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Orientation of Signed Graphs*

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A graph with signed arcs is *oriented* by directing each end of each arc in accordance with a sign-compatibility rule. We prove that the regions of the hyperplane representation of a signed graph Σ, as well as the vertices of the convex hull of all degree vectors of orientations of Σ, are in natural one-to-one correspondence with the cyclic orientations of Σ. The proof uses the oriented matroid of a signed graph. For use elsewhere, we also develop the relationships between orientations and hyperplane representations of a signed graph and those of its double covering graph.

1. INTRODUCTION

In this paper we develop a theory of orientation of signed graphs (graphs where each arc is labelled + or −). The chief purpose is to generalize a simple but profound observation of Curtis Greene's: the acyclic orientations of a graph Γ are in natural one-to-one correspondence with the regions of an associated arrangement of hyperplanes $H[\Gamma]$ (see Proposition 4.3 below). With this observation one can count the regions of the arrangement by using Stanley's theorem on the number of acyclic orientations [10], or conversely count the acyclic orientations from a general formula for the number of regions in an arrangement [13, 14]. More significantly, Greene's observation leads to new results on the numbers of orientations of various types; for instance, the number of acyclic orientations with a single specified source [5, 6, 14]. Since the associated arrangement generalizes to signed graphs [15, 16], where it consists of certain hyperplanes determined by equations of the forms $x_i = \pm x_j$ and $x_i = 0$, it seemed to me that Greene's correspondence should generalize as well. But it was necessary first to find the right definitions of orientation and acyclicity for signed graphs; then the desired generalization turned out to be surprisingly hard to prove. Greene's proof takes a few lines: our extension to signed graphs requires a few pages and the theory of oriented matroids. Even so, the analogs of the ancillary results on restricted acyclic orientations are not immediate corollaries. The machinery necessary for their proof (in [6, Section 9]) is developed here in Sections 5 and 6.

To orient a signed graph Σ one assigns a direction to each end of an arc—what Edmonds calls 'bidirecting' the arc—so that the two arrows on a positive arc agree (this is like ordinary graph orientation), but those on a negative arc are opposed. A 'cycle' is then a matroid circuit with no source or sink. This generalizes ordinary oriented cycles, since the signed-graphic matroid $G(\Sigma)$ generalizes the polygon matroid of a graph [16].

Our first proof of the generalized Greene's correspondence, in earlier versions of this paper, relied entirely on the double covering graph $\tilde{\Sigma}$ (Section 5), which has unsigned arcs; through it we can reduce signed to ordinary orientation by establishing a relationship between cycles in Σ and in $\tilde{\Sigma}$. Following the suggestion of a referee, I have simplified the proof and considerably strengthened the theory by first establishing

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the connections among oriented signed graphs, the standard real vector representation, and oriented matroids (Section 3). From them we deduce relationships among orientations of Σ, their 'net degree' vectors (the *net degree* of a vertex is its indegree less its outdegree), and the zonotope generated by the representing vectors of Σ, which we dub the 'acyclotope'; and, by dualizing, the crucial theorems about the hyperplane arrangement $H[\Sigma]$ (Section 4). In Section 5 we show how cycles in Σ and $\tilde{\Sigma}$ are related. This is interesting both in its own right, as an oriented strengthening of the circuit relationship of [16, Section 6], and as enabling us to prove Proposition 6.1, which is needed in [6, Section 9]. There we regard the arrangement $H[\Sigma]$, which resides in $\mathbb{R}^N$ (where N is the node set), as a cross-section of $H[\tilde{\Sigma}]$, which sits in $\mathbb{R}^N \times \mathbb{R}^N$. Proposition 6.1 states that the regions of $H[\tilde{\Sigma}]$ that meet the cross-section are just those corresponding to orientations of $\tilde{\Sigma}$ obtained by lifting acyclic orientations of Σ.

Our results are here proved only for finite graphs (in part because the oriented matroid theory of [2] is finite). Since $G(\Sigma)$ is finitary I do not doubt that they will hold in general (when they are meaningful). But I prefer not to tackle here the issues raised by infinities.

Signed graph orientation is interesting for other reasons than the geometric connection emphasized here. For instance, it enriches Edmonds' theory of bidirected flows (see [8, Section 6.3]). Because bidirecting a graph is the same as orienting node-arc incidences, it determines arc signs and therefore a matroid $G(\Sigma)$. One can show by deletion-and-contraction arguments that, if $t(x, y)$ is the Tutte polynomial of $G(\Sigma)$, then $|t(0, 1-n)|$ is the number of nowhere-zero bidirected flows with values in a finite abelian group of odd order n. This partly extends a result for ordinary graphs [12, Theorem XI] and regular matroids [4, Theorem III]. Although the method of proof is familiar, the result is a new one because $G(\Sigma)$ need not be binary [16]. (Integer-valued bidirected flows have been studied recently by Bouchet, who proved a sufficient-width theorem [3].)

Another point of interest is that an arc two-coloring of a graph Γ can be regarded as an orientation of the all-negative graph $-\Gamma$, by calling the colors 'introverted' and 'extroverted' (the two ways to orient a negative arc). Let us call an *alternating cycle* either an even circle with alternating colors or a pair of odd circles connected only by a simple path of length at least 0 and colored so there is no monochromatic node. Since this is a cycle in the oriented all-negative graph, by Corollary 3.7 the union of all alternating cycles is closed in the even-cycle matroid of Γ. Moreover, whatever one can prove about acyclic orientations translates into a statement about two-colorings with no alternating cycles. For instance, their number is a known function of Γ [18, Section 8.9].

A third outgrowth of signed graph orientation is that it gives us the right language with which to define a notion of a line graph of a digraph in such a way that it has the nice matrix properties of ordinary line graphs. One finds also that the properties of Hoffman's generalized line graphs result from their being line graphs of special oriented signed graphs. This topic will be treated elsewhere.

Finally, Stanley's theorem on the number of compatible pairs of colorings and acyclic orientations generalizes to signed graphs [17].

Incidental Notes. (1) References [6] and [17] cite results from this paper, the identifications of which have been changed since the early versions. The changes are as follows: 2.1 is now 5.1, 2.4 is 3.7, 3.1 is now 3.10, 4.2 is now 4.4, 4.5 is 4.6.

(2) I take this opportunity to correct misleading remarks in [6]. The 'Note' on page 124 should have said that the basic results of the article [6] as a whole date from 1975

and were announced by Greene in [5] and Zaslavsky in [14]. The results on signed graphs of [6, Section 9] date from 1979–1981, after the first (1979) version of the present paper. Also, the statement which I inserted on page 102 that the task of extending 'most of our results' to oriented matroids is 'straightforward' is a very considerable exaggeration. Indeed, it is a fairly complex task just to formulate many of the results in terms of oriented matroids.

