Signed Graphs and Geometry

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Contents

INTRODUCTION

These notes and lectures are a personal introduction to signed graphs, concentrating on the aspects that have been most persistently interesting to me. They are just a few corners of the theory; I am leaving out a great deal. The emphasis is on the way signed graphs arise naturally from geometry, especially from the geometry of the classical root systems. Most of the properties I discuss generalize those of unsigned graphs, but the constructions and proofs are often more complicated.

The arrangement of the notes is topical, not historical. In the lectures I will talk about the historical development, but in the notes the purpose is to provide a printed reference, with a few proofs.

For a fairly comprehensive list of articles on signed graphs, generalizations, and related work see [BSG]; for (much of the) terminology see [Glo]. The principal reference for most of the more elementary properties of signed graphs treated here is Zaslavsky (1982a). A simple introduction to the hyperplane geometry is Zaslavsky (1981a). Citations in the style [BSG, Name (yeara)] refer to author Name's item (yeara) in [BSG].Many of my articles can be downloaded from my Web site,

`http://www.math.binghamton.edu/zaslav/Tpapers/`

Now, bon voyage! *Suffa yathra.*

1. GRAPHS

In these lectures all graphs are finite.

A graph is $\Gamma = (V, E)$, where $V := V(\Gamma)$ is the *vertex set* and $E := E(\Gamma)$ is the *edge set*. Notation:

- $n := |V|$, called the *order* of Γ.
- $V(e)$ is the multiset of vertices of the edge e.
- If $S \subseteq E$, $V(S)$ is the set of endpoints of edges in S.
- If $X \subseteq V$, its *complement* is $X^c := V \setminus X$.

Edges and edge sets:

- We allow multiple edges as well as loops and oddballs called half and loose edges.
- There are four kinds of edge: A *link* has two distinct endpoints. A *loop* has two equal endpoints. A *half edge* has one endpoint. A *loose edge* has no endpoints.
- An *ordinary edge* is a link or a loop. An *ordinary graph* is a graph in which every edge is a link or a loop. A *link graph* is a graph whose edges are links.
- The set of loose edges of Γ is $E_0(\Gamma)$. The set of ordinary edges of Γ is $E_* := E_*(\Gamma)$.
- Edges are *parallel* if they have the same endpoints. A *simple graph* is a link graph with no parallel edges.
- If $S \subseteq E$, $S^c := E \setminus S$ is its *complement*.
- $E(X, Y)$, where $X, Y \subseteq V$, is the set of edges with one endpoint in X and the other in Y. (Every such edge must be a link or, if $X \cap Y \neq \varnothing$, a loop.)
- A *cut* or *cutset* is an edge set $E(X, X^c)$ that is nonempty.

Vertices and vertex sets in Γ: Let $X \subseteq V$.

- An *isolated vertex* is a vertex that has no incident edges; i.e., a vertex of degree 0.
- X is *stable* or *independent* if $E{:}X = \varnothing$.

Degrees and regularity:

- The *degree* of a vertex v, $d(v) := d_\Gamma(v)$, is the number of edges of which v is an endpoint, but a loop counts twice.
- Γ is *regular* if every vertex has the same degree. If that degree is k, it is *k-regular*.

Walks, trails, paths, circles:

- A *walk* is a sequence $v_0e_1v_1 \cdots e_lv_l$ where $V(e_i) = \{v_{i-1}, v_i\}$ and $l \geq 0$. Its *length* is l. A walk may be written $e_1e_2 \cdots e_l$ or $v_0v_1 \cdots v_l$.
- A *closed walk* is a walk where $l \geq 1$ and $v_0 = v_l$.
- A *trail* is a walk with no repeated edges.
- A *path* or *open path* is a trail with no repeated vertex, or the graph of such a trail (technically, the latter is a *path graph*), or the edge set of a path graph.
- A *closed path* is a closed trail with no repeated vertex other than that $v_0 = v_l$. (Thus, a closed path is not a path.)
- A *circle* (also called 'cycle', 'polygon', etc.) is the graph, or the edge set, of a closed path. Equivalently, it is a connected, regular graph with degree 2, or its edge set.
- $\mathcal{C} = \mathcal{C}(\Gamma)$ is the class of all circles in Γ.

Examples:

- K_n is the complete graph of order n. K_X is the complete graph with vertex set X.
- K_n^c is the edgeless graph of order n.
- Γ^c is the complement of Γ, if Γ is simple.
- P_l is a path of length l (as a graph or edge set).
- C_l is a circle of length l (as a graph or edge set).
- $K_{r,s}$ is the complete bipartite graph with r left vertices and s right vertices. $K_{X,Y}$ is the complete bipartite graph with left vertex set X and right vertex set Y.
- The empty graph, $\varnothing := (\varnothing, \varnothing)$, has no vertices and no edges. It is not connected.

Types of subgraph: In Γ, let $X \subseteq V$ and $S \subseteq E$.

- A *component* (or *connected component*) of Γ is a maximal connected subgraph, excluding loose edges. An *isolated vertex* is a component that has one vertex and no edges. A loose edge is not a component.
- $c(\Gamma)$ is the number of components of Γ. $c(S)$ is short for $c(V, S)$.
- A *spanning subgraph* is $\Gamma' \subseteq \Gamma$ such that $V' = V$.
- $\Gamma|S := (V, S)$. This is a spanning subgraph.
- $S{:}X := \{e \in S : \varnothing \neq V(e) \subseteq X\} = (E{:}X) \cap S$. We often write $S{:}X$ as short for the subgraph $(X, S{:}X)$.
- The *induced subgraph* $\Gamma{:}X$ is the subgraph $\Gamma{:}X := (X, E{:}X)$. An induced subgraph has no loose edges. We often write $E{:}X$ as short for $(X, E{:}X)$.
- $\Gamma \setminus S := (V, E \setminus S) = \Gamma|S^c$.
- $\Gamma \setminus X$ is the subgraph with

$$V(\Gamma \setminus X) := X^c \text{ and } E(\Gamma \setminus X) := \{e \in E \mid V(e) \subseteq V \setminus X\}.$$

  We say X is *deleted* from Γ. $\Gamma \setminus X$ includes all loose edges, if there are any (unlike $\Gamma{:}X^c$, which has no loose edges).

Graph structures and types:

- A *theta graph* is the union of 3 internally disjoint paths that have the same endpoints.

- A *block* of Γ is a maximal subgraph without isolated vertices or loose edges, such that every pair of edges is in a circle together. The simplest kind of block is $(\{v\}, \{e\})$ where e is a loop or half edge at vertex v. A loose edge or isolated vertex is not in any block.
- Γ is *inseparable* if it has only one block or it is an isolated vertex.
- A *cutpoint* is a vertex that belongs to more than one block.

Let T be a maximal forest in Γ. If $e \in E_* \setminus T$, there is a unique circle $C_e \subseteq T \cup \{e\}$. The *fundamental system of circles* for Γ, with respect to T, is the set of all circles C_e for $e \in E_* \setminus T$. The *set sum* or *symmetric difference* of two sets A, B is denoted by $A \oplus B := (A \setminus B) \cup (B \setminus A)$.

Proposition 1.1. *Choose a maximal forest T. Every circle in Γ is the set sum of fundamental circles with respect to T.*

Proof. $C = \bigoplus_{e \in C \setminus T} C_T(e)$. □

2. Signed Graphs

A *signed graph* $\Sigma = (\Gamma, \sigma) = (V, E, \sigma)$ is a graph Γ together with a function σ that assigns a sign, $\sigma(e) \in \{+, -\}$, to each ordinary edge (link or loop) in Γ. σ is called the *signature* (or *sign function*). A half or loose edge does not get a sign. Thus, the signature is $\sigma : E_* \to \{+, -\}$. Notation:

- $|\Sigma|$ is the *underlying graph* Γ.
- $E^+ := \sigma^{-1}(+) = \{e \in E : \sigma(e) = +\}$. The *positive subgraph* is $\Sigma^+ := (V, E^+)$.
- $E^- := \sigma^{-1}(+) = \{e \in E : \sigma(e) = -\}$. The *negative subgraph* is $\Sigma^- := (V, E^-)$.
- $+\Gamma := (\Gamma, +)$ is an *all-positive* signed graph (every ordinary edge is $+$). $e \in E_*(\Gamma)$ becomes $+e \in +E = E(+\Gamma)$.
- $-\Gamma := (\Gamma, -)$ is an *all-negative* signed graph (every ordinary edge is $-$). $e \in E_*(\Gamma)$ becomes $-e \in -E = E(-\Gamma)$.
- $\pm\Gamma := (+\Gamma) \cup (-\Gamma)$. $E(\pm\Gamma) = \pm E := (+E) \cup (-E)$. This is the *signed expansion* of Γ.
- $\Sigma^\bullet := \Sigma$ with a half edge or negative loop attached to every vertex that does not have one. $\Sigma^\bullet$ is called a *full* signed graph.
- $\Sigma^\circ := \Sigma$ with a negative loop attached to every vertex that does not have one.

Equivalent notations for the sign group: $\{+, -\}$, $\{+1, -1\}$, or $\mathbb{Z}_2 := \{0, 1\}$ modulo 2.

Signed graphs Σ_1 and Σ_2 are *isomorphic*, written $\Sigma_1 \cong \Sigma_2$, if there is an isomorphism between their underlying graphs that preserves the signs of edges.

2.1. Balance.

Signs and balance:

- The *sign of a walk*, $\sigma(W)$, is the product of the signs of its edges, including repeated edges.
- The *sign of an edge set*, $\sigma(S)$, is the product of the signs of its edges, without repetition.
- The sign of a circle, $\sigma(C)$, is the same whether the circle is treated as a walk or as an edge set.
- The *class of positive circles* is

$$\mathcal{B} = \mathcal{B}(\Sigma) := \{C \in \mathcal{C}(|\Sigma|) : \sigma(C) = +\}.$$

- Σ is *balanced* if it has no half edges and every circle in it is positive. Similarly, any subgraph or edge set is balanced if it has no half edges and every circle in it is positive.
- A circle is balanced if and only if it is positive. However, in general, a walk cannot be balanced because it is not a graph or edge set.
- A *negative digon* is a circle of length 2 (i.e., a pair of parallel edges) that has one positive edge and one negative edge.
- $b(\Sigma)$ is the number of components of Σ (omitting loose edges) that are balanced. $b(S)$ is short for $b(\Sigma|S)$.
- $\pi_{\mathrm{b}}(\Sigma) := \{V(\Sigma') : \Sigma' \text{ is a balanced component of } \Sigma\}$. Then $b(\Sigma) = |\pi_{\mathrm{b}}(\Sigma)|$. $\pi_{\mathrm{b}}(S)$ is short for $\pi_{\mathrm{b}}(\Sigma|S)$.
- $V_0(\Sigma)$ is the set of vertices of unbalanced components of Σ. Formally, $V_0(\Sigma) := V \setminus \bigcup_{W \in \pi_{\mathrm{b}}(\Sigma)} W$. $V_0(S)$ is short for $V_0(\Sigma|S)$.

