Reference Notes for Lectures on Signed Graphs and Geometry

by

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Workshop on Signed Graphs, Set-Valuations and Geometry

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16–19 August 2010

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[1]I deeply appreciate the generous labors of Sr. Dr. Germina K.A. and Professor A.M. Mathai in organizing the workshop at Pala and Mananthavady and inviting me to give a lecture series, and especially their kind care and their wish to show the beauties of Kerala to 'a traveller from a distant land'.

Introduction

These notes and the lectures are a personal introduction to signed graphs, concentrating on the aspects that have been most persistently interesting to me. This is just a few corners of the theory; I am leaving out a great deal. The emphasis is on the way signed graphs arise naturally from geometry, especially from the geometry of the classical root systems.

The arrangement of the notes is topical, not historical. In the lectures I will talk about the historical development, but in the notes the purpose is to provide a printed reference for most of the definitions, theorems, and examples, and possibly some proofs.

The principal reference for most of the properties of signed graphs treated here is [4]. A simple introduction to the hyperplane geometry is [B, Zaslavsky (1981a)]. Many of my articles can be downloaded from my Web site,

`http://www.math.binghamton.edu/zaslav/Tpapers/`

Bon voyage! *Suffa yathra.*

1. Graphs

In these lectures all graphs are finite.

A graph is $\Gamma = (V, E)$, where $V := V(\Gamma)$ is the *vertex set* and $E := E(\Gamma)$ is the *edge set*. Notation:

- $n := |V|$, called the *order* of Γ.
- $V(e)$ is the multiset of vertices of the edge e.
- If $S \subseteq E$, $V(S)$ is the set of endpoints of edges in S.

Edges and edge sets:

- There are four kinds of edge: A *link* has two distinct endpoints. A *loop* has two equal endpoints. A *half edge* has one endpoint. A *loose edge* has no endpoints. The set of loose edges of Γ is $E_0(\Gamma)$.
- An *ordinary edge* is a link or a loop. An *ordinary graph* is a graph in which every edge is ordinary. A *link graph* is a graph whose edges are links.
- The set of loose edges of Γ is $E_0(\Gamma)$. The set of ordinary edges of Γ is $E_* := E_*(\Gamma)$.
- Edges are *parallel* if they have the same endpoints. A *simple graph* is a link graph with no parallel edges.
- If $S \subseteq E$, $S^c := E \setminus S$ is its *complement*.
- $E(X, Y)$, where $X, Y \subseteq V$, is the set of edges with one endpoint in X and the other in Y.
- A *cut* or *cutset* is an edge set $E(X, X^c)$ that is nonempty.

The *degree* of a vertex v, $d(v) := d_\Gamma(v)$, is the number of edges of which v is an endpoint, but a loop counts twice. Γ is *regular* if every vertex has the same degree. If that degree is k, it is *k-regular*.

Walks, trails, paths, circles:

- A *walk* is a sequence $v_0 e_1 v_1 \cdots e_l v_l$ where $V(e_i) = \{v_{i-1}, v_i\}$ and $l \geq 0$. Its *length* is l. It may be written $e_1 e_2 \cdots e_l$ or $v_0 v_1 \cdots v_l$.
- A *closed walk* is a walk where $l \geq 1$ and $v_0 = v_l$.
- A *trail* is a walk with no repeated edges.
- A *path* or *open path* is a trail with no repeated vertex, or the graph of such a trail (technically, the latter is a *path graph*), or the edge set of a path graph.

- A *closed path* is a closed trail with no repeated vertex other than that $v_0 = v_l$. (A closed path is not a path.)
- A *circle* (also called 'cycle', 'polygon', etc.) is the graph, or the edge set, of a closed path. Equivalently, it is a connected, regular graph with degree 2.
- $\mathcal{C} = \mathcal{C}(\Gamma)$ is the class of all circles in Γ.

Examples:

- K_n is the complete graph of order n. K_X is the complete graph with vertex set X.
- K_n^c is the edgeless graph of order n.
- Γ^c is the complement of Γ, if Γ is simple.
- P_l is a path of length l (as a graph or edge set).
- C_l is a circle of length l (as a graph or edge set).
- $K_{r,s}$ is the complete bipartite graph with r left vertices and s right vertices. $K_{X,Y}$ is the complete bipartite graph with left vertex set X and right vertex set Y.
- The empty graph, $\varnothing := (\varnothing, \varnothing)$, has no vertices and no edges. It is not connected.

Types of subgraph: In Γ, let $X \subseteq V$ and $S \subseteq E$.