2. Fundamental Definitions and Lemmas

Graphs. An (*unsigned*) *graph* Γ consists of a node set $N(\Gamma)$ and an arc set $E(\Gamma)$. (We often write merely N, E and also $\Gamma=(N, E)$.) Multiple arcs are allowed. We assume finiteness. Besides *links* (two distinct endpoints) and *loops* (two coincident endpoints) we need *half arcs* (one endpoint) and *loose arcs* (no endpoints; here regarded as a kind of balanced loop). We write the *incidence* between an arc e and its endpoint v as (v, e). (The two incidences of a loop at v are not distinguished by the notation, but that will not cause any problems.) We denote the set of incidences by $I(\Gamma)$. Other useful notations are as follows:

$X^c = N\backslash X$ for a node set X,
$S^c = E\backslash X$ for an arc set S,
$E^* =$ the set of arcs excluding half arcs,
$\Gamma{:}X = (X, E{:}X)$, the subgraph induced by the node set X, where $E{:}X$ is the set of arcs having their endpoints in X (but not loose arcs),
$E:\langle X, Y\rangle =$ the set of arcs having one endpoint in X and the other in Y, where $X, Y \subseteq N$.

Signed graphs. We summarize basic concepts of signed graphs from [16]. A *signed graph* $\Sigma=(\Gamma, \sigma)$ consists of an underlying unsigned graph $\Gamma=(N, E)$ and a *sign mapping* $\sigma\colon E^* \to \{\pm\}$ such that $\sigma \mid \{\text{loose arcs}\} = +$. We write $\Sigma{:}X$ for the subgraph induced by a node set X. The product of arc signs along a path P is denoted by $\sigma(P)$. A circle C is *balanced* if $\sigma(C)=+$; an arc set S is *balanced* if it contains no unbalanced circle or half arc (these are called *unbalanced figures*). We set

$\pi_b(S) = \{B \subseteq N\colon B \neq \varnothing \text{ is the node set of a balanced component of } S\}$,
$N_u(S) =$ the set of nodes of unbalanced components of S,
$b(S) = \#\pi_b(S)$.

(Here S means the subgraph (N, S).) For restrictions and contractions of Σ, see below.

The *signed-graphic* (or *bias*) *matroid* $G(\Sigma)$ is the matroid on E the *circuits* of which are the balanced circles, and the arc sets consisting either of two unbalanced figures having one node and no arcs in common, or of two disjoint unbalanced figures and a simple connecting path meeting each figure once, at an endpoint. The two unbalanced types are unified if we view the former as having a connecting path of length zero. A *bond* (cocircuit) of $G(\Sigma)$ is any arc set of the form

$$D = (E{:}\langle X, Y\rangle) \cup (D{:}X), \tag{2.1}$$

for which $\Sigma{:}(X \cup Y)$ is a connected component of Σ, X is non-empty and disjoint from Y, $\Sigma{:}X$ is connected, $D^c{:}X$ is a maximal balanced arc set in $\Sigma{:}X$, and $b(\Sigma{:}Y) = b(\Sigma{:}(X \cup Y))$. The roles of X and Y are interchangeable if $\Sigma{:}(X \cup Y)$ is balanced, but not otherwise.

The *rank* function of $G(\Sigma)$ is $\operatorname{rk} S = n - b(S)$. The lattice of closed arc sets of $G(\Sigma)$ is denoted by $\operatorname{Lat} G(\Sigma)$.

Switching Σ by a switching function $\nu: N \to \{\pm\}$ means replacing σ by σ^{ν}, defined by $\sigma^{\nu}(e) = \nu(v)\,\sigma(e)\,\nu(w)$, where v and w are the two (possibly coincident) endpoints of e. The switched graph is written Σ^{ν}. Switching does not alter balance or the matroid. We call ν a *potential* for Σ if Σ^{ν} is (balanced and) all positive, since then $\sigma(e) = \nu(v)\,\nu(w)$ for each arc e (where v and w are its endpoints).

Orientation. A *bidirection* of an unsigned graph Γ is any mapping $\tau: I(\Gamma) \to \{\pm\}$. The interpretation of τ is this: if $\tau(v, e) = +$, the incidence (v, e) points into the node v; if $\tau(v, e) = -$, (v, e) points away from v. The possible orientations of arcs are illustrated in Figure 1. The sign mapping σ given by

$$\sigma(e) = -\tau(v, e)\,\tau(w, e) \tag{2.2}$$

whenever e has the two (possibly coincident) endpoints v and w, determines a signed graph $\Sigma(\tau)$. Conversely, given a signed graph Σ we call any bidirection τ satisfying (2.2) an *orientation of* Σ, and we call (Σ, τ) an *oriented signed graph*. We write $\tau{:}X$ to mean $\Sigma{:}X$ oriented by τ.

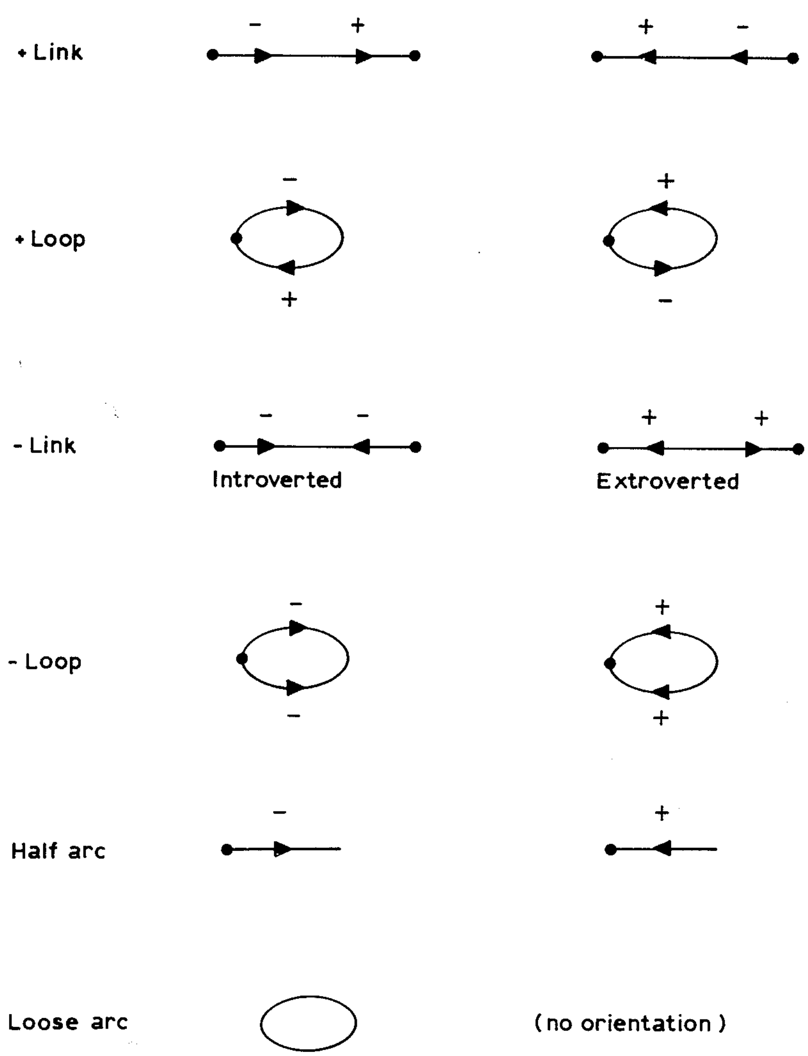

FIGURE 1. Types of oriented arcs.