Types of vertex and edge in Σ:

- A *balancing vertex* is a vertex v such that $\Sigma \setminus v$ is balanced although Σ is unbalanced.
- A *partial balancing edge* is an edge e such that $\Sigma \setminus e$ has more balanced components than does Σ.
- A *total balancing edge* is an edge e such that $\Sigma \setminus e$ is balanced although Σ is not balanced. A total balancing edge is a partial balancing edge, but a partial balancing edge may not be a total balancing edge.

Proposition 2.1. *An edge e is a partial balancing edge of Σ if and only if it is either*

(i) *an isthmus between two components of $\Sigma \setminus e$, of which at least one is balanced, or*
(ii) *a negative loop or half edge in a component Σ' such that $\Sigma' \setminus e$ is balanced, or*
(iii) *a link with endpoints v, w, which is not an isthmus, such that every vw-path in $\Sigma \setminus e$ has sign opposite to that of e.*

Lemma 2.2. *Σ is balanced if and only if every block is balanced.*

A *bipartition* of a set X is an unordered pair $\{X_1, X_2\}$ such that $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$. X_1 or X_2 could be empty.

Theorem 2.3 (Harary's Balance Theorem (1953a))**.** *Σ is balanced $\iff$ it has no half edges and there is a bipartition $V = V_1 \mathbin{\dot\cup} V_2$ such that $E^- = E(V_1, V_2)$.*

Corollary 2.4. *$-\Gamma$ is balanced if and only if Γ is bipartite.*

Thus, balance is a generalization of bipartiteness.

2.2. Switching.

A *switching function* for Σ is a function $\zeta : V \to \{+, -\}$. The *switched signature* is $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has endpoints v, w. The *switched signed graph* is $\Sigma^\zeta := (|\Sigma|, \sigma^\zeta)$. We say Σ is *switched by ζ*. Note that $\Sigma^\zeta = \Sigma^{-\zeta}$.

If $X \subseteq V$, *switching Σ by X* (or simply *switching X*) means reversing the sign of every edge in the cutset $E(X, X^c)$. The switched graph is Σ^X. This is the same as Σ^ζ where $\zeta(v) := -$ if and only if $v \in X$. Switching by ζ or X is the same operation with different notation. Note that $\Sigma^X = \Sigma^{X^c}$.

Proposition 2.5. (i) *Switching leaves the signs of all closed walks, including all circles, unchanged. Thus, $\mathcal{B}(\Sigma^\zeta) = \mathcal{B}(\Sigma)$.*

(ii) *If* $|\Sigma_1| = |\Sigma_2|$ *and* $\mathcal{B}(\Sigma_1) = \mathcal{B}(\Sigma_2)$*, then there exists a switching function* ζ *such that* $\Sigma_2 = \Sigma_1^\zeta$.

Proof of (i). Let ζ be a switching function and let $W = v_0e_0v_1e_1v_2\cdots v_{n-1}e_{n-1}v_0$ be a closed walk. Then

$$\begin{aligned}\sigma^\zeta(W) &= \big[\zeta(v_0)\sigma(e_0)\zeta(v_1)\big]\big[\zeta(v_1)\sigma(e_1)\zeta(v_2)\big]\dots\big[\zeta(v_{n-1})\sigma(e_{n-1})\zeta(v_0)\big]\\ &= \sigma(e_0)\sigma(e_1)\cdots\sigma(e_{n-1}) = \sigma(W).\end{aligned}$$

□

Proof of (ii). We may assume Σ_1 is connected. Pick a spanning tree T and list the vertices in such a way that v_i is always adjacent to a vertex in $\{v_1,\dots,v_{i-1}\}$ (for $i > 1$). Let t_i be the unique tree edge connecting v_i to $\Sigma{:}\{v_1,\dots,v_{i-1}\}$.

We define a switching function ζ:

$$\zeta(v_i) = \begin{cases} +, & \text{if } i = 1,\\ \sigma_1(t_i)\sigma_2(t_i)\zeta(v_j), & \text{if } i > 1, \text{ where } v_j \text{ is the endpoint of } t_i \text{ that is not } v_i.\end{cases}$$

Now it is easy to show that $\Sigma_1^\zeta = \Sigma_2$. □

Signed graphs Σ_1 and Σ_2 are *switching equivalent*, written $\Sigma_1 \sim \Sigma_2$, if they have the same underlying graph and there exists a switching function ζ such that $\Sigma_1^\zeta \cong \Sigma_2$. The equivalence class of Σ,

$$[\Sigma] := \{\Sigma' : \Sigma' \sim \Sigma\},$$

is called its *switching class*.

Similarly, Σ_1 and Σ_2 are *switching isomorphic*, written $\Sigma_1 \simeq \Sigma_2$, if Σ_1 is isomorphic to a switching of Σ_2. The equivalence class of Σ is called its *switching isomorphism class*.

Proposition 2.6. *Switching equivalence,* $\sim$*, is an equivalence relation on signatures of a given underlying graph.*

Switching isomorphism, $\simeq$*, is an equivalence relation on signed graphs.*

Corollary 2.7. Σ *is balanced if and only if it has no half edges and it is switching equivalent to* $+|\Sigma|$.

Proof of Harary's Balance Theorem. Σ has the form stated in the theorem $\iff$ it is $(+|\Sigma|)^{V_1}$ $\iff$ it is a switching of $+|\Sigma|$ $\iff$ (by Proposition 2.5) it is balanced. □

2.3. Deletion, contraction, and minors.

R, S denote subsets of E. A *component of* S means a component of (V, S).

The *deletion* of S (or, the *deletion of* Σ *by* S) is the signed graph $\Sigma \setminus S := (V, S^c, \sigma|_{S^c})$.

The *contraction* of S (or, the *contraction of* Σ *by* S) is a signed graph Σ/S, to be defined next.

2.3.1. *Contracting an edge e.*

If e is a positive link, delete e and identify its endpoints; do not change any edge signs. (This is the same as contracting a link in an unsigned graph.) If e is a negative link, switch Σ by a switching function ζ, chosen so e is positive in Σ^ζ; then contract e as a positive link. The choice of ζ does not matter, up to switching.

Lemma 2.8. *In a signed graph* Σ *any two contractions of a link* e *are switching equivalent. The contraction of a link in a switching class is a well defined switching class.*

To contract a positive loop or a loose edge e, just delete e.

If e is a negative loop or half edge and v is the vertex of e, delete v and e, but not any other edges. Any other edges at v lose their endpoint v. A loop or half edge at v becomes a loose edge. A link with endpoints v, w becomes a half edge at w.

2.3.2. *Contracting an edge set S.*

The edge set and vertex set of Σ/S are

$$E(\Sigma/S) := E \setminus S, \quad V(\Sigma/S) := \pi_{\mathrm{b}}(\Sigma|S) = \pi_{\mathrm{b}}(S).$$

This means we identify all the vertices of each balanced component so they become a single vertex. For $f \in E(\Sigma/S)$, the endpoints are given by the rule

$$V_{\Sigma/S}(f) = \{W \in \pi_{\mathrm{b}}(S) : w \in V_\Sigma(f) \text{ and } w \in W \in \pi_{\mathrm{b}}(S)\}.$$

(For instance, suppose f is a loop at w in Σ, so that $V_\Sigma(f) = \{w, w\}$. If $w \in W \in \pi_{\mathrm{b}}(S)$, then W is a repeated vertex in $V_{\Sigma/S}(f)$ so f is a loop in Σ/S. If $w \in V_0(S)$, then $V_{\Sigma/S}(f) = \varnothing$ so f is a loose edge in Σ/S.) To define the signature of Σ/S, first switch Σ to Σ^ζ so every balanced component of S is all positive. Then $\sigma_{\Sigma/S}(e) := \sigma^\zeta(e)$.

Lemma 2.9. (a) *Given $S \subseteq E(\Sigma)$, all contractions Σ/S (by different choices of how to switch Σ) are switching equivalent. Any switching of one contraction Σ/S is another contraction and any contraction Σ^ζ/S of a switching of Σ is a contraction of Σ.*

(b) *If $|\Sigma_1| = |\Sigma_2|$, $S \subseteq E$ is balanced in both Σ_1 and Σ_2, and Σ_1/S and Σ_2/S are switching equivalent, then Σ_1 and Σ_2 are switching equivalent.*

(c) *For $e \in E$, $[\Sigma/e]$ and $[\Sigma/\{e\}]$ are essentially the same switching class.*

Part (a) means that the switching class $[\Sigma/S]$ is uniquely defined, even though the signed graph Σ/S is not unique. Part (c) means that $[\Sigma/e] = [\Sigma/\{e\}]$ except for details of notation.

2.3.3. *Minors.*

A *minor* of Σ is any contraction of any subgraph.

Theorem 2.10. *A minor of a minor is a minor. Thus, the result of any sequence of deletions and contractions of edge and vertex sets of Σ is a minor of Σ.*

Proof. See Zaslavsky (1982a), Proposition 4.2. □

2.4. **Frame circuits.**

A *frame circuit* of Σ is a subgraph, or edge set, that is either a positive circle or a loose edge, or a pair of negative circles that intersect in precisely one vertex and no edges (this is a *tight handcuff circuit*), or a pair of disjoint negative circles together with a minimal path that connects them (this is a *loose handcuff circuit*). We regard a tight handcuff circuit as having a connecting path of length 0 (it is the common vertex of two the circles). A half edge and a negative loop are equivalent in everything that concerns frame circuits; a 'negative circle' in the definition may be a half edge.

In $+\Gamma$ (if Γ has no half edges), a frame circuit is simply a circle or a loose edge.