- A *component* (or *connected component*) of Γ is a maximal connected subgraph, excluding loose edges. An *isolated vertex* is a component that has one vertex and no edges.
- $c(\Gamma)$ is the number of components of Γ (excluding loose edges). $c(S)$ is short for $c(V, S)$.
- A *spanning subgraph* is $\Gamma' \subseteq \Gamma$ such that $V' = V$.
- $\Gamma|S := (V, S)$. This is a spanning subgraph.
- $S{:}X := \{e \in S : \varnothing \neq V(e) \subseteq X\} = (E{:}X \cap S$. We often write $S{:}X$ as short for the subgraph $(X, S{:}X)$.
- The *induced subgraph* $\Gamma{:}X$ is the subgraph $\Gamma{:}X := (X, E{:}X)$. An induced subgraph has no loose edges. We often write $E{:}X$ as short for $(X, E{:}X)$.
- $\Gamma \setminus S := (V, E \setminus S)$.
- $\Gamma \setminus X$ is the subgraph with

$$V(\Gamma \setminus X) := V \setminus X \text{ and } E(\Gamma \setminus X) := \{e \in E \mid V(e) \subseteq V \setminus X\}.$$

  We say X is *deleted* from Γ. $\Gamma \setminus X$ includes all loose edges, if there are any.

Vertices and vertex sets in Γ: Let $X \subseteq V$.

- An *isolated vertex* is a vertex that has no incident edges; i.e., a vertex of degree 0.
- X is *stable* or *independent* if $E{:}X = \varnothing$.
- X is a *clique* if every pair of its vertices is adjacent.
- X^c denotes $V \setminus X$.

Graph structures and types:

- A *theta graph* is the union of 3 internally disjoint paths that have the same endpoints.
- A *block* of Γ is a maximal subgraph without isolated vertices or loose edges, such that every pair of edges is in a circle together. The simplest kind of block is $(\{v\}, \{e\})$ where e is a loop or half edge at vertex v. A loose edge or isolated vertex is not in any block.
- Γ is *inseparable* if it has only one block or it is an isolated vertex.
- A *cutpoint* is a vertex that belongs to more than one block.

Let T be a maximal forest in Γ. If $e \in E_* \setminus T$, there is a unique circle $C_e \subseteq T \cup \{e\}$. The *fundamental system of circles* for Γ, with respect to T, is the set of all circles C_e for $e \in E_* \setminus T$. The *set sum* or *symmetric difference* of two sets A, B is denoted by $A \oplus B := (A \setminus B) \cup (B \setminus A)$.

Proposition 1.1. *Choose a maximal forest T. Every circle in Γ is the set sum of fundamental circles with respect to T.*

Proof. $C = \bigoplus_{e \in C \setminus T} C_T(e)$. □

2. Signed Graphs

A *signed graph* $\Sigma = (\Gamma, \sigma) = (V, E, \sigma)$ is a graph Γ together with a function σ that assigns a sign, $\sigma(e) \in \{+, -\}$, to each ordinary edge (link or loop) in Γ. A half or loose edge does not get a sign. Thus, the *sign function* (or *signature*) is $\sigma : E_* \to \{+, -\}$. Notation:

- $|\Sigma|$ is the *underlying graph* Γ.
- $E^+ := \sigma^{-1}(+) = \{e \in E : \sigma(e) = +\}$. The *positive subgraph* is $\Sigma^+ := (V, E^+)$.
- $E^- := \sigma^{-1}(+) = \{e \in E : \sigma(e) = -\}$. The *negative subgraph* is $\Sigma^- := (V, E^-)$.
- $+\Gamma := (\Gamma, +)$, an *all-positive* signed graph (every ordinary edge is $+$). $e \in E_* = E_*(\Gamma)$ becomes $+e \in +E = E(+\Gamma)$.
- $-\Gamma := (\Gamma, -)$, an *all-negative* signed graph (every ordinary edge is $-$). $e \in E_*$ becomes $-e \in -E = E(-\Gamma)$.
- $\pm\Gamma := (+\Gamma) \cup (-\Gamma)$. $E(\pm\Gamma) = \pm E := (+E) \cup (-E)$. This is the *signed expansion* of Γ.
- $\Sigma^\bullet := \Sigma$ with a half edge or negative loop attached to every vertex that does not have one. $\Sigma^\bullet$ is called a *full* signed graph.
- $\Sigma^\circ := \Sigma$ with a negative loop attached to every vertex that does not have one.
- If Δ is a simple graph, then $K_\Delta := (-\Delta) \cup (+\Delta^c)$, with underlying graph $|K_\Delta| = K_n$. This is a *signed complete graph*.