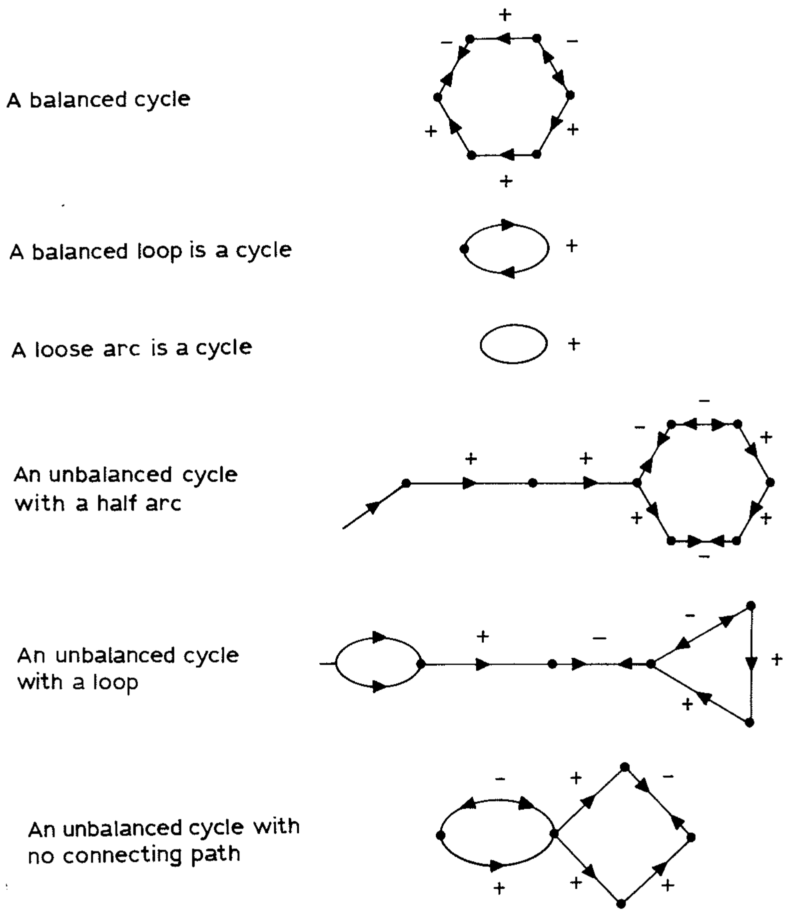

FIGURE 2. Cycles of oriented signed graphs. Positive arcs are marked (usually) by single arrows, and negative arcs by double arrows.

A *cycle* in an oriented signed graph Σ is a matroid circuit of $G(\Sigma)$ that has neither a source nor a sink. The *cyclic part* $C(\tau)$ of an orientation τ is the union of all cycles; τ is *acyclic* if $C(\tau)=\varnothing$. Cycles are shown in Figure 2.

A *cocycle* of τ is a bond D for which D^c:X has a potential $\nu_X: X \to \{\pm\}$ such that $\tau(v, e)=\nu_X(v)$ for every incidence (v, e) with $v \in X$, $e \in D$. Equivalently, in some switching of Σ every arc of D points into X wherever it meets X.

Switching τ by $\nu: N \to \{\pm\}$ means replacing τ by τ^ν, defined by

$$\tau^\nu(v, e)=\nu(v)\,\tau(v, e).$$

Then $\Sigma(\tau^\nu)=\Sigma(\tau)^\nu$. Switching does not affect cyclicity.

A *restriction* subgraph $\Sigma \mid S=(N, S, \sigma \mid S)$ of an oriented signed graph (Σ, τ), where $S \subseteq E$, is oriented by restricting τ. We write this orientation $\tau \mid S$ or $\tau \backslash S^c$.

A *contraction* Σ/S is oriented by τ in a more complicated way. To construct the contracted orientation τ/S, first switch (Σ, τ) until every balanced component of S consists entirely of positive arcs. Then discard $N_u(S)$ (but not any arcs); this may reduce some arcs to half arcs or loose arcs. Then coalesce all the nodes in each balanced component of S. Finally, discard the arcs of S. This defines both the contracted signed graph Σ/S and the contracted orientation τ/S, not uniquely but up to switching. That is all we need since switching does not affect cyclicity.

The oriented incidence matrix. The *incidence matrix* of τ is the $N \times E$ matrix $M(\tau) = (m_{ve}; v \in N, e \in E)$ the entries of which are

$$m_{ve} = \begin{cases} +1 & \text{if } e \text{ enters } v, \\ -1 & \text{if } e \text{ leaves } v, \text{ except} \\ \pm 2 & \text{if } e \text{ is a negative loop at } v \ (+2 \text{ if entering}, -2 \text{ if leaving}), \text{ and} \\ 0 & \text{if } e \text{ is a positive loop or not incident with } v. \end{cases}$$

This matrix is an oriented incidence matrix of Σ in the sense of [16, Section 8A].

The column of $M(\tau)$ labelled by e is a vector $x_\tau(e) \in \mathbb{R}^N$, which depends only on Σ up to sign (i.e. vector orientation). We call the mapping $x_\tau: E \to \mathbb{R}^N$ or its image the *vector representation* of τ. Theorem 8B.1 of [16] says that the linear dependence matroid of $x_\tau(E)$, which is clearly independent of τ, equals $G(\Sigma)$.

Oriented matroids. We shall require some properties of oriented matroids. For notation we refer to [2]. Let M be an oriented matroid on the point set E. The union of all cycles (positive circuits) is $C(M)$. For $e \in E$, M_e denotes M with e reoriented (signs reversed in every signature).