Proposition 2.11. *Σ contains a loose handcuff circuit if and only if there is a component of Σ that contains two disjoint negative circles.*

Proposition 2.12. *Let $e \in E$ be an edge in an unbalanced component Σ' of Σ. Then e is contained in a frame circuit if and only if e is not a partial balancing edge.*

Proof. If e is in a frame circuit C, then Σ' contains C. If e is an isthmus of C, then $\Sigma' \setminus e$ contains both negative circles of C; if $\Sigma' \setminus e$ is disconnected, each of its two components contains one of those negative circles. Therefore, e is not a partial balancing edge. If e belongs to a circle in C, then $\Sigma' \setminus e$ is connected. Suppose C is unbalanced; then $C \setminus e$ is unbalanced so $\Sigma' \setminus e$ is unbalanced; thus, e is not a partial balancing edge.

Suppose to the contrary that C is a positive circle. As there is a negative circle D in Σ', for e to be a partial balancing edge it must belong to D; we show this leads to a contradiction. If $C \cup D \setminus e$ were balanced, it could be switched to be all positive and then, as D is negative, e would be negative in the switched graph, but that would contradict the positivity of C. Thus, $C \cup D \setminus e$ is unbalanced; therefore it contains a negative circle, so $\Sigma' \setminus e$ is unbalanced and e is not a partial balancing edge.

Conversely, suppose e is not a partial balancing edge; we produce a frame circuit C containing e. Then $\Sigma' \setminus e$ is unbalanced, so it has a negative circle D. If e is an unbalanced edge (a half edge or negative loop) at v, there is a path P in Σ' from v to D; then $C = D \cup P \cup e$.

If e is a balanced edge, it is a link with endpoints v, w. If it is an isthmus, then $\Sigma' \setminus e$ has two components, both unbalanced (by Proposition 2.1), so C is a negative circle in each of those components together with a connecting path (which must contain e). If e is not an isthmus, it lies in a circle C'. If C' is positive, let $C = C'$. But suppose C' is negative; then there are three subcases, depending on how many points of intersection C' has with D. If there are no such points, take a minimal path P connecting C' to D and let $C = D \cup P \cup C'$. If there is just one such point, $C = D \cup C'$. If there are two or more such points, take P to be a maximal path in C' that contains e and is internally disjoint from D. Then $P \cup D$ is a theta graph in which D is negative; hence one of the two circles containing P is positive, and this is the circuit C. $\square$

This suggests vertex-disjoint negative circles are important, which is true. There is an important theorem about when they do not exist.

Theorem 2.13 (Slilaty (2007a)). *Σ has no two vertex-disjoint negative circles if and only if one or more of the following is true:*

(1) *Σ is balanced,*
(2) *Σ has a balancing vertex,*
(3) *Σ embeds in the projective plane, or*
(4) *Σ is one of a few exceptional cases.*

We will not discuss embedding in the projective plane, which is a large topic in itself; see Zaslavsky (1993a) Archdeacon and Debowsky (2005a).

2.5. **Closure and closed sets.**

The *balance-closure* of an edge set R is

$$\operatorname{bcl}(R) := R \cup \{e \in R^c : \exists \text{ a positive circle } C \subseteq R \cup e \text{ such that } e \in C\} \cup E_0(\Sigma).$$

The *closure* of an edge set S is

$$\operatorname{clos}(S) := \big(E{:}V_0(S)\big) \cup \Big(\bigcup_{i=1}^{k} \operatorname{bcl}(S_i)\Big) \cup E_0(\Sigma),$$

where $S_1, \ldots, S_k$ are the balanced components of S.

An edge set is *closed* if it equals its own closure: $\operatorname{clos} S = S$. We write

$$\operatorname{Lat}\Sigma := \{S \subseteq E : S \text{ is closed}\}.$$

When partially ordered by set inclusion, $\operatorname{Lat}\Sigma$ is a lattice.

A half edge and a negative loop are equivalent in everything that concerns closure.

The usual closure operator in a graph Γ is the same as closure in $+\Gamma$.

Lemma 2.14. $\operatorname{bcl}(R)$ *is balanced if and only if* R *is balanced. Furthermore,* $\operatorname{bcl}(\operatorname{bcl} R) = \operatorname{bcl}(R) = \operatorname{clos}(R)$.

Lemma 2.15. *For an edge set* S, $\pi_{\mathrm{b}}(\operatorname{clos} S) = \pi_{\mathrm{b}}(\operatorname{bcl} S) = \pi_{\mathrm{b}}(S)$ *and* $V_0(\operatorname{clos} S) = V_0(\operatorname{bcl} S) = V_0(S)$.

Let E be any set; its *power set* $\mathcal{P}(E)$ is the class of all subsets of E. A function $J : \mathcal{P}(E) \to \mathcal{P}(E)$ is an *(abstract) closure operator on* E if it has the three properties

(C1) $J(S) \supseteq S$ for every $S \subseteq E$ (increase).
(C2) $R \subseteq S \implies J(R) \subseteq J(S)$ (isotonicity).
(C3) $J(J(S)) = J(S)$ (idempotence).

Theorem 2.16. *The operator* clos *on subsets of* $E(\Sigma)$ *is an abstract closure operator.*

Proof. The definition makes it clear that clos is increasing and isotonic. What remains to be proved is that $\operatorname{clos}(\operatorname{clos}(S)) = \operatorname{clos}(S)$. For simplicity we ignore loose edges.

Let $\pi_{\mathrm{b}}(S) = \{B_1, \dots, B_k\}$; thus, $S{:}B_i$ is balanced. By the definition of closure and Lemma 2.15,

$$\begin{aligned}\operatorname{clos}(\operatorname{clos} S) &= \big(E{:}V_0(\operatorname{clos} S)\big) \cup \bigcup_{i=1}^{k} \operatorname{bcl}\big((\operatorname{clos} S){:}B_i\big) = \big(E{:}V_0(S)\big) \cup \bigcup_{i=1}^{k} \operatorname{bcl}\big((\operatorname{bcl} S){:}B_i\big)\\ &= \big(E{:}V_0(S)\big) \cup \bigcup_{i=1}^{k} \operatorname{bcl}(S{:}B_i) = \operatorname{clos} S. \qquad \square\end{aligned}$$

The closure operator of a signed graph has an additional property, the exchange property, whose theory is the theory of matroids. That is, $\operatorname{clos}_\Sigma$ is a matroid closure. Matroids are too complicated to go into here; see [1]. One aspect of matroid closure we do need is:

Theorem 2.17. *For* $S \subseteq E$,

$$\operatorname{clos} S = S \cup \{e \notin S : \exists \text{ a frame circuit } C \text{ such that } e \in C \subseteq S \cup e\}.$$

Proof. We treat a half edge as if it were a negative loop, and for simplicity we neglect loose edges.

Necessity. We want to prove that if $e \in \operatorname{clos} S$, a frame circuit C exists. Let S' be the component of $S \cup \{e\}$ that contains e.

If S' is contained in one of the sets $\operatorname{bcl} S_i$, then C exists by the definition of balance-closure.

Assume $S' \subseteq E{:}V_0(S)$. Then $S' \setminus e$ consists of one or two components of $S{:}V_0(S)$. Every such component is unbalanced, so S' is unbalanced and e is not a partial balancing edge of it. By Proposition 2.12, e is contained in a frame circuit $C \subseteq S \cup \{e\}$.

Sufficiency. Assuming a circuit C exists, we want to prove that $e \in \operatorname{clos} S$.

If C is balanced, $e \in \operatorname{bcl} S \subseteq \operatorname{clos} S$.

If C is unbalanced, the component S' of $S \cup \{e\}$ that contains e is unbalanced. By Proposition 2.12, e is not a partial balancing edge of $S \cup \{e\}$; therefore $S' \setminus e$ has only unbalanced components. It follows that $V(S') \subseteq V_0(S)$, so $e \in C \subseteq E{:}V_0(S) \subseteq \operatorname{clos} S$. □

2.6. **Orientation; bidirected graphs.**

Bidirected graphs were introduced by Edmonds and first published in a paper on matching theory, Edmonds and Johnson (1970a). Later, Zaslavsky (1991b) found that they are oriented signed graphs.

An orientation of an ordinary graph gives a direction to each edge. An *orientation* of a signed graph Σ gives a direction to each end of each edge. If e is positive, the directions at the two ends of e must agree in pointing from one endpoint to the other. If e is negative, the directions at the two ends of e must disagree; that is, they both point towards the middle of e (an *introverted* edge) or both away from the middle (an *extraverted* edge).

A *bidirected graph* is a graph in which each end of each edge has an independent direction. Thus, an oriented signed graph is a bidirected graph. Formally, a bidirected graph B (read 'Beta') is a pair (Γ, τ) where Γ is a graph and τ is a function from edge ends to $\{+,-\}$. If e has endpoints v, w and we write (v, e) for the end of edge e at vertex v, then $\tau(v, e) = +$ if the end is directed towards v and $= -$ if the end is directed away from v. As a consequence, the two directions on e agree when $\tau(v, e) = -\tau(w, e)$ (which may at first sight seem peculiar). A bidirected graph has an edge signature:

$$\sigma_{\mathrm{B}}(e) = -\tau(v, e)\tau(w, e).$$

That is, if the directions at the two ends agree, the edge is positive; if they disagree, the edge is negative. Thus, bidirected graphs and oriented signed graphs are exactly the same thing.

We write $|\mathrm{B}|$ for the underlying graph of B and Σ_{B} for the signed graph $(|\mathrm{B}|, \sigma_{\mathrm{B}})$. B is switched by the rule $\mathrm{B}^\zeta := (|\mathrm{B}|, \tau^\zeta)$ where $\tau^\zeta(v, e) := \tau(v, e)\zeta(v)$.

Lemma 2.18. *Σ_{B} and $\Sigma_{\mathrm{B}^\zeta}$ are switching equivalent; in fact, $\Sigma_{\mathrm{B}^\zeta} = (\Sigma_{\mathrm{B}})^\zeta$.*

Proof. Let e have endpoints v, w. Then

$$\begin{aligned} \sigma_{\mathrm{B}}^\zeta(e) &= \zeta(v)\sigma(e)\zeta(w) = \zeta(v)[-\tau(v, e)\tau(w, e)]\zeta(w) \\ &= -[\tau(v, e)\zeta(v)][\tau(w, e)\zeta(w)] = \sigma_{\mathrm{B}^\zeta}(e). \end{aligned}$$ □

In an orientation τ of a signed graph, a vertex v is a *source* if $\tau(v, e) = +$ for every edge end (v, e) at v. It is a *sink* if $\tau(v, e) = -$ for every edge end at v. An orientation τ of Σ is called *acyclic* if for every frame circuit $C \subseteq \Sigma$, the oriented subgraph $(\Sigma|C, \tau|_C)$ has a source or a sink, where $\tau|_C$ denotes the restriction to the edge ends (v, e) in $\Sigma|C$.