Equivalent notations for the sign group: $\{+, -\}$, $\{+1, -1\}$, or $\mathbb{Z}_2 := \{0, 1\}$ modulo 2.

2.1. **Balance and switching.**

2.1.1. *Balance.*

Signs and balance:

- The *sign of a walk*, $\sigma(W)$, is the product of the signs of its edges, including repeated edges.
- The *sign of an edge set*, $\sigma(S)$, is the product of the signs of its edges, without repetition.
- The sign of a circle, $\sigma(C)$, is the same whether the circle is treated as a walk or as an edge set.
- The *class of positive circles* is
$$\mathcal{B} = \mathcal{B}(\Sigma) := \{C \in \mathcal{C}(|\Sigma|) : \sigma(C) = +\}.$$
- Σ is *balanced* if it has no half edges and every circle in it is positive. Similarly, any subgraph or edge set is balanced if it has no half edges and every circle in it is positive.
- A circle is balanced if and only if it is positive. However, in general, a walk cannot be balanced because it is not a graph or edge set.
- $b(\Sigma)$ is the number of components of Σ (omitting loose edges) that are balanced.
- $\pi_{\mathrm{b}}(\Sigma) := \{V(\Sigma') : \Sigma' \text{ is a balanced component of } \Sigma\}$. Then $b(\Sigma) = |\pi_{\mathrm{b}}(\Sigma)|$.

- $V_0(\Sigma)$ is the set of vertices of unbalanced components of Σ. Formally, $V_0(\Sigma) := V \setminus \bigcup_{W \in \pi_{\mathrm{b}}(\Sigma)} W$.
- $\pi_{\mathrm{b}}(S)$ is short for $\pi_{\mathrm{b}}(\Sigma|S)$. $V_0(S)$ is short for $V_0(\Sigma|S)$.

Types of vertices and edges in Σ:

- A *balancing vertex* is a vertex v such that $\Sigma \setminus v$ is balanced although Σ is unbalanced.
- A *partial balancing edge* is an edge e such that $\Sigma \setminus e$ has more balanced components than does Σ.
- A *total balancing edge* is an edge e such that $\Sigma \setminus e$ is balanced although Σ is not balanced. A total balancing edge is a partial balancing edge, but a partial balancing edge may not be a total balancing edge.

Proposition 2.1. *An edge e is a partial balancing edge of Σ if and only if it is either*

(i) *an isthmus between two components of $\Sigma \setminus e$, of which at least one is balanced, or*
(ii) *a negative loop or half edge in a component Σ' such that $\Sigma' \setminus e$ is balanced, or*
(iii) *a link with endpoints v, w, such that every vw-path in $\Sigma \setminus e$ has sign opposite to that of e.*

Lemma 2.2. *Σ is balanced if and only if every block is balanced.*

A *bipartition* of a set X is an unordered pair $\{X_1, X_2\}$ such that $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$. X_1 or X_2 could be empty.

Theorem 2.3 (Harary's Balance Theorem [1])**.** *Σ is balanced $\iff$ it has no half edges and there is a bipartition $V = V_1 \cup\!\!\!\cdot\, V_2$ such that $E^- = E(V_1, V_2)$.*

Corollary 2.4. *$-\Gamma$ is balanced if and only if Γ is bipartite.*

2.1.2. *Switching.*

A *switching function* for Σ is a function $\zeta : V \to \{+, -\}$. The *switched signature* is $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has endpoints v, w. The *switched signed graph* is $\Sigma^\zeta := (|\Sigma|, \sigma^\zeta)$. We say Σ is *switched by ζ*. Note that $\Sigma^\zeta = \Sigma^{-\zeta}$.

If $X \subseteq V$, *switching Σ by X* means reversing the signs of every edge in the cutset $E(X, X^c)$. The switched graph is Σ^X. (This is the same as Σ^ζ where $\zeta(v) := -$ if and only if $v \in Z$. Switching by ζ or X is the same operation, with different notation.) Note that $\Sigma^X = \Sigma^{X^c}$.