LEMMA 2.1. *$C(M^\perp) = E \setminus C(M)$. Also, $C(M)$ is closed in the underlying matroid.*

PROOF. For the first statement see [7, p. 233]. Since $C(M)$ is the intersection of complements of cocycles, it is closed. □

LEMMA 2.2. *For any $S \subseteq E$ we have $C(M/S) \supseteq C(M) \setminus S$. If $S \subseteq C(M)$, then $C(M/S) = C(M) \setminus S$.*

PROOF. For the first statement see [2, Proposition 4.4]. For the second, by [2, Theorem 4.3] we have $C((M/S)^\perp) = C(M^\perp \setminus S)$, which equals $C(M^\perp)$ since $S \cap C(M^\perp) = \varnothing$. Dualizing by Lemma 2.1, $C(M/S) = C(M) \setminus S$. □

LEMMA 2.3. (a) *If $e \in C(M) \cap C(M_e)$, then $C(M) = C(M_e)$ and $C(M \setminus e) = C(M/e) = C(M) \setminus e$.*
(b) *If $e \in C(M) \setminus C(M_e)$, then $C(M \setminus e) = C(M_e) \subseteq C(M) \setminus e$ and $C(M/e) = C(M) \setminus e$.*
(c) *If $e \notin C(M) \cup C(M_e)$, then $C(M) = C(M_e) = C(M \setminus e) = C(M/e)$.*

This result strengthens [7, Lemma 3.1.1].

PROOF. If C is a cycle in M and C_e is one in M_e and both contain e, then $(C \cup C_e) \setminus e$ is a union of cycles by [2, Axiom (II)]. Part (a) follows from this and Lemma 2.2. Part (b) is obvious, given Lemma 2.2. In part (c) only $C(M/e) = C(M)$ requires comment. If X is a cycle in M/e, then X or $X + e$, whichever is a circuit, is a cycle in M. From this (c) follows. □

3. ORIENTATION AND GEOMETRY

An orientation τ of Σ defines a natural oriented matroid structure upon E which agrees with that implied by positive dependence in the vector representation. This is our fundamental geometrical result.

We can describe an oriented matroid by its circuit signature or its bond signature [2]. Let C be a circuit of Σ, and ξ any cyclic orientation of C. We define

$$C^+ = \{f \in C: \tau \,|\, f = \xi \,|\, f\}, \qquad C^- = \{f \in C: \tau \,|\, f \neq \xi \,|\, f\}.$$

The names C^+ and C^- are interchanged by reversing ξ but the bipartition $\{C^+, C^-\}$ is unique. The *circuit signature* of τ is

$$\mathcal{O}(\tau) = \{(C^+, C^-): C \text{ is a circuit and } \xi \text{ is a cyclic orientation of } C\}.$$

LEMMA 3.1. *Let e, f be arcs of a circuit C and let P be a simple path in $C \setminus \{e, f\}$ from an endpoint v of e to an endpoint w of f. Suppose $e \in C^\varepsilon$ and $f \in C^\delta$. Then $\delta\varepsilon = -\sigma(P)\, \tau(v, e)\, \tau(w, f)$.*

PROOF. Let us switch so that P is all positive and $\tau(v, e) = +$ and let us choose ξ so that $\varepsilon = +$. Then ξ directs P from v to w, so $f \in C^+$ precisely when $\tau(w, f) = -$. □

Let D be a bond of Σ as in (2.1) and let ν_X be a potential for $D^c{:}X$. We define

$$D^\delta = \{f \in D: \nu_X(v)\tau(v, f) = \delta \text{ for } (v, f) \in I(\Gamma) \text{ and } v \in X\},$$

where $\delta = +$ or $-$. The *bond signature* of τ is

$$\mathcal{O}^\perp(\tau) = \{(D^+, D^-): D \text{ is a bond and } \nu_X \text{ a potential for } D^c{:}X\}.$$

LEMMA 3.2. *For each bond D, $\{D^+, D^-\}$ is well defined.*

PROOF. For $f \in D{:}X$ we might have an inconsistency from choosing the other end of f. Say v, w are the endpoints of f. Then $\nu_X(v)\nu_X(w) = -\sigma(f) = \tau(v, f)\tau(w, f)$. It follows that δ is independent of the endpoint selected.

In case both $\Sigma{:}X$ and $\Sigma{:}Y$ are balanced we could have reversed the roles of X and Y, giving the criterion

$$D^\varepsilon = \{f \in D: \nu_Y(y)\tau(y, f) = \varepsilon \text{ for } (y, f) \in I(\Gamma) \text{ and } y \in Y\}.$$

But in this case $\Sigma{:}(X \cup Y)$ is balanced. Thus

$$\begin{aligned}\delta\varepsilon &= \nu_X(v)\nu_Y(y)\, \tau(v, f)\tau(y, f)\\ &= -\nu_X(v)\nu_Y(y)\sigma(e),\end{aligned}$$

which is $-$ if we choose ν to be a potential for $\Sigma{:}(X \cup Y)$ and $\nu_X = \nu|_X$, $\nu_Y = \nu|_Y$. Therefore the $\{D^+, D^-\}$ determined by X and Y are one and the same. □

THEOREM 3.3. *Let τ be an orientation of Σ. The oriented matroid on $G(\Sigma)$ implied by the vector representation x_τ has circuit signature $\mathcal{O}(\tau)$ and bond signature $\mathcal{O}^\perp(\tau)$.*

PROOF. The proof has two parts. The easier half is to show that $\mathcal{O}(\tau)$ is the right circuit signature.

Let C be a circuit of Σ and let a linear dependence relation for $x_\tau(C)$ be

$$\sum_{e \in C} \alpha_e x_\tau(e) = 0.$$

A cycle has the positive dependency given by $\alpha_e = +1$ except $\alpha_e = +2$ if C is unbalanced and contains e in its connecting path. If C is not a cycle, we see that $C^+ = \{e: \alpha_e > 0\}$ and $C^- = \{e: \alpha_e < 0\}$, as required, by reorienting it to be a cycle, calculating α_e, and negating α_e on the reoriented arcs.

The second half of the theorem now follows from Lemma 3.4 and the uniqueness of the dual [2, Theorem 2.2].

LEMMA 3.4. *$\mathcal{O}^{\perp}(\tau)$ is the orthogonal dual of $\mathcal{O}(\tau)$ in the sense of oriented matroids.*

PROOF. We show that $\mathcal{O} = \mathcal{O}(\tau)$ and $\mathcal{O}' = \mathcal{O}^{\perp}(\tau)$ satisfy property (III) of [2, Theorem 2.2]. The property is that, if C and D are a circuit and bond having 2 or 3 common elements, then

$$(C^+ \cap D^+) \cup (C^- \cap D^-) \neq \varnothing \tag{III$_1$}$$

and

$$(C^+ \cap D^-) \cup (C^- \cap D^+) \neq \varnothing. \tag{III$_2$}$$

Let D be given by (2.1) with $B = D^c : X$ balanced. Say $e \in C \cap D$ with endpoint $v \in X$. We may switch so that B is all positive and $\tau(v, e) = +$; then $e \in C^+ \cap D^+$ and (III$_1$) is satisfied.