3. GEOMETRY AND MATRICES

In this section we write the vertex set as $V = \{v_1, v_2, \dots, v_n\}$. $\mathbf{F}$ denotes any field. The most important field is $\mathbb{R}$, the real number field.

3.1. **Vectors for edges.**

We have a signed graph Σ of order n. For each edge e there is a vector $\mathbf{x}(e) \in \mathbf{F}^n$, whose definition is, for the four types of edge:

$$
\begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \\ \mp\sigma(e) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\qquad
\begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \mp \sigma(e) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\qquad
\begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\qquad
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

a link $e{:}v_iv_j$, a loop e at v_i, a half edge e at v_i, a loose edge.

These vectors are well defined only up to sign, i.e., the negative of $\mathbf{x}(e)$ is another possible choice of $\mathbf{x}(e)$. We make an arbitrary choice $\mathbf{x}(e)$ for each edge e, which does not affect the linear dependence properties. The choice amounts to choosing an orientation of Σ, because if we orient Σ as $\mathrm{B} = (|\Sigma|, \tau)$, and if we define

$$\eta(v,e) := \sum_{\text{incidences } (v,e)} \tau(v,e), \tag{3.1}$$

then $\mathbf{x}(e)_v$ is precisely equal to $\eta(v,e)$, even for a loop. Conversely, if we choose $\mathbf{x}(e)$ first and then define τ to orient Σ, one can show that τ satisfies (3.1).

For a set $S \subseteq E$, define $\mathbf{x}(S) := \{\mathbf{x}(e) : e \in S\}$.

Theorem 3.1. *Let S be an edge set in Σ and consider the corresponding vector set $\mathbf{x}(S)$ in the vector space $\mathbf{F}^n$ over a field $\mathbf{F}$.*

(1) When $\operatorname{char}\mathbf{F} \neq 2$, $\mathbf{x}(S)$ is linearly dependent if and only if S contains a frame circuit.
(2) When $\operatorname{char}\mathbf{F} = 2$, $\mathbf{x}(S)$ is linearly dependent if and only if S contains a circle or a loose edge.

The theorem is implicit in Zaslavsky (1982a), Theorem 8B.1. The proof, which we omit, is neither very short nor very long.

Corollary 3.2. *The minimal linearly dependent subsets of $\mathbf{x}(E)$ are the sets $\mathbf{x}(C)$ where C is a frame circuit in Σ.*

The proofs of the next results are short. Define a set $S \subseteq E(\Sigma)$ to be *independent* if the vectors in $\mathbf{x}(S)$ are linearly independent (and distinct from each other).

Corollary 3.3. *A set $S \subseteq E(\Sigma)$ is independent if and only if it does not contain a frame circuit.*

The vector subspace generated by a set $X \subseteq \mathbf{F}^n$ is denoted by $\langle X \rangle$. We write

$$\mathcal{L}_{\mathbf{F}}(\Sigma) := \{\langle X \rangle : X \subseteq \mathbf{x}(E)\}.$$

When partially ordered by set inclusion, $\mathcal{L}_{\mathbf{F}}(\Sigma)$ is a lattice.

Corollary 3.4. *For $S \subseteq E(\Sigma)$, $\mathbf{x}(E) \cap \langle \mathbf{x}(S) \rangle = \mathbf{x}(\operatorname{clos} S)$. Thus, $\mathcal{L}_{\mathbb{R}}(\Sigma) \cong \operatorname{Lat} \Sigma$.*

The *rank* of $S \subseteq E$ is defined to be

$$\operatorname{rk} S := n - b(S).$$

The rank of Σ is $\operatorname{rk} \Sigma := \operatorname{rk} E = n - b(\Sigma)$.

Theorem 3.5. *Let $S \subseteq E$. Then $\dim \langle \mathbf{x}(S) \rangle = \operatorname{rk} S$.*

Proof. The proof is simplest when expressed in terms of the frame matroid (Section 3.3), so I omit it; see Zaslavsky (1982a), Theorem 8B.1 and following remarks. The essence of the proof is using Corollary 3.3 to compare the minimum number of edges required to generate S by closure in Σ to the minimum number of vectors $\mathbf{x}(e)$ required to generate $\langle \mathbf{x}(S) \rangle$. □

3.2. The incidence matrix.

The *incidence matrix* $\mathrm{H}(\Sigma)$ (read 'Eta of Sigma') is a $V \times E$ matrix (thus, it has n rows and m columns where $m := |E|$) in which the column corresponding to edge e is the column vector $\mathbf{x}(e)$.

Theorem 3.6. *Over a field whose characteristic is not 2, the rank of $\mathrm{H}(\Sigma)$ is $\operatorname{rk} \Sigma = n - b(\Sigma)$ and, for $S \subseteq E$, the rank of $\mathrm{H}(\Sigma|S)$ is $\operatorname{rk} S$.*

Proof. The column rank is the dimension of the span of the columns corresponding to S, which is the span of $\mathbf{x}(S)$. Apply Theorem 3.5. □

3.3. Matroid.

The *frame matroid* $G(\Sigma)$ is an abstract way of describing all the previous characteristics of a signed graph: linearly dependent edge sets, minimal dependencies, rank, closure, and closed sets. See Zaslavsky (1982a), Section 5, for more information. For matroid theory, consult [1].

I mention matroids here because in $G(\Sigma)$ we have a notion of *independent edge set*; it means a set of edges whose columns in $\mathrm{H}(\Sigma)$ are linearly independent.

3.4. The adjacency and Kirchhoff (Laplacian) matrices.

Assume (to avoid complications) that Σ is a signed link graph, i.e., every edge is a link.

The *adjacency matrix* is $A(\Sigma) = (a_{ij})_{n \times n}$ defined by $a_{ii} := 0$, and $a_{ij} :=$ (the number of positive edges $v_i v_j$) $-$ (the number of negative edges $v_i v_j$) if $i \neq j$. An important fact about the adjacency matrix is that it does not change if a parallel pair of edges, one positive and one negative, is deleted from Σ (this is *cancellation of a negative digon*). A signed link graph is *reduced* if it has no such parallel pairs. Up to isomorphism there is a unique reduced signed graph $\bar{\Sigma}$ with the same adjacency matrix as Σ.

The *Kirchhoff* or *Laplacian matrix* of Σ is $K(\Sigma) := D(|\Sigma|) - A(\Sigma)$, where $D(|\Sigma|)$, called the *degree matrix*, is the diagonal matrix whose diagonal element $d_{ii} = d_{|\Sigma|}(v_i)$.

Some examples:

- $A(-\Sigma) = -A(\Sigma)$.
- $A(+\Gamma) = A(\Gamma)$, the adjacency matrix of Γ, and $K(+\Gamma) = D(\Gamma) - A(\Gamma)$, the Kirchhoff or Laplacian matrix of Γ.

- $A(-\Gamma) = -A(\Gamma)$, and $K(-\Gamma) = D(\Gamma) + A(\Gamma)$, the so-called 'signless Laplacian matrix' of Γ, which has recently been studied intensively.

Proposition 3.7. *For a signed link graph Σ, $K(\Sigma) = \mathrm{H}(\Sigma)\mathrm{H}(\Sigma)^{\mathrm{T}}$.*

The proof is a straightforward calculation.

Theorem 3.8. *For a signed link graph Σ, the eigenvalues of $A(\Sigma)$ are real and the eigenvalues of $K(\Sigma)$ are real and non-negative.*

The proof follows standard lines based on the fact that $\mathrm{H}(\Sigma)\mathrm{H}(\Sigma)^{\mathrm{T}}$ is positive semidefinite.

The eigenvalues of $A(\Sigma)$ are usually called the *eigenvalues of* Σ. Those of $K(\Sigma)$ are usually called the *Laplacian eigenvalues* of Σ.

Theorem 3.9 (Matrix-Tree Theorem for Signed Graphs)**.** *Let $b_i :=$ the number of sets of n independent edges in Σ that contain exactly i circles. Then* $\det K(\Sigma) = \sum_{i=0}^{n} 4^i b_i$.

The proof uses the Cauchy-Binet Theorem in the same way as it is used to prove the Matrix-Tree Theorem for ordinary graphs. Note that the i circles must all be negative for the edge set to be independent. Chaiken has proved a generalization to signed digraphs Chaiken (1982a).

3.5. **Arrangements of hyperplanes.**

An *arrangement of hyperplanes* in $\mathbb{R}^n$, $\mathcal{H} = \{h_1, h_2, \dots, h_m\}$, is a finite set of hyperplanes. A *region* of $\mathcal{H}$ is a connected component of the complement, $\mathbb{R}^n \setminus \left(\bigcup_{k=1}^{m} h_k\right)$. We write $r(\mathcal{H}) :=$ the number of regions. The *intersection lattice* is the family $\mathcal{L}(\mathcal{H})$ of all subspaces that are intersections of hyperplanes in $\mathcal{H}$, partially ordered by reverse inclusion, $s \le t \iff t \subseteq s$. The *characteristic polynomial* of $\mathcal{H}$ is

$$p_{\mathcal{H}}(\lambda) := \sum_{\mathcal{S} \subseteq \mathcal{H}} (-1)^{|\mathcal{S}|} \lambda^{\dim \mathcal{S}}, \tag{3.2}$$

where $\dim \mathcal{S} := \dim \left(\bigcap_{h_k \in \mathcal{S}} h_k\right)$.

