LOCAL MULTARY QUASIGROUPS AND TOPOLOGICAL BIASED GRAPHS

Thomas Zaslavsky

(Binghamton University, Binghamton, N.Y., U.S.A.)

E-mail address: `zaslav@math.binghamton.edu`

Abstract. We propose an approach through graph theory that may allow extending factorization theorems about multary quasigroups to local multary quasigroups.

A multary (or more precisely n-ary) *quasigroup* on a set $\mathbf{Q}$ is a function $f : \mathbf{Q}^n \to \mathbf{Q}$ such that the equation $f(x_1, x_2, \ldots, x_n) = x_0$ has a unique solution for each x_i if the values of all other variables x_j for $0 \le j \le n$, $j \ne i$, are given. One can restate this as the following property (Q):

There exist functions $f_i : \mathbf{Q}^n \to \mathbf{Q}$ for $i = 1, \ldots, n$ such that

$$
\begin{aligned}
&f(x_1, x_2, \ldots, x_n) = x_0 \\
\iff &f_i(x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n) = x_i,
\end{aligned}
$$

where $\hat{x}_i$ means the variable x_i is omitted.

One can put a topological structure on $\mathbf{Q}$ and require f and the f_i to be continuous; that defines a *continuous* quasigroup. A *local* quasigroup [1] is a partially defined continuous quasigroup; that is, f is defined on an open domain $D \subseteq \mathbf{Q}^n$ and each f_i is defined on an open domain $D_i \subseteq \mathbf{Q}^n$ and each function has property (Q) in some neighborhood of any point of its domain. (Usually, $\mathbf{Q}$ is a manifold. It may have a differentiable structure; then f and the f_i should be differentiable and one says that f defines a local differentiable quasigroup. One may have different manifolds $\mathbf{Q}_i$ for each variable x_i, but of the same dimension.)

An n-ary quasigroup has a combinatorial representation as a "biased expansion" of a circle (or cycle, or circuit) graph of length $n+1$; that is a graph with $n+1$ vertices and $(n+1)|\mathbf{Q}|$ edges and additional structure to be explained later. I described this representation in [2] and (with proofs) in [3] and used it to prove that, if f has enough factorizations—specifically, if its factorization graph is 3-connected—then it is an iterated group operation (up to isotopy, i.e., separately renaming each of the variables). Furthermore, if every ternary retract quasigroup, obtained from f by fixing all but three of the independent variables, is isotopic to an iterated group, then so is f.

My question is how to apply the method of biased expansion graphs to deduce properties of local quasigroups. In particular, can one prove a local analog of the 3-connected factorization theorem or the ternary retract theorem? My idea is that one can model a continuous quasigroup by defining a topology on a biased expansion graph, then generalize the definition to topological biased graphs in order to model local quasigroups.

It is time to define the graph theory terms precisely. A *biased graph* is a pair $(\Gamma, \mathcal{B})$ consisting of a graph Γ and a list $\mathcal{B}$ of circles in Γ such that, if two circles are in $\mathcal{B}$ and their intersection is an open path of length at least 1, then their symmetric difference (which is a circle) also lies in $\mathcal{B}$. A *biased expansion* of a graph Δ is a biased graph $(\Gamma, \mathcal{B})$ together with a projection mapping $p : \Gamma \to \Delta$ such that p is bijective on vertices and surjective on edges and, if C is a circle in Δ and e is an edge in C, and if $\tilde{P}$ is any path in Γ that projects bijectively onto $C \setminus e$, then there is a unique edge $\tilde{e} \in p^{-1}(e)$ such that $\tilde{P} \cup \{\tilde{e}\}$ is a circle in $\mathcal{B}$. In particular, if Δ is a circle $e_0 e_1 \cdots e_n$, then one can identify $p^{-1}(e_i)$ with the possible values of the variable x_i in such a way that a circle $\tilde{e}_0 \tilde{e}_1 \cdots \tilde{e}_n \in \mathcal{B}$ which simply covers Δ corresponds with an $(n+1)$-tuple that solves the equation $f(x_1, x_2, \ldots, x_n) = x_0$. (See [2, 3].) This makes a biased expansion model of the quasigroup.

A subgraph of Γ is called *balanced* if every circle in it belongs to $\mathcal{B}$. The idea for modelling local quasigroups is that points of the domain D correspond to maximal balanced subgraphs of $(\Gamma, \mathcal{B})$. One puts a topology on the set of maximal balanced subgraphs in a way that is analogous to how the topology on D respects the local quasigroup operation f. I will try to make this more precise in the talk.

References

[1] V.V. Goldberg, *Local differentiable quasigroups and webs.* In: O. Chein, H.O. Pflugfelder, and J.D.H. Smith, eds., "Quasigroups and Loops: Theory and Applications", Ch. X, pp. 263–311. Sigma Ser. Pure Math., Vol. 8. Heldermann, Berlin, 1990.

[2] Thomas Zaslavsky, *Quasigroup associativity and biased expansion graphs*, Electron. Res. Announc. Amer. Math. Soc. **12** (2006), 13–18.

[3] Thomas Zaslavsky, *Associativity in multary quasigroups: The way of biased expansions*, submitted. `arXiv:math.CO/0411268v2`.