There must be a second arc $f \in C \cap D$ connected to e at v by a simple path $P \subseteq B$. (If not, e would have to cut C into halves. Each half would be unbalanced. The half containing v would have to lie in B. This is impossible.) Let w be the node at which P meets f. By Lemma 3.1, $f \in C^{-\tau(w,f)}$. Since also $f \in D^{\tau(w,f)}$, (III$_2$) is satisfied. □

Restrictions and contractions of oriented matroids are defined in [2, Section 4].

THEOREM 3.5. *Let τ orient Σ and let $S \subseteq E$. Then $\mathcal{O}(\tau \mid S) = \mathcal{O}(\tau) \mid S$ and $\mathcal{O}(\tau/S) = \mathcal{O}(\tau)/S$.*

PROOF. The restriction part is trivial. We prove the part concerning contractions by the linear algebra model. We assume by switching that each balanced component of S is positive. Let $\pi = \pi_b(S)$ and define $L: \mathbb{R}^N \to \mathbb{R}^{\pi}$ by

$$L(x)_B = \sum_{v \in B} x_B \qquad \text{for } B \in \pi.$$

Obviously, L is linear. Its kernel is spanned by $x_\tau(S)$. Now we appeal to a lemma of oriented matroid representation theory. For $E_1 \subseteq \mathbb{R}^n$, let $\mathcal{O}(E_1)$ be the circuit signature of the oriented matroid of positive dependence.

LEMMA 3.6. *Let $E_1 \subseteq \mathbb{R}^n$ and let $L: \mathbb{R}^n \to \mathbb{R}^p$ be linear. Suppose* Ker L *is spanned by $S_1 \subseteq E_1$. Then $\mathcal{O}(L(E_1 \backslash S_1)) = \mathcal{O}(E_1)/S_1$.*

PROOF. Each circuit C_1 of $L(E_1 \backslash S_1)$ has the form $L(C_1' \backslash S_1)$ for some circuit C_1' of E_1. The signature of $C_1' \backslash S_1$ in $\mathcal{O}(E_1)/S_1$ is that of C_1' restricted to $C_1' \backslash S_1$. We compare this to $\mathcal{O}(L(E_1 \backslash S_1))$. Let the linear dependence of C_1 be

$$\sum_{e \in C_1' \backslash S_1} \alpha_e L(e) = 0$$

and that of C_1' be

$$\sum_{e \in C_1' \backslash S_1} \beta_e e + \sum_{f \in C_1' \cap S_1} \beta_f f = 0.$$

Applying L to the latter expression, $\sum_e \beta_e L(e) = 0$. Thus the β_e are constant multiples of the α_e, which implies that C_1 has the same signature as $C_1' \backslash S_1$. □

The theorem follows upon setting $E_1 = x_\tau(E)$ (with other appropriate identifications) and observing that $L(E_1 \backslash S_1) = x_{\tau/S}(E \backslash S)$, the vector set representing τ/S. □

From Theorems 3.3 and 3.5 and the lemmas in Section 2 we draw several conclusions.

COROLLARY 3.7. *The cyclic part of an orientation τ of the signed graph Σ is closed in $G(\Sigma)$. Its complement is the union of all cocycles.*

COROLLARY 3.8. *For any $S \subseteq E$ we have $C(\tau/S) \supseteq C(\tau)/S$.*

COROLLARY 3.9. *If $S \subseteq C(\tau)$, then $C(\tau/S) = C(\tau)/S$. In particular, $\tau/C(\tau)$ is acyclic.*

COROLLARY 3.10. *Let τ be an orientation of the signed graph Σ and let $e \in E$. Let τ_e be τ with the orientation of e reversed. Then:*
(a) *If $e \in C(\tau) \cap C(\tau_e)$, then*

$$C(\tau) = C(\tau_e), \qquad C(\tau \backslash e) = C(\tau) \backslash e, \qquad C(\tau/e) = C(\tau)/e.$$

(b) *If $e \in C(\tau) \backslash C(\tau_e)$, then*

$$C(\tau \backslash e) = C(\tau_e) \subseteq C(\tau) \backslash e, \qquad C(\tau/e) = C(\tau)/e.$$

(c) *If $e \notin C(\tau) \cup C(\tau_e)$, then*

$$C(\tau) = C(\tau_e) = C(\tau \backslash e) = C(\tau/e).$$

4. GEOMETRY OF ORIENTATIONS OF A SIGNED GRAPH

Score vectors and the acyclotope. Let $S(e)$ be the line segment between $x_\tau(e)$ and $-x_\tau(e)$ in $\mathbb{R}^N$, and let

$$Z[\Sigma] = \sum_{e \in E} S(e).$$

Such a vector sum of line segments is called a *zonotope* (a reference is [9]). For $Z[\Sigma]$ we prefer the vivid name *acyclotope* because it is the convex hull of the net degree vectors of the acyclic orientations of Σ (Corollary 4.7). (The *permutohedron,* the convex hull of all permutations of $(1, 2, \ldots, n)$, is therefore a half-sized translate of $Z[+K_n]$. And the vertices of $Z[\pm K_n^\circ]$, where $\pm K_n^\circ$ is the signed graph with all possible links and negative loops, are the permutations of $(\pm 2, \pm 4, \ldots, \pm 2n)$.)

Our theorem is a relationship between faces of $Z[\Sigma]$ and the *net degree vector* (or *score vector*)

$$d(\tau) = \sum_{e \in E} x_\tau(e)$$

of an orientation τ, so called because its v-component is the indegree less the outdegree at v. Suppose ξ acyclically orients a contraction Σ/T; let

$$f(\Sigma/T, \xi) = \operatorname{conv}\{d(\tau)\colon \tau \text{ orients } \Sigma \text{ and agrees with } \xi \text{ on all arcs } e \notin T\}.$$

(For this definition it is necessary to use the same switching throughout in contracting τ onto Σ/T.)

THEOREM 4.1. *The mapping $(\Sigma/T, \xi) \rightarrow f(\Sigma/T, \xi)$ is a one-to-one correspondence between the acyclically oriented contractions of Σ and the faces of $Z[\Sigma]$, under which* $\dim f(\Sigma/T, \xi) = \operatorname{rk} T$.

If τ orients Σ, then the smallest face of $Z[\Sigma]$ which contains $d(\tau)$ is $f(\tau) = f(\Sigma/C(\tau), \tau/C(\tau))$, the dimension of which equals $\operatorname{rk} C(\tau)$.