Theorem 3.10 ([2, Theorem A])**.** *We have $r(\mathcal{H}) = (-1)^n p_{\mathcal{H}}(-1)$.*

A signed graph Σ, with edge set $\{e_1, e_2, \dots, e_m\}$, gives rise to a hyperplane arrangement

$$\mathcal{H}[\Sigma] := \{h_1, h_2, \dots, h_m\}$$

where h_k is the solution set of the equation $\mathbf{x}(e_k) \cdot \mathbf{x} = 0$; i.e.,

$$h_k = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}(e_k) \cdot \mathbf{x} = 0\}.$$

(By $\cdot$ I mean the usual inner product, or 'dot product', $\mathbf{y} \cdot \mathbf{x} := y_1 x_1 + \cdots + y_n x_n$.) In terms of the graph,

$$h_k \text{ has the equation } \begin{cases} x_j = \sigma(e_k) x_i, & \text{if } e_k \text{ is a link or loop with endpoints } v_i, v_j, \\ x_i = 0, & \text{if } e_k \text{ is a half edge or a negative loop at } v_i, \\ 0 = 0, & \text{if } e_k \text{ is a loose edge or a positive loop.} \end{cases}$$

(The last equation has the solution set $\mathbb{R}^n$, so it is not truly a hyperplane, but I allow it under the name 'degenerate hyperplane'.)

Lemma 3.11. *Let $\mathcal{S} = \{h_{i_1}, \dots, h_{i_l}\}$ be the subset of $\mathcal{H}[\Sigma]$ that corresponds to the edge set $S = \{e_{i_1}, \dots, e_{i_l}\}$. Then* $\dim \bigcap \mathcal{S} = b(S)$.

Proof. Apply vector space duality to Theorem 3.5. □

Theorem 3.12. *$\mathcal{L}(\mathcal{H}[\Sigma])$, $\mathcal{L}_{\mathbb{R}}(\Sigma)$, and $\operatorname{Lat}\Sigma$ are all isomorphic.*

Proof. The isomorphism between $\mathcal{L}(\mathcal{H}[\Sigma])$ and $\mathcal{L}_{\mathbb{R}}(\Sigma)$ is standard vector-space duality. The isomorphism $\mathcal{L}_{\mathbb{R}}(\Sigma) \cong \operatorname{Lat}\Sigma$ is in Corollary 3.4. □

The regions of $\mathcal{H}[\Sigma]$ are in bijective correspondence with the acyclic orientations of Σ. For an orientation τ define

$$R(\tau) := \big\{\mathbf{x} \in \mathbb{R}^n : \tau(v_i, e)x_i + \tau(v_j, e)x_j > 0 \text{ for every edge } e, \text{ where } V(e) = \{v_i, v_j\}\big\}.$$

Theorem 3.13. *$R(\tau)$ is nonempty if and only if τ is acyclic.*

There are two proofs in Zaslavsky (1991b).

4. Coloring

We color a signed graph from a color set

$$\Lambda_k := \{\pm 1, \pm 2, \ldots, \pm k\} \cup \{0\}$$

or a zero-free color set

$$\Lambda_k^* := \Lambda_k \backslash \{0\} = \{\pm 1, \pm 2, \ldots, \pm k\}.$$

A *k-coloration* (or *k-coloring*) of Σ is a function $\gamma : V \to \Lambda_k$. A coloration is *zero free* if it does not use the color 0. Coloring of signed graphs comes from Zaslavsky (1982b, 1982c).

A coloration γ is *proper* if it satisfies all the properties

$$\begin{cases} \gamma(v_j) \neq \sigma(e)\gamma(v_i), & \text{for a link or loop } e \text{ with endpoints } v_i, v_j, \\ \gamma(v_i) \neq 0, & \text{for a half edge } e \text{ with endpoint } v_i, \end{cases}$$

and there are no loose edges. (Note that these conditions for a proper coloration are opposite to the equations of the hyperplanes h_k.)

4.1. Chromatic polynomials.

There are two chromatic polynomials of a signed graph. For an integer $k \geq 0$, define

$$\chi_\Sigma(2k+1) := \text{ the number of proper } k\text{-colorations,}$$

and

$$\chi_\Sigma^*(2k) := \text{ the number of proper zero-free } k\text{-colorations.}$$

(Beck and Zaslavsky (2006a) explains exactly why there are two chromatic polynomials of a signed graph when one is enough for ordinary graphs.)

Theorem 4.1. *The chromatic polynomials have the properties of*

(i) *Unitarity:*

$$\chi_\emptyset(2k+1) = 1 = \chi_\emptyset^*(2k) \text{ for all } k \geq 0.$$

(ii) *Switching Invariance: If $\Sigma \sim \Sigma'$, then*

$$\chi_\Sigma(2k+1) = \chi_{\Sigma'}(2k+1) \text{ and } \chi_\Sigma^*(2k) = \chi_{\Sigma'}^*(2k).$$

(iii) *Multiplicativity: If Σ is the disjoint union of Σ_1 and Σ_2, then*

$$\chi_\Sigma(2k+1) = \chi_{\Sigma_1}(2k+1)\chi_{\Sigma_2}(2k+1) \quad \text{and} \quad \chi_\Sigma^*(2k) = \chi_{\Sigma_1}^*(2k)\chi_{\Sigma_2}^*(2k).$$

(iv) *Deletion-Contraction: If e is a link, a positive loop, or a loose edge, then*

$$\chi_\Sigma(2k+1) = \chi_{\Sigma\setminus e}(2k+1) - \chi_{\Sigma/e}(2k+1)$$

and

$$\chi^*_\Sigma(2k) = \chi^*_{\Sigma\setminus e}(2k) - \chi^*_{\Sigma/e}(2k).$$

Outline of Proof. The hard part is the deletion-contraction property. The proof is similar to the usual proof for ordinary graphs: count proper colorations of $\Sigma\setminus e$. If e is a link, switch so it is positive. Then a proper coloration of $\Sigma\setminus e$ gives unequal colors to the endpoints of e and is a proper coloration of Σ, or it gives the same color to the endpoints and it corresponds to a proper coloration of Σ/e. If e is a half edge or a negative loop, there are two cases depending on whether the endpoint gets a nonzero color or is colored 0. ☐

Theorem 4.2. *$\chi_\Sigma(\lambda)$ is a polynomial function of $\lambda = 2k+1 > 0$; specifically,*

$$\chi_\Sigma(\lambda) = \sum_{S\subseteq E} (-1)^{|S|}\lambda^{b(S)}. \tag{4.1}$$

Also, $\chi^*_\Sigma(\lambda)$ is a polynomial function of $\lambda = 2k \geq 0$. Specifically,

$$\chi^*_\Sigma(\lambda) = \sum_{S\subseteq E:\, balanced} (-1)^{|S|}\lambda^{b(S)}. \tag{4.2}$$

Proof. Apply Theorem 4.1 and induction on $|E|$ and n. ☐

Therefore, we can extend the range of λ to all of $\mathbb{R}$. In particular, we can evaluate $\chi_\Sigma(-1)$. This lets us draw an important connection between geometry and coloring of a signed graph.

Lemma 4.3. $\chi_\Sigma(\lambda) = p_{\mathcal{H}[\Sigma]}(\lambda)$.

Proof. Compare the summation expressions, (4.1) and (3.2), for the two polynomials, and note that by Lemma 3.11 $b(S) = \dim \mathcal{S}$ if $\mathcal{S} \subseteq \mathcal{H}[\Sigma]$ corresponds to the edge set S. ☐

Theorem 4.4. *The number of acyclic orientations of Σ and the number of regions of $\mathcal{H}[\Sigma]$ are both equal to $(-1)^n\chi_\Sigma(-1)$.*

To compute the chromatic polynomial it is often easiest to get the zero-free polynomial first and use

Theorem 4.5 (Zero-Free Expansion Identity)**.** *The chromatic and zero-free chromatic polynomials are related by*

$$\chi_\Sigma(\lambda) = \sum_{W\subseteq V:\, stable} \chi^*_{\Sigma\setminus W}(\lambda-1).$$

Proof. Let $\lambda = 2k+1$. For each proper k-coloration γ there is a set $W := \{v \in V : \gamma(v) = 0\}$, which must be stable. The restricted coloration $\gamma|_{V\setminus W}$ is a zero-free proper k-coloration of $\Sigma \setminus W$. This construction is reversible. ☐

4.2. **Chromatic numbers.**

The *chromatic number* of Σ is

$$\chi(\Sigma) := \min\{k : \exists \text{ a proper } k\text{-coloration}\},$$

and the *zero-free chromatic number* is

$$\chi^*(\Sigma) := \min\{k : \exists \text{ a zero-free proper } k\text{-coloration}\}.$$

Thus, $\chi(\Sigma) = \min\{k \geq 0 : \chi_\Sigma(2k+1) \neq 0\}$ and $\chi^*(\Sigma) = \min\{k \geq 0 : \chi^*_\Sigma(2k) \neq 0\}$.

Almost any question about chromatic numbers of signed graphs is open. The little I know about the graphs with a given value of a chromatic number is in Zaslavsky (1984a), where I studied complete signed graphs with largest or smallest zero-free chromatic number.

5. Examples

The standard basis vectors of $\mathbb{R}^n$ are

$$\mathbf{b}_1 = (1, 0, \ldots, 0),\ \mathbf{b}_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \mathbf{b}_n = (0, \ldots, 0, 1).$$

5.1. **Full signed graphs.**

In this example Σ is a signed graph with no half or loose edges or negative loops, $\Sigma^\bullet$ is Σ with a half edge at every vertex, and Σ° is Σ with a negative loop at every vertex. Whether a half edge or negative loop is added makes little difference, because each is an unbalanced edge. Write f_i for the unbalanced edge added to v_i.

- *Balance*: The balanced subgraphs in $\Sigma^\bullet$ are the same as those of Σ.
- *Closed sets*: An edge set in $\Sigma^\bullet$ is closed if and only if it consists of the induced edge set $E(\Sigma^\bullet){:}W$ together with a balanced, closed subset of $E(\Sigma){:}W^c$, for some vertex set $W \subseteq V$. Σ° is similar.
- *Vectors*: $\mathbf{x}(E(\Sigma^\bullet))$ is $\mathbf{x}(E(\Sigma))$ together with the unit basis vectors $\mathbf{b}_i$ of $\mathbb{R}^n$. $\mathbf{x}(E(\Sigma^\circ))$ is $\mathbf{x}(E(\Sigma))$ together with the vectors $2\mathbf{b}_i$.
- *Hyperplane arrangement*: $\mathcal{H}[\Sigma^\bullet] = \mathcal{H}[\Sigma^\circ]$, and they equal $\mathcal{H}[\Sigma]$ together with all the coordinate hyperplanes $x_i = 0$.
- *Chromatic polynomials*: $\chi^*_{\Sigma^\bullet}(\lambda) = \chi^*_{\Sigma^\circ}(\lambda) = \chi^*_\Sigma(\lambda)$. $\chi_{\Sigma^\bullet}(\lambda) = \chi_{\Sigma^\circ}(\lambda) = \chi^*_\Sigma(\lambda - 1)$ by Theorem 4.5, since the only stable set is $W = \varnothing$.
- *Chromatic numbers*: $\chi(\Sigma^\bullet) = \chi(\Sigma^\circ) = \chi^*(\Sigma)$ since the unbalanced edges prevent the use of color 0.

