A_{n-1} \cup \{x \in \mathbb{R}^n : x = \pm(b_j + b_i) \text{ for } i < j\}, \\ B_n &\cong D_n \cup \{b_i : i \leq n\}, \\ C_n &\cong D_n \cup \{2b_i : i \leq n\}, \end{aligned}$$

and $E_6,\ E_7,\ E_8$, where

$$E_6 \subset E_7 \subset E_8 \cong D_n \cup \{\tfrac{1}{2}(\pm b_1 \pm \cdots \pm b_8) : \text{ evenly many signs are } -\}.$$

Theorem 3.1 (Cameron, Goethals, Seidel, and Shult)**.** *Any graph with eigenvalues ≥ -2 is negatively represented by the angles of a subset of a root system D_r for some $r \in \mathbb{Z}$, or E_8.*

Negative angle representation of Γ:

$$\psi : V \to \mathbb{R}^d \text{ such that } \psi(v_i) \cdot \psi(v_j) = \begin{cases} -2a_{ij} & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases}$$

(The factor 2 is merely a normalization.)

4. What is a Signed Graph?

Signed graph:

$$\Sigma = (\Gamma, \sigma) \text{ where } \sigma : E \to \{+, -\}.$$

Examples:

$+\Gamma = \Gamma$ with all edges positive.
$-\Gamma = \Gamma$ with all edges negative.
$\pm\Delta = \Delta$ with all edges both positive and negative (2 edges for each original edge).

Underlying graph: $|\Sigma| := \Gamma$.

Positive and negative circles: Product of the edge signs.

Adjacency matrix:

$$A(\Sigma) := (a_{ij})_{i,j \leq n} \text{ with } a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are positively adjacent,} \\ -1 & \text{if } v_i, v_j \text{ are negatively adjacent,} \\ 0 & \text{if not adjacent,} \\ 0 & \text{if } i = j. \end{cases}$$

Reduced signed graph $\bar{\Sigma}$:
Delete every pair of parallel edges with opposite sign.
No effect on $A(\Sigma)$.

Incidence matrix:

$$\mathrm{H}(\Sigma) = (\eta_{ik})_{i \leq n, k \leq m} \text{ where } \eta_{ik} = \begin{cases} \pm 1 & \text{if } v_i, e_k \text{ are incident,} \\ 0 & \text{if not,} \end{cases}$$

in such a way that the the two nonzero elements of the column of e_k satisfy

$$\eta_{ik}\eta_{jk} = -\sigma(e_k).$$

Kirchhoff ('Laplacian') matrix of Σ:

$$K(\Sigma) := \mathrm{H}(\Sigma)\mathrm{H}(\Sigma)^{\mathrm{T}} = D(\Sigma) - A(\Sigma)$$

where $D(\Sigma) := \mathrm{diag}(d(v_i))_i$ is the degree matrix of the underlying graph of Σ.

5. What is the Line Graph of a Signed Graph?

Oriented signed graph: (Σ, η) where $\eta : V \times E \to \{+,-\} \cup \{0\}$ satisfies

$$\eta(v_i, e_k)\eta(v_j, e_k) = -\sigma(e_k) \text{ if } e_k{:}v_iv_j,$$

$$\eta(v_i, e_k) = 0 \text{ if } v_i \text{ and } e_k \text{ are not incident.}$$

Meaning:

$+$ denotes an arrow pointing into the vertex.
$-$ denotes an arrow pointing out of the vertex.

Bidirected graph B: Every end of every edge has an independent arrow, or, $\mathrm{B} = (\Gamma, \eta)$.
(Due to Edmonds.)

An oriented signed graph is a bidirected graph.
A bidirected graph is an oriented signed graph.

(Due to Zaslavsky.)

Line graph of Σ: $\Lambda(\Sigma) = (L(|\Sigma|), \eta_\Lambda)$ where

$$\eta_\Lambda(e_k, e_k e_l) = \eta(v_i, e_k)$$

and v_i is the vertex common to e_k and e_l.
That is:

(1) Orient Σ (arbitrarily).
(2) Construct $L(|\Sigma|)$.
(3) Treat each edge end in L as the end in Σ with the arrow turned around so it remains into, or out of, the vertex.

Proposition 5.1. *The circle signs in* $\Lambda(\Sigma)$ *are independent of the arbitrary orientation.*

Reduced line graph: $\bar{\Lambda}(\Sigma)$.

Theorem 5.2. $\mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma) = 2I - A(\Lambda(\Sigma)) = 2I - A(\bar{\Lambda}(\Sigma))$.

Corollary 5.3. *All eigenvalues of* $\Lambda(\Gamma)$*, or* $\bar{\Lambda}(\Gamma)$*, are* ≤ 2.

6. What Does It All Mean?

First Answer::

Theorem 6.1 (Cameron, Goethals, Seidel, and Shult, reinterpreted)**.** *Any signed graph with eigenvalues* ≤ 2 *is represented by the angles of a subset of a root system* D_r *for some* $r \in \mathbb{Z}$*, or* E_8*.*

Angle representation of Σ:

$$\psi : V \to \mathbb{R}^d \text{ such that } \psi(v_i) \cdot \psi(v_j) = \begin{cases} 2a_{ij} & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases}$$

(The factor 2 is merely a normalization.)

Second Answer::

Theorem 6.2. *A signed graph with eigenvalues* ≤ 2 *is either the line graph of a signed graph, or one of the finitely many signed graphs with an angle representation in* E_8*.*

Those generalized line graphs are line graphs.
$\Sigma(r_1, \ldots, r_n) := \Sigma$ with r_i negative digons attached to v_i.

Proposition 6.3. $-L(\Gamma; r_1, \ldots, r_n) = \bar{\Lambda}(-\Gamma(r_1, \ldots, r_n))$.

Mantra:

The proper context for line graphs is signed graphs.

7. What *Are* Those Line Graphs of Signed Graphs?

Theorem 7.1 (Chawathe & G.R. Vijayakumar)**.** *The signed graphs represented by angles in D_n for some n are those in which no induced subgraph is one of a certain finite list of signed graphs of order up to* 6*.*

Theorem 7.2 (G.R. Vijayakumar)**.** *The signed graphs represented by angles in E_8 are those in which no induced subgraph is one of a certain finite list of signed graphs of order up to* 10*.*

What ***graphs*** are (reduced) line graphs of signed graphs?

(1) The reduced line graphs that are all negative are $-\Delta$ for $\Delta =$ generalized line graph.
Eigenvalues of $-\Delta$ are ≤ 2; of Δ are ≥ -2.
Forbidden induced subgraphs (S.B. Rao, Singhi, and Vijayan, without signed graphs).

(2) The reduced line graphs that are all positive, $+\Delta$, are much fewer and less interesting.
Eigenvalues of $+\Delta$ are ≤ 2.

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