PROOF. Suppose that we forget signed graphs momentarily and redefine Σ to be a set of vectors $x_e \in \mathbb{R}^N$ indexed by E, $G(\Sigma)$ to be the linear dependence matroid, $Z[\Sigma]$ the zonotope $\sum_{e\in E} \operatorname{conv}(x_e, -x_e)$, Σ/T the projection of $\Sigma \setminus \{x_e : e \in T\}$ onto a subspace complementary to $\operatorname{lin}\{x_e : e \in T\}$; furthermore, we regard τ as a mapping $E \to \{\pm\}$, $d(\tau) = \sum_{e\in E} \tau_e x_e$, and $C(\tau) =$ the set of $e \in E$ such that $\tau_e x_e$ is positively dependent on the other vectors $\tau_f x_f$. Then the theorem is a standard property of zonotopes. Specializing it to a signed graph Σ, with $\tau_e x_e$ defined as $x_\tau(e)$, we see by the results of Section 3 that the revised definitions agree with the original, graphical ones. For example, the revised $C(\tau)$ is the same set as the graphical $C(\tau)$. Theorem 4.1 follows. □

COROLLARY 4.2. *The vertices of $Z[\Sigma]$ are the net degree vectors of the acyclic orientations of $\Sigma/0$ (Σ with its balanced loops removed). All of these net degree vectors are distinct from each other and from all $d(\tau)$ where τ is not acyclic.*

This corollary generalizes half of a characterization of the net degree vectors of orientations of an ordinary graph Γ (unpublished; cited in [11, Example 3.1]). Let d be the (unoriented) degree vector of Γ; then the net degree vectors are the integral points y in the acyclotope of Γ (that is, $Z[+\Gamma]$) for which $y \equiv d(\operatorname{mod} 2)$. (This result follows from Greene's observation about $H[\Gamma]$ and the total unimodularity of the oriented incidence matrix of Γ.) In the extension to signed graphs the net degree vectors are further restricted depending on which half arcs are present. The details will appear elsewhere.

Hyperplanes, regions and faces. The *arrangement of hyperplanes* of Σ is the set

$$H[\Sigma] = \{h(e) : e \in E\},$$

where if $e{:}vw$ is a link or a loop then $h(e)$ is defined by $x_v = \sigma(e)x_w$; if $e{:}v$ is a half arc its equation is $x_v = 0$, and if e is a balanced loop, $h(e)$ is the whole space (we grant it honorary hyperplane status as the 'degenerate hyperplane'). Since $h(e) = x_\tau(e)^\perp$, the set $\operatorname{Lat} H[\Sigma]$ of all intersections of subsets of $H[\Sigma]$, ordered by reverse inclusion, is a geometric lattice isomorphic to $\operatorname{Lat} G(\Sigma)$ under the mapping $h: E \to (\mathbb{R}^N)^*$.

The *regions* of $H[\Sigma]$ are the connected components of $\mathbb{R}^N \setminus (\bigcup H[\Sigma])$; its *faces* are the regions of all the induced arrangements

$$H[\Sigma]_t = \{h \cap t :\ h \in H[\Sigma],\ h \not\supseteq t\}$$

for $t \in \operatorname{Lat} H[\Sigma]$. The faces of $Z[\Sigma]$ and $H[\Sigma]$ are in one-to-one order-inverting correspondence, k-dimensional faces of one corresponding to $(n-k)$-dimensional faces of the other. Given an orientation τ of Σ, the hyperplanes are oriented; the positive half-space of e is

$$h_\tau(e) = \{x \in \mathbb{R}^N :\ x_\tau(e) \cdot x > 0\}.$$

(This is void if $h(e)$ is degenerate.) Let us write

$$R(\tau) = \bigcap_{e\in E} h_\tau(e).$$

Contrariwise, for a region R let $\alpha(R)$ give an incidence (v, e) the positive sign if e is a link to w and $x_v > \sigma(e)x_w$ in R, or if e is a negative loop or half arc at v and $x_v > 0$ in R, and otherwise the negative sign. Clearly each $R(\tau)$ is either a region or void. Curtis Greene's fundamental observation is as follows:

PROPOSITION 4.3 [5, 6, 14]. *If Γ is an unsigned (i.e. all-positive) graph oriented by τ,* *then $R(\tau)$ is a region of $H[\Gamma]$ iff τ is acyclic. The correspondence between regions and acyclic orientations is a bijection.*

Greene's proof depends on the fact that the inequalities defining $R(\tau)$ are directional, of the form $x_v < x_w$, just like orientations of unsigned (i.e. positive) arcs. Thus one has an easy proof by ordering the nodes. For signed graphs that is not possible. In order to treat them we dualize Theorem 3.3, appealing to the relationship between positive independence of vectors and, dually, regions of the arrangement of hyperplanes; equivalently, we dualize Theorem 4.1. Here, then, is our main theorem:

THEOREM 4.4. *If Σ is a signed graph oriented by τ, then $R(\tau)$ is a region of $H[\Sigma]$ iff τ is acyclic. The correspondence between regions and acyclic orientations is a bijection. The inverse of the mapping $\tau \to R(\tau)$ is $R \to \alpha(R)$.*

PROOF (concluded). The last statement follows from the obvious fact that $\alpha(R(\tau)) = \tau$. □

We can characterize the faces of $H[\Sigma]$ once we describe its flats.

LEMMA 4.5. *Let $t \in \mathrm{Lat}\, H[\Sigma]$ and let $T = \{e \in E\colon h(e) \supseteq t\}$. For each $B \in \pi_b(T)$, let $\nu_B\colon B \to \{\pm\}$ be a potential for $T{:}B$. Then t is described by the equations*

$$x_v = 0 \qquad \textit{for } v \in N_u(T),$$

and, for each $B \in \pi_b(T)$,

$$\nu_B(v)x_v = \textit{constant } x_B \qquad \textit{for } v \in B.$$

Also, $\dim t = b(T)$.

Now let $T \subseteq E$ and let τ be an orientation of Σ/T. If T is closed, then τ determines an orientation of $\Sigma \mid T^c$, whence $h_\tau(e)$ is a well-defined half-space in $\mathbb{R}^N$ if $e \notin T$. We can therefore define

$$F_T(\tau) = \left[\bigcap_{e \in T} h(e)\right] \cap \left[\bigcap_{e \notin T} h_\tau(e)\right],$$

which is $\varnothing$ if T is not closed.

COROLLARY 4.6. *There is a one-to-one correspondence between the k-dimensional faces of $H[\Sigma]$ and the acyclic orientations of contractions Σ/T, where T is a closed arc set with k balanced components.*

Precisely: each face F of $H[\Sigma]$ is an $F_T(\tau)$ for some closed T and acyclic orientation τ of Σ/T. Both T and τ are unique, and $b(T) = \dim F$. Furthermore, $F_T(\tau) \supseteq F_{T'}(\tau')$ iff $T \subseteq T'$ and $\tau/(T' \setminus T) = \tau'$.