5.2. **All-positive signed graphs.**

Assume Γ is a graph with no half or loose edges. $+\Gamma$ has almost exactly the same properties as its underlying graph.

- *Balance*: Every subgraph is balanced. $b(S) = c(S)$ for all $S \subseteq E$.
- *Closed sets*: S is closed $\iff$ every edge whose endpoints are connected by S is in S. Closure in $+\Gamma$ is identical to the usual closure in Γ, and the closed sets in $+\Gamma$ are the same as in Γ.
- *Vectors*: If e has endpoints v_i, v_j, then $\mathbf{x}(e) = \pm(\mathbf{b}_j - \mathbf{b}_i)$. All $\mathbf{x}(e) \in$ the subspace $x_1 + \cdots + x_n = 0$.

  If $\Gamma = K_n$ and one takes both signs, the set of vectors is the classical root system

$$A_{n-1} := \{\mathbf{b}_j - \mathbf{b}_i : i, j \leq n,\ i \neq j\}.$$

Thus, $\mathbf{x}(E)$ for any graph is a subset of A_{n-1}.

- *Incidence matrix*: $\mathrm{H}(+\Gamma)$ is the 'oriented incidence matrix' of Γ.
- *Hyperplane arrangement*: If e_k has endpoints v_i, v_j, then h_k has equation $x_i = x_j$. All $h_k \supseteq$ the line $x_1 = \cdots = x_n$.

  Take $\Gamma = K_n$; then $\mathcal{H}[+K_n] = \mathcal{A}_{n-1}$, the hyperplane arrangement dual to A_{n-1}.
- *Chromatic polynomials*: $\chi_{+\Gamma}(\lambda) = \chi^*_{+\Gamma}(\lambda) = \chi_\Gamma(\lambda)$, the chromatic polynomial of Γ.
- *Chromatic numbers*: $\chi(+\Gamma) = \lfloor \chi(\Gamma)/2 \rfloor$ and $\chi^*(+\Gamma) = \lceil \chi(\Gamma)/2 \rceil$.

5.3. **All-positive, full signed graphs.**

The signed graph $+\Gamma^\bullet$ is closely related to $\Gamma + v_0$, which consists of Γ and an extra vertex v_0 which is adjacent to all of V by edges $v_0 v_i$. There is a natural bijection $\alpha : E(+\Gamma^\bullet) \to E(\Gamma + v_0)$ by $\alpha(e) := e$ if $e \in E(\Gamma)$ and $\alpha(f_i) := v_0 v_i$.

- *Balance*: S is balanced if and only if it does not contain any half edge f_i.
- *Closed sets*: S is closed $\iff$ $\alpha(S)$ is closed in $\Gamma + v_0$.
- *Chromatic polynomials*: $\chi_{+\Gamma}(\lambda) = \chi^*_{+\Gamma}(\lambda - 1) = \chi_\Gamma(\lambda - 1)$.
- *Chromatic numbers*: $\chi(+\Gamma^\bullet) = \chi^*(+\Gamma^\bullet) = \lceil \chi(\Gamma)/2 \rceil$.

5.4. **All-negative signed graphs.**

Assume Γ is a graph with no unbalanced edges. $-\Gamma$ is very interesting.

- *Balance*: A subgraph is balanced $\iff$ it is bipartite. $b_{-\Gamma}(S) =$ the number of bipartite components of S (including isolated vertices).
- *Closed sets*: S is closed if the union of its non-bipartite components is an induced subgraph.
- *Vectors*: If e has endpoints v_i, v_j, then $\mathbf{x}(e) = \mathbf{b}_i + \mathbf{b}_j$ (or its negative).
- *Incidence matrix*: $\mathrm{H}(-\Gamma)$ is the 'unoriented incidence matrix' of Γ.
- *Hyperplane arrangement*: h_k has equation $x_i + x_j = 0$ if e_k has endpoints v_i, v_j. Also, $r(\mathcal{H}[-\Gamma]) = \sum_{F \in \mathrm{Lat}\,\Gamma} |\chi_{\Gamma/F}(-\frac{1}{2})|$.
- *Chromatic polynomials*: $\chi^*_{-\Gamma}(\lambda) = \sum_{F \in \mathrm{Lat}\,\Gamma} \chi_{\Gamma/F}(\frac{1}{2}\lambda)$ (Zaslavsky (1982c), Theorem 5.2). $\chi_{-\Gamma}(\lambda)$ has not seemed interesting.
- *Chromatic numbers*: $\chi^*(-\Gamma) =$ the largest size of a matching in the complement of a contraction of Γ (Zaslavsky (1982c), page 299). $\chi(-\Gamma)$ has not yet seemed interesting.

5.5. **Signed expansion graphs.**

The properties of $\pm\Gamma$ and $\pm\Gamma^\bullet$ are closely related to those of Γ.

- *Balance*: Each balanced set $S \subseteq E(\Gamma)$ gives $2^{n-b(S)}$ balanced subsets of $E(\pm\Gamma)$ by switching $+S$.
- *Hyperplane arrangement*: $r(\mathcal{H}[\pm\Gamma^\bullet]) = 2^n(-1)^n \chi_\Gamma(-1) = 2^n |\chi_\Gamma(-1)|$ and

$$r(\mathcal{H}[\pm\Gamma]) = \sum_{W \subseteq V\text{: stable in }\Gamma} (-2)^{n-|W|} |\chi_{\Gamma \setminus W}(-1)|.$$

- *Chromatic polynomials*: $\chi_{\pm\Gamma^\bullet}(\lambda) = 2^n \chi_\Gamma(\frac{1}{2}(\lambda - 1))$, $\chi^*_{\pm\Gamma}(\lambda) = 2^n \chi_\Gamma(\frac{1}{2}\lambda)$, and

$$\chi_{\pm\Gamma}(\lambda) = \sum_{W \subseteq V\text{: stable in }\Gamma} 2^{n-|W|} \chi_{\Gamma \setminus W}(\tfrac{1}{2}(\lambda - 1)).$$

- *Chromatic numbers*: $\chi(\pm\Gamma^\bullet) = \chi^*(\pm\Gamma) = \chi(\Gamma)$, the chromatic number of Γ, and $\chi(\pm\Gamma) = \chi(\Gamma) - 1$.

5.6. **Complete signed expansion graphs.**

The signed expansions $\pm K_n$, called the *complete signed link graph*, and $\pm K_n^\bullet$, called the *complete signed graph*, have very simple properties.

- *Closed sets*: $\mathrm{Lat}(\pm K_n^\bullet) \cong$ the lattice of signed partial partitions of V (Dowling (1973b)).
- *Vectors*: $\mathbf{x}(E(\pm K_n)) = \{\pm(\mathbf{b}_j - \mathbf{b}_i), \pm(\mathbf{b}_j + \mathbf{b}_i) : i \neq j\}$ where we take either $+$ or $-$ for each vector. $\mathbf{x}(E(\pm K_n^\bullet)) = \{\pm(\mathbf{b}_j - \mathbf{b}_i), \pm(\mathbf{b}_j + \mathbf{b}_i) : i \neq j\} \cup \{\pm\mathbf{b}_i\}$ (if f_i is a half edge; but $\pm 2\mathbf{b}_i$ if f_i is a negative loop) where we take either $+$ or $-$ for each vector.

  If we take both signs for each vector we get the classical root systems
  $$D_n := \{\pm(\mathbf{b}_j - \mathbf{b}_i), \pm(\mathbf{b}_j + \mathbf{b}_i) : i \neq j\}$$
  from $\pm K_n$ (where we take both $+$ and $-$ signs), and
  $$B_n := D_n \cup \{\pm\mathbf{b}_i\} \text{ and } C_n := D_n \cup \{\pm 2\mathbf{b}_i\}$$
  from $\pm K_n^\bullet$ (the former if all f_i are half edges, the latter if they are negative loops).
- *Hyperplane arrangement*: $\mathcal{H}[\pm K_n^\bullet] = \mathcal{B}_n = \mathcal{C}_n$ and $\mathcal{H}[\pm K_n] = \mathcal{D}_n$, the duals of B_n, C_n, and D_n. The numbers of regions are $2^n n!$ and $2^{n-1}n!$, respectively.
- *Chromatic polynomials*:
  $\chi_{\pm K_n^\bullet}(\lambda) = (\lambda-1)(\lambda-3)\cdots(\lambda-2n+1)$,
  $\chi_{\pm K_n}(\lambda) = (\lambda-1)(\lambda-3)\cdots(\lambda-2n+3)\cdot(\lambda-n+1)$, and
  $\chi^*_{\pm K_n}(\lambda) = \chi^*_{\pm K_n^\bullet}(\lambda) = \lambda(\lambda-2)\cdots(\lambda-2n+2)$.
- *Chromatic numbers*: $\chi(\pm K_n^\bullet) = \chi^*(\pm K_n^\bullet) = \chi^*(\pm K_n) = n$ and $\chi(\pm K_n) = n-1$.

6. LINE GRAPHS

In this section all graphs are link graphs.

The *line graph* of $\Gamma = (V, E)$ is the graph $\Lambda(\Gamma)$ of adjacency of edges in Γ. Its vertices are the edges of Γ, and two edges are adjacent if they have a common endpoint. When $e, f \in E$ are parallel, in $\Lambda(\Gamma)$ they are doubly adjacent.

6.1. **Bidirected line graphs and switching classes.**

The *line graph of a bidirected graph* B is a bidirection of the line graph of $|\mathrm{B}|$. We write $\Lambda(\mathrm{B}) := (\Lambda(|\mathrm{B}|), \tau_\Lambda)$, where τ_Λ is the bidirection. To define $\tau_\Lambda(ef)$, where $ef \in E(\Lambda(|\mathrm{B}|))$, let v be the vertex at which e, f are adjacent. Then we define
$$\tau_\Lambda(e, ef) := \tau(v, e).$$

This definition implies that, given a signed graph Σ, to define a line graph we first must orient Σ as B, then take the line graph $\Lambda(\mathrm{B})$. Different orientations of Σ give different bidirected line graphs $\Lambda(\mathrm{B})$, which may have different signed graphs $\Sigma_{\Lambda(\mathrm{B})}$.