Conversely, assuming that τ orients Σ/T, then $F_T(\tau) \neq \varnothing$ iff τ is acyclic. Then $F_T(\tau)$ is a face of $H[\Sigma]$ the dimension of which is $b(T)$.

COROLLARY 4.7. *Let τ be an orientation of Σ and let $T \subseteq E$. If T is closed and contains $C(\tau)$, then $F_T(\tau/T)$ is a face of $H[\Sigma]$, the dimension of which is $b(T)$. Otherwise, $F_T(\tau/T) = \varnothing$.*

PROOF. The first part is a special case of the second part of Corollary 4.6. The second follows from Corollary 3.8. □

5. Double Covering of an Oriented Signed Graph

The *signed covering* $\tilde{\Sigma}$ of a signed graph $\Sigma=(\Gamma, \sigma)$ (defined in [15] or see [1, Definition 19.1]) has node set $\tilde{N}=\pm N=\{\pm\}\times N$; its *covering projection* $p: \tilde{\Sigma}\to\Gamma$ is a graph homomorphism, 2-to-1, and locally an isomorphism, with $p(\pm v)=v$. $\tilde{\Sigma}$ is uniquely determined by the additional requirement that, if $\tilde{e}$ has endpoints δv and εw, then $\delta\varepsilon=\sigma(p(\tilde{e}))$. The *double covering* $\tilde{\Gamma}$ is the same, except that one forgets the signs on $\tilde{N}$. In [16, Section 6] we explored the connection between the matroids of $\tilde{\Gamma}$ and Σ.

If Σ is oriented by τ, then $\tilde{\Sigma}$ is oriented by $\tilde{\tau}$ determined by the rule

$$\tilde{\tau}(\varepsilon v, \tilde{e})=\varepsilon\tau(v, p(\tilde{e})).$$

We call $(\tilde{\Sigma}, \tilde{\tau})$ the *signed covering of* (Σ, τ). Notice that $\tilde{\tau}$ is an ordinary orientation of $\tilde{\Sigma}$, which is all positive (or in essence unsigned). In this section we show how the cyclicity of τ is related to that of $\tilde{\tau}$.

A double covering has a canonical involutory automorphism *, defined by $p(x^*)=p(x)$ and $x^*\neq x$. We call S^* (the image of a set S of nodes and arcs) the *opposite* of S. If $\tilde{e}$ is an arc oriented $\tilde{v}\to\tilde{w}$, then $\tilde{e}^*$ is oriented $\tilde{w}^*\to\tilde{v}^*$. Conversely, suppose that $\tilde{\Gamma}$ is any graph having an involutory automorphism * with no fixed points, and $\tilde{\tau}$ is an orientation of $\tilde{\Gamma}$ which is reversed by *. Then $\tilde{\Gamma}$ is a double covering of a signed graph Σ and the fact that $\tilde{\tau}$ is reversed allows us to orient Σ so that $\tilde{\tau}$ is the lift of that orientation.

Theorem 5.1. *Let τ be an orientation of the signed graph Σ, and let $(\tilde{\Sigma}, \tilde{\tau})$ be the signed covering. Then $C(\tilde{\tau})=p^{-1}(C(\tau))$. In particular, $\tilde{\tau}$ is acyclic iff τ is.*

The proof depends on the following lemma.

Lemma 5.2. *If C is a cycle in $\tilde{\tau}$, then its projection $p(C)$ is a union of cycles of τ.*

Proof. Let us consider an arc $e\in C$. We must show that $p(e)$ lies in a cycle of $\Sigma(\tau)$. We proceed by gradually transforming C until we have a cycle C' containing e, the projection of which is a circuit, hence a cycle. We shall treat only the case where C contains no half arcs. (We can always reduce to that case by changing the half arcs to negative loops.) We may assume that $p(C)$ is not a balanced circle, since then the lemma is trivial.

To find out how to transform C we look at the Gauss code of $p(C)$. A *double path* (in C) will mean a pair of maximal paths in C having the same projection; that is, which are of the form

$$P: \quad \varepsilon_0 v_0, e_1, \varepsilon_1 v_1, e_2, \ldots, e_r, \varepsilon_r v_r;$$

$$P^*: \quad -\varepsilon_r v_r, e_r^*, \ldots, e_2^*, -\varepsilon_1 v_1, e_1^*, -\varepsilon_0 v_0;$$

oriented from left to right, with $r\geqslant 0$. If we name all the double paths, a *Gauss code* of C is a sequence of names in the cyclic order in which they are encountered as one travels once around C. Note that each name appears just twice. The *order* of C is the number of names in its Gauss code.

If C has order 1, $p(C)$ is an unbalanced circuit in $\Sigma(\tau)$. Then the lemma is obvious. We shall show that, when the order is at least 2, we can rearrange parts of C or their opposites to create a cycle which contains e and has order smaller than that of C. The lemma will then follow by descent.

First we solve a special case. We call a Gauss code $a\cdots z$ *normalized* if e belongs to the path denoted by a or z or to the path which wraps around from the final z to the initial a.

The split case. If the Gauss code, normalized or not, has the form

$$ab \cdots b(\cdots)a \cdots$$

where e is not in the portion of C denoted by $(\cdots)$, write $C = APB \cdots B^*QA^* \cdots$, where A and A^* constitute the double path named a, and B and B^* constitute that named b. Since $e \notin Q$, replacing Q by P^* transforms C into a cycle C_1 which contains e. Then C_1 has smaller order than C because a and b have been combined into a double path $\{APB, (APB)^*\}$.

The general case. Suppose that C has normalized Gauss code $a \cdots a \cdots$. If it is $aa \cdots$ (thus $aa(\cdots)z \cdots z$ normalized, which is $zaa(\cdots)z \cdots$ unnormalized) or $aa_1 \cdots a_1(\cdots)a \cdots$, the split case applies. That leaves us only with Gauss codes of the form

$$aa_1 \cdots a \cdots a_1 \cdots.$$

If a does not succeed a_1, again either the split case applies or the code has the form

$$aa_1a_2 \cdots a \cdots a_1 \cdots a_2 \cdots.$$

Continuing in this fashion we see that the Gauss code, if not reducible, must be of the form

$$a(a_1a_2 \cdots a_i)a \cdots a_1 \cdots a_2 \cdots \quad \cdots \quad \cdots a_i \cdots.$$

Write $C = A \cdots A_i^*RA^* \cdots A_iQ$, so $e \in A \cup A_i \cup Q$. Then the cycle AR^*A_iQ has reduced order since, at the very least, A and A_i are no longer in double paths. That completes the proof. □

Proof of Theorem 5.1. To show $C(\tilde{\tau}) \supseteq p^{-1}(C(\tau))$, suppose that C is a cycle in τ. One can see (cf. the proof of Lemma 6.6 in [16]) that $p^{-1}(C)$ is the union of two circles which project to C. It is clear from the definition that both are cycles; so $p^{-1}(C) \subseteq C(\tilde{\tau})$.

To show that $C(\tilde{\tau}) \subseteq p^{-1}(C(\tau))$, it is enough to show that $p(C(\tilde{\tau})) \subseteq C(\tau)$; this is a consequence of Lemma 5.2. □

Corollary 5.3. *An acyclically oriented signed graph with finitely many nodes has a source or a sink.*

We cannot conclude that it has a source *and* a sink even when there are no half arcs.

Remark. Theorem 5.1 gives a graph-theoretic proof of the first half of Corollary 3.7. Because $\tilde{\Sigma}$ is positive, $C(\tilde{\tau})$ is closed in $G(\tilde{\Sigma})$. Then $C(\tau)$ is closed by [16, Theorem 6.5(i)].

6. Double Covering of the Geometric Representation of a Signed Graph

An alternative approach to the results of Sections 3 and 4 is to deduce them from the ordinary graphical geometry of the double covering graph. This is the method employed in [6, Section 9] to count certain restricted acyclic orientations of a signed graph. Here we develop the geometrical machinery needed in [6].

Because $\tilde{\Sigma}$ is a positive (or unsigned) graph with node set $\pm N$, it is represented by an arrangement of hyperplanes $H[\tilde{\Sigma}]$ in $\mathbb{R}^{\pm N}$, the regions of which, by Proposition 4.3, are all the sets $\tilde{R}(\alpha)$ for which α is an acyclic orientation of $\tilde{\Sigma}$. Let the co-ordinates in $\mathbb{R}^{+N}$ be x_v^+ and let those in $\mathbb{R}^{-N}$ be x_v^- for $v \in N$. For $x \in \mathbb{R}^{\pm N}$ we write $x = (x^+, x^-)$. Let

$$s = \{x \in \mathbb{R}^{\pm N}: x^+ + x^- = 0\}.$$

Now if e lifts to $\tilde{e}$ and $\tilde{e}^*$,

$$h(\tilde{e}) \cap s = h(\tilde{e}^*) \cap s = \{x \in s:\ x^+ \in h(e)\}.$$

Thus $H[\tilde{\Sigma}]_s$ is essentially identical to $H[\Sigma]$, only having two copies of each hyperplane in $H[\Sigma]$, if we embed $\mathbb{R}^N$ in $\mathbb{R}^{\pm N}$ as the subspace s by the mapping $y \to (y, -y)$. Our problem is to characterize how s cross-sections $H[\tilde{\Sigma}]$.

Proposition 6.1. *Regard $H[\Sigma]$ as lying in $s \subseteq \mathbb{R}^{\pm N}$. Then the regions of $H[\Sigma]$ are the non-void intersections of s with the regions of $H[\tilde{\Sigma}]$. Let α be an acyclic orientation of $\tilde{\Sigma}$. Then $s \cap \tilde{R}(\alpha) \neq \emptyset$ iff $\alpha = \tilde{\tau}$, the lift of an orientation τ of Σ; and then $s \cap \tilde{R}(\alpha) = R(\tau)$ and τ is acyclic.*

Proof. The first assertion is true because no $h(\tilde{e})$ contains s. To see this observe that $h(\tilde{e})$ is defined by an equation in at most two variables. The only such equations satisfied by s are $x_v^- = -x_v^+$, which are not equations of a positive arc or a half arc. But $\tilde{e}$ is such an arc. The desired conclusion follows.

Suppose in the second assertion that α is not a lift of an orientation of Σ. From our discussion of the involution * in Section 5, we know that this means that there are arcs $\tilde{e}$ and $\tilde{e}^*$ which are not oppositely oriented. Suppose that the arcs are links (the other cases are similar), say $\tilde{e}: +v \to \varepsilon w$ and $\tilde{e}^*: -v \to -\varepsilon w$, where $\varepsilon = \sigma(p(\tilde{e}))$. Then $\tilde{R}(\alpha)$ lies in the half-spaces defined by

$$x_v^+ < x_w^\varepsilon \qquad \text{and} \qquad x_v^- < x_w^{-\varepsilon},$$

which in s are the complementary half-spaces $x_v^+ < \varepsilon x_w^+$ and $-x_v^+ < -\varepsilon x_w^+$. So $s \cap \tilde{R}(\alpha) = \emptyset$.

Now suppose that $\alpha = \tilde{\tau}$. By Theorem 5.1, τ is acyclic. By the first assertion $s \cap \tilde{R}(\alpha)$ is a region of $H[\Sigma]$ if it is not void; by inspection that region can only be $R(\tau)$. So all that remains is to prove $s \cap \tilde{R}(\alpha) \neq \emptyset$.

Let Z_1 be the set of sources of α (non-void because $\tilde{\Sigma}$ is ordinary and α is acyclic). Then $-Z_1$ is the set of sinks. Let us agree that, when both $+v$ and $-v$ are sources (and sinks), we put only one in Z_1. Now inductively define

$$Z_{i+1} = Z_i \cup \{\text{sources of } \alpha{:}(\pm Z_i)^c\},$$

with the same convention on ambiguous cases. Since α orients the induced subgraph $\tilde{\Sigma}:(\pm Z_i)^c$ acyclically, this process will not end until $\pm N$ is exhausted. Let $Z = \lim Z_i$. Then Z and $-Z$ partition $\pm N$.

To simplify the rest of the proof, switch $\tilde{\Sigma}$ so that $Z = +N$. In the switched graph an arc with endpoints of both signs is directed $+v \to -w$.

Now let x^+ be any vector such that $x_v^+ < x_w^+$ wherever there is an arc $\tilde{e}: +v \to +w$. The existence of x^+ is guaranteed by the acyclicity of $\alpha^+ = \alpha{:}(+N)$; we simply choose $x^+ \in R(\alpha^+)$, a region of the arrangement $H[\tilde{\Sigma}{:}(+N)]$. By subtracting a suitable multiple of $(1, 1, \ldots, 1)$ we can even make all co-ordinates of x^+ negative. Let $x = (x^+, -x^+)$. Thus $x \in s$ and $x \in \tilde{R}(\alpha)$. So we have proved $s \cap \tilde{R}(\alpha) \neq \emptyset$, completing the proof of the proposition. □

Theorem 4.4 is an easy consequence (that was our original proof of the theorem).

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