Lemma 6.1. *Any orientations of any two switchings of Σ have line graphs that are switching equivalent.*

Proof. We assume there are no parallel edges; the proof is not much different if there are any.

Let Σ^ζ be a switching of Σ and let τ and τ' be orientation functions of Σ and Σ^ζ, respectively, giving bidirected graphs B and B' on the underlying graph $\Gamma := |\Sigma|$. Then

$\tau(v,e)\tau(w,e) = -\sigma(e)$ and $\tau'(v,e)\tau'(w,e) = -\zeta(v)\sigma(e)\zeta(w)$ for each edge e with $V(e) = \{v,w\}$.

Let $\Lambda := \Lambda(\mathrm{B})$ and $\Lambda' := \Lambda(\mathrm{B}')$; they have the same underlying graph $\Lambda(\Gamma)$. Suppose e, f are adjacent at v. In the line graph, $\tau_\Lambda(e, ef) = \tau(v,e)$. Thus,

$$\sigma_\Lambda(ef) = -\tau_\Lambda(e,ef)\tau_\Lambda(f,ef) = -\tau(v,e)\tau(v,f)$$

and, similarly, $\sigma'_\Lambda(ef) = -\tau'(v,e)\tau'(v,f)$.

Let $W := e_0e_1\cdots e_{l-1}e_l$, where $e_l = e_0$, be a closed walk in Λ. Thus, e_{i-1}, e_i have a common vertex v_i in Γ. Then

$$\begin{aligned} \sigma_\Lambda(W) &= \sigma_\Lambda(e_0e_1)\cdots\sigma_\Lambda(e_{l-1}e_l) \\ &= \big[-\tau(v_1,e_0)\tau(v_1,e_1)\big]\big[-\tau(v_2,e_1)\tau(v_2,e_2)\big]\cdots\big[-\tau(v_l,e_{l-1})\tau(v_l,e_l)\big] \\ &= (-)^l\tau(v_1,e_0)\big[\tau(v_1,e_1)\tau(v_2,e_1)\big]\cdots\big[\tau(v_{l-1},e_{l-1})\tau(v_l,e_{l-1})\big]\tau(v_l,e_l). \end{aligned} \tag{6.1}$$

Now there are two cases.

If all $v_i = v_1$, then $\sigma(W) = (-)^l\tau(v_1,e_0)\tau(v_l,e_l) = (-)^l$ since $e_0 = e_l$ and $v_l = v_1$.

Otherwise, not all v_i are the same vertex. A consecutive pair v_{i-1}, v_i may be the same or different. If they are the same, the factor $[\tau(v_{l-1},e_{i-1})\tau(v_i,e_{i-1})] = +$, and also $W' := e_0e_1\cdots e_{i-2}e_i\cdots e_{l-1}e_l$ is a walk in Λ. Then $\sigma_\Lambda(W) = -\sigma_\Lambda(W')$. In this way we can reduce W by eliminating consecutive equal vertices while negating the sign of the walk. Similarly, if $v_1 = v_l$ we can eliminate v_l and e_l from the reduced walk. Let $W'' = f_0f_1\cdots f_m$ be the walk in Λ that results after all these reductions and let w_i be the common vertex of f_{i-1}, f_i. W'' has positive length and $f_0 = f_m$, so W'' is a closed walk and it has sign $(-)^{l-m}\sigma(W)$. Furthermore, $V(f_i) = \{w_i, w_{i+1}\}$ for $0 < i < m$. Define w_0 so that $V(f_0) = \{w_0, w_1\}$. Now $w_0f_0w_1f_1\cdots f_{m-1}w_m$ is a walk in Γ. Because $f_0 = f_m$ and, by the construction of W'', $w_1 \neq w_m$, it must be true that $w_0 = w_m$. Therefore, $W_0 := w_0f_0w_1\cdots f_{m-1}w_m$ is a closed walk of length m in Γ.

Now we evaluate $\sigma(W'')$. From (6.1),

$$\begin{aligned} \sigma_\Lambda(W'') &= \sigma_\Lambda(f_0f_1)\cdots\sigma_\Lambda(f_{m-1}f_m) \\ &= (-)^m\tau(w_1,f_0) \\ &\qquad \cdot\big[\tau(w_1,f_1)\tau(w_2,f_1)\big]\cdots\big[\tau(w_{m-1},f_{m-1})\tau(w_m,f_{m-1})\big]\tau(w_m,f_m) \\ &= (-)^m\tau(w_0,f_0)\tau(w_1,f_0) \\ &\qquad \cdot\big[\tau(w_1,f_1)\tau(w_2,f_1)\big]\cdots\big[\tau(w_{m-1},f_{m-1})\tau(w_m,f_{m-1})\big] \\ &= \sigma(W_0). \end{aligned} \tag{6.2}$$

Therefore,

$$\sigma(W) = (-)^{l-m}\sigma(W_0) \tag{6.3}$$

in the case that not all the vertices v_i are the same vertex. When all v_i are the same, we can take W_0 to be a trivial walk (length $m = 0$) and once again we have the same formula.

We prove the lemma by observing that $\sigma(W_0)$ and hence also $\sigma(W)$ is not affected by the choice of orientation and is not altered by switching Σ. Therefore Λ and Λ' have the same positive circles. By Proposition 2.5(ii), Λ and Λ' are switching equivalent. ∎

The line graph of a signed graph Σ cannot be a signed graph, because reorienting an edge of a bidirected graph corresponds to switching the corresponding vertex in its line graph. Therefore, $\Lambda(\Sigma)$ must be a switching class of signatures of $\Lambda(|\Sigma|)$.

Theorem 6.2. *The line graph of a switching class of signed graphs is a well defined switching class of signed graphs.*

Proof. The theorem means that if two signed graphs are switching equivalent, and if each one is oriented arbitrarily, the signed graphs of the line graphs of the two oriented signed graphs are switching equivalent. That is Lemma 6.1. □

In view of this theorem we may write $\Lambda([\Sigma])$ to denote the switching class of line graphs of the signed graphs in the switching class $[\Sigma]$. I sometimes refer to *a line graph* of Σ, meaning any signed graph in the switching class $\Lambda([\Sigma])$.

There is one circumstance in which there is a well defined signed line graph: an all-negative signature. We can say that $\Lambda(-\Gamma) = -\Lambda(\Gamma)$ because of:

Proposition 6.3. *If Γ is a link graph, then $\Lambda([-\Gamma]) = [-\Lambda(\Gamma)]$.*

Proof. Orient $-\Gamma$ so every edge is extraverted; that is, $\tau(v, e) \equiv +$. Then in $\Lambda(-\Gamma, \tau)$, every edge is extraverted; thus, the signed graph underlying $\Lambda(-\Gamma, \tau)$ has all negative edges. □

On the other hand, if Σ is all positive, its line graph cannot usually be made to be all positive or all negative. Thus, all-negative signed graphs are special. Indeed, in connection with line graphs the best way to think of an ordinary graph Γ is as $-\Gamma$, not $+\Gamma$ as in most other respects.

6.2. **Adjacency matrix and eigenvalues.**

The adjacency matrix of the line graph of an ordinary graph is computed directly from the incidence matrix of the graph. The same is true for signed graphs.

Theorem 6.4. *For a bidrected link graph Σ, $A(\Lambda(\Sigma)) = 2I - \mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma)$.*

The statement implies that the orientation of Σ used to calculate $\Lambda(\Sigma)$ does not affect the values in $A(\Lambda(\Sigma))$.

Proof. The (j, j) entry of $\mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma)$ is the sum over all vertices of $\eta(v_i, e_j)^2 = 1$, therefore it equals 2.

The (j, k) entry of $\mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma)$ for $j \neq k$ is the sum over all vertices of $\eta(v_i, e_j)\eta(v_i, e_k)$. By Equation (3.1) this is 0 if e_j and e_k are not adjacent, and if they are adjacent at v_m then it is $\tau(v_i, e_j)\tau(v_i, e_k) = -\sigma(e_j e_k)$.

Thus, the off-diagonal entries of $\mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma)$ are those of $-A(\Lambda(\Sigma))$ and the diagonal entries all equal 2. □

We can interpret Theorem 6.4 as saying that the inner product of representation vectors $\mathbf{x}(e_j)$ and $\mathbf{x}(e_k)$ equals 2 if $j = k$ and $-\sigma(e_j e_k)$ if $j \neq k$. A matrix of inner products is known as a *Gram matrix*; thus, $2I - A(\Lambda(\Sigma))$ is a Gram matrix of vectors with length $\sqrt{2}$.

Corollary 6.5. *All the eigenvalues of a line graph of a signed graph are ≤ 2.*

Proof. A matrix of the form $M^{\mathrm{T}}M$ has non-negative real eigenvalues. Apply Proposition 6.4. □

In unsigned graph theory the eigenvalues of a line graph are ≥ -2. Corollary 6.5 is the generalisation to signed graphs, because in what concerns line graphs, an unsigned graph should be taken as all negative, and the eigenvalues of $-\Sigma$ are the negatives of those of Σ (since $A(-\Sigma) = -A(\Sigma)$).

6.3. **Reduced line graphs and induced non-subgraphs.**
If Σ has a pair of parallel edges e, f, one being positive and the other negative, then in $\Lambda(\Sigma)$ there is a double edge ef that forms a negative digon. Therefore, the (e, f) entry of $A(\Lambda(\Sigma))$ equals 0, and correspondingly, in the reduced line graph $\bar{\Lambda}(\Sigma)$, the vertices e and f are not adjacent.

A well known theorem of Beineke is that a simple graph is a line graph if and only if it has no induced subgraph that is one of nine particular graphs, all of order at most 6. Chawathe and Vijayakumar (1990a) found the analogous 49 excluded induced switching classes for signed simple graphs that are reduced line graphs of signed graphs.

7. Angle Representations

In this section all signed graphs have underlying graphs that are simple. For a non-zero vector $\mathbf{y}$, $\hat{\mathbf{y}}$ denotes the unit vector in the same direction: $\hat{\mathbf{y}} := \|\mathbf{y}\|^{-1}\mathbf{y}$.

An *angle representation* of Σ is a mapping $\boldsymbol{\rho} : V \to \mathbb{R}^d$, for some dimension d, such that

$$\hat{\boldsymbol{\rho}}(v) \cdot \hat{\boldsymbol{\rho}}(w) = \frac{a_{vw}}{\nu} = \begin{cases} 0, & \text{if } vw \text{ is not an edge and } v \neq w, \\ +1/\nu, & \text{if } vw \text{ is a positive edge, and} \\ -1/\nu, & \text{if } vw \text{ is a negative edge,} \end{cases}$$

for some positive constant ν. Equivalently, the representing vectors $\boldsymbol{\rho}(v), \boldsymbol{\rho}(w)$ of adjacent vertices v, w make an angle of $\theta = \arccos(1/\nu) \in [0, \pi/2]$ if $\sigma(vw) = +$, $\pi - \theta$ if $\sigma(vw) = -$, and $\pi/2$ if $vw \notin E$. When $X \subseteq \mathbb{R}^d$, we call $\boldsymbol{\rho}$ an *angle representation in X* if $\operatorname{Im} \boldsymbol{\rho} \subseteq X$. As the length of $\boldsymbol{\rho}(v)$ has no role in the definition, one still has an angle representation after multiplying any $\boldsymbol{\rho}(v)$ by any positive real number. Thus, for instance, one can simply assume all the representing vectors have a particular desired length such as 1 or 2.

Switching v in Σ corresponds to replacing $\boldsymbol{\rho}(v)$ by $-\boldsymbol{\rho}(v)$ in the angle representation.

A generalization of the Gram-matrix interpretation of Theorem 6.4 is a *Gramian angle representation* of Σ. That is an angle representation such that

$$\boldsymbol{\rho}(v) \cdot \boldsymbol{\rho}(w) = a_{vw}$$

for every pair of distinct vertices. It follows by comparing the two definitions that $\|\boldsymbol{\rho}(v)\| \cdot \|\boldsymbol{\rho}(w)\| = \nu$ for adjacent vertices. An *anti-Gramian angle representation* of Σ is a Gramian angle representation of $-\Sigma$. Vijayakumar uses anti-Gramian representations (Vijayakumar (1987a) et al.). Example 7.3 shows why one wants them.

Proposition 7.1. *In a Gramian angle representation of a connected signed simple graph Σ:*
(1) *If Σ is not bipartite, all representing vectors $\boldsymbol{\rho}(v)$ have the same length $\sqrt{\nu}$.*
(2) *If Σ is bipartite with color classes V_1 and V_2, then $\|\boldsymbol{\rho}(v)\| = \alpha_1$ if $v \in V_1$ and $\|\boldsymbol{\rho}(v)\| = \alpha_2$ if $v \in V_2$, where α_1, α_2 are positive real numbers whose product is ν. Then $\boldsymbol{\rho}'$ defined by $\boldsymbol{\rho}'(v) = \hat{\boldsymbol{\rho}}(v)\sqrt{\nu}$ is an angle representation in which all representing vectors have the same length.*

Idea of Proof. Apply the equation $\|\boldsymbol{\rho}(v)\|\|\boldsymbol{\rho}(w)\| = \nu$ for an edge vw, propagated around an odd circle if there is one, and an even circle if there is not. □

Call a Gramian angle representation in which all vectors have the same length *normalized*. Then the Gram matrix of the representing vectors is $A(\Sigma) + \nu I$. Henceforth we assume all Gramian representations are normalized. By Proposition 7.1 any Gramian angle representation becomes normalized if we replace $\boldsymbol{\rho}(v)$ by $\sqrt{\nu}\hat{\boldsymbol{\rho}}(v)$.

Theorem 7.2. *A signed simple graph Σ has a Gramian (or, anti-Gramian) angle representation with constant ν if and only if the eigenvalues of Σ are $\geq -\nu$ (respectively, $\leq \nu$).*

Proof. This proof is based on the treatment of equiangular lines by Seidel et al. (see Seidel (1976a, 1995a), e.g., or Godsil and Royle (2001a)).

We consider a normalized Gramian angle representation. The Gram matrix $A(\Sigma) + \nu I$ has an eigenvalue $\lambda + \nu$ for each eigenvalue λ of $A(\Sigma)$. As a Gram matrix has non-negative eigenvalues, every $\lambda \geq -\nu$.

Now assume Σ has eigenvalues $\geq -\nu$. The matrix $A(\Sigma) + \nu I$ is positive semidefinite and symmetric. It follows by matrix theory that $A(\Sigma)+\nu I$ is the Gram matrix of vectors $\mathbf{v}_i \in \mathbb{R}^n$ for $v_i \in V$, i.e., $a_{ij} + \nu\delta_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$ for all i, j. Then $\boldsymbol{\rho}(v_i) := \mathbf{v}_i$ is a (normalized) Gramian angle representation of Σ with constant ν. □

Example 7.3. The mapping $\mathbf{x} : E(\Sigma) \to \mathbb{R}^n$ of Section 3.1, which gives a vector representation of Σ, gives an anti-Gramian angle representation of $\bar{\Lambda}(\Sigma)$. We take $\boldsymbol{\rho} := \mathbf{x}$, since $V(\bar{\Lambda}(\Sigma)) = E(\Sigma)$. The constant $\nu = 2$ and the angle $\theta = \pi/3$. Every vector $\mathbf{x}(e)$ has the same length, $\sqrt{2}$, and the inner products are $+1$ if $\sigma_\Lambda(ef) = -$, in which case the angle between $\mathbf{x}(e)$ and $\mathbf{x}(f)$ is $\pi/3$, and -1 if $\sigma_\Lambda(ef) = +$, in which case the angle between $\mathbf{x}(e)$ and $\mathbf{x}(f)$ is $2\pi/3$.

The vectors $\mathbf{x}(e)$ are some of the vectors of the root system D_n mentioned in Section 5.6, and the angle representation of $\bar{\Lambda}(\pm K_n)$ is all of D_n. The treatment of $\mathbf{x}$ as an angle representation of a reduced line signed graph is implicit in Cameron, Goethals, Seidel, and Shult (1976a), but the proper signed-graphic treatment only came later, in Zaslavsky (1979a, 1984c, 2010b).

Cameron, Goethals, et al. used Gramian angle representations of unsigned graphs to classify the graphs Γ whose eigenvalues are ≥ -2. They obtained the all-positive and all-negative cases of the following theorem. (The all-positive case corresponds, in our terminology, to a Gramian representation of $-\Gamma$, and the all-negative case to a Gramian representation of $+\Gamma$, since the theorem concerns anti-Gramian representations.)

The root system E_8 is defined by

$$E_8 := D_8 \cup \left\{\tfrac{1}{2}(\varepsilon_1, \ldots, \varepsilon_8) \in \mathbb{R}^8 : \varepsilon_i \in \{\pm 1\},\ \varepsilon_1 \cdots \varepsilon_8 = +1\right\}.$$

Theorem 7.4. *An anti-Gramian angle representation of Σ with $\nu = 2$ is a vector representation of a reduced line graph $\bar{\Lambda}(\Sigma)$, or else $|V(\Sigma)| \leq 184$ and the representation is in E_8.*

Proof. As Vijayakumar (1987a) observed, Cameron, Goethals, et al. (1976a) implies that an anti-Gramian angle representation of Σ having $\nu = 2$ is, after choosing the appropriate coordinate system, either in D_n for some $n > 0$ or in E_8. If the representation is in D_n, then there is a signed graph Σ' with vertex set $E(\Sigma)$ whose vector representation $\mathbf{x} : E \to \mathbb{R}^n$ is the same as $\boldsymbol{\rho}$, and it is easy to verify that Σ' is a reduced line graph of Σ.

If the representation is in E_8, the order of Σ cannot be greater than the number of pairs of opposite vectors in E_8, which is 184 (because $|D_n| = n(n-1)$ and the number of choices for $(\varepsilon_1, \ldots, \varepsilon_8)$ is 2^7). □

Corollary 7.5. *A signed simple graph has all eigenvalues ≤ 2 if and only if it is a reduced line graph of a signed graph or it has order ≤ 184 and has an anti-Gramian angle representation in E_8.*

Proof of Sufficiency. Vectors in D_n or E_8 have angles $\pi/3$, $2\pi/3$, and $\pi/2$, therefore an angle representation in D_n or E_8 has $\nu = 2$. □

In particular, the number of signed simple graphs with all eigenvalues ≤ 2 that are not reduced line graphs of signed graphs is finite.

Example 7.6. Let Γ be a simple graph with $V = \{v_1, \ldots, v_n\}$. A *cocktail party graph* CP_m is $K_{2m} \setminus M$ where M is a perfect matching. A *generalized line graph* $\Lambda(\Gamma; m_1, \ldots, m_n)$, where $m_i \in \mathbb{Z}_{\geq 0}$, is the disjoint union $\Lambda(\Gamma) \cupdot \mathrm{CP}_{m_1} \cupdot \cdots \cupdot \mathrm{CP}_{m_n}$ with additional edges edges from every vertex in CP_{m_i} to every $v_i v_j \in V(\Lambda(\Gamma))$ (Hoffman (1977a)). (It is the line graph if all $m_i = 0$.) Hoffman found that a generalized line graph has least eigenvalue ≥ -2, just like a line graph. Then Cameron, Goethals, et al. proved that there are no other graphs with least eigenvalue ≥ -2 except a handful that have Gramian angle representations in E_8.

This is a consequence of Corollary 7.5. We deduce it by showing how $\Lambda(\Gamma; m_1, \ldots, m_n)$ is a reduced line graph of a signed graph. Let $\Gamma(m_1, \ldots, m_n)$ be $-\Gamma$ with m_i negative digons attached to v_i. The other vertex of each negative digon is a new vertex; thus, $\Gamma(m_1, \ldots, m_n)$ has order $n + m_1 + \cdots + m_n$ and $|E| + 2(m_1 + \cdots + m_n)$ edges. Then $\bar{\Lambda}(\Gamma(m_1, \ldots, m_n)) = -\Lambda(\Gamma; m_1, \ldots, m_n)$. The eigenvalue property of $\Lambda(\Gamma; m_1, \ldots, m_n)$ follows immediately from Theorem 7.2.

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