Proposition 2.5. (i) *Switching leaves the signs of all circles unchanged. That is, $\mathcal{B}(\Sigma^\zeta) = \mathcal{B}(\Sigma)$.*
(ii) *If $|\Sigma_1| = |\Sigma_2|$ and $\mathcal{B}(\Sigma_1) = \mathcal{B}(\Sigma_2)$, then there exists a switching function ζ such that $\Sigma_2 = \Sigma_1^\zeta$.*

Proof of (i). Let ζ be a switching function and let $C = v_0e_0v_1e_1v_2 \cdots v_{n-1}e_{n-1}v_0$ be a circle. Then

$$\begin{aligned}\sigma^\zeta(C) &= \big(\zeta(v_0)\sigma(e_0)\zeta(v_1)\big)\big(\zeta(v_1)\sigma(e_1)\zeta(v_2)\big)\ldots\big(\zeta(v_{n-1})\sigma(e_{n-1})\zeta(v_0)\big)\\ &= \sigma(e_0)\sigma(e_1)\cdots\sigma(e_{n-1}) = \sigma(C).\end{aligned}$$

□

Proof of (ii). We may assume Σ_1 is connected. Pick a spanning tree T and list the vertices in such a way that v_i is always adjacent to a vertex in $\{v_1, \ldots, v_{i-1}\}$ (for $i > 1$). Let t_i be the unique tree edge connecting v_i to $\Sigma{:}\{v_1, \ldots, v_{i-1}\}$.

We define a switching function ζ:

$$\zeta(v_i) = \begin{cases} + & \text{if } i = 1, \\ \sigma_1(t_i)\sigma_2(t_i)\zeta(v_j) & \text{if } i > 1, \text{ where } v_j \text{ is the endpoint of } t_i \text{ that is not } v_i. \end{cases}$$

Now it is easy to show that $\Sigma_1^\zeta = \Sigma_2$. □

Signed graphs Σ_1 and Σ_2 are *switching equivalent*, written $\Sigma_1 \sim \Sigma_2$, if they have the same underlying graph and there exists a switching function ζ such that $\Sigma_1^\zeta = \Sigma_2$. The equivalence class of Σ,

$$[\Sigma] := \{\Sigma' : \Sigma' \sim \Sigma\},$$

is called its *switching class*.

Proposition 2.6. *Switching equivalence, $\sim$, is an equivalence relation on signatures of a given underlying graph.*

Corollary 2.7. *Σ is balanced if and only if it has no half edges and it is switching equivalent to $+|\Sigma|$.*

Proof of Harary's Balance Theorem. Σ has the form stated in the theorem $\iff$ it is $(+|\Sigma|)^{V_1} \iff$ it is a switching of $+|\Sigma| \iff$ (by Proposition 2.5) it is balanced. □

2.2. **Deletion, contraction, and minors.**

R, S denote subsets of E. A *component of* S means a component of (V, S).

The *deletion* of S is the signed graph $(V, S^c, \sigma|_{S^c})$.

2.2.1. *Contracting an edge* e.

If e is a positive link, delete e and identify its endpoints (this is how to contract a link in an unsigned graph); do not change any edge signs. If e is a negative link, switch Σ by a switching function ζ, chosen so e is positive link in Σ^ζ; then contract e as a positive link.

Lemma 2.8. *In a signed graph Σ any two contractions of a link e are switching equivalent. The contraction of a link in a switching class is a well defined switching class.*

To contract a positive loop or a loose edge e, just delete e.

If e is a negative loop or half edge and v is the vertex of e, delete v and e, but not any other edges. Any other edges at v lose their endpoint v. A loop or half edge at v becomes a loose edge. A link with endpoints v, w becomes a half edge at w.

2.2.2. *Contracting an edge set* S.

The edge set and vertex set of Σ/S are

$$E(\Sigma/S) := E \setminus S, \quad V(\Sigma/S) := \pi_{\mathrm{b}}(\Sigma|S) = \pi_{\mathrm{b}}(S).$$

For $w \in V$, define This means we identify all the vertices of each balanced component so they become a single vertex. For $e \in E(\Sigma/S)$, the endpoints are given by the rule

$$V_{\Sigma/S}(e) = \{W \in \pi_{\mathrm{b}}(S) : w \in V_\Sigma(e) \text{ and } w \in W \in \pi_{\mathrm{b}}(S)\}.$$

(If e is a loop at w in Σ, then w is a repeated vertex in $V_\Sigma(e)$; if $w \in W \in \pi_{\mathrm{b}}(S)$, then W is a repeated vertex in $V_{\Sigma/S}(e)$ so e is a loop in Σ/S. If $w \in V_\Sigma(e) \cap V_0(S)$, then w disappears from $V_{\Sigma/S}(e)$. To define the signature of Σ/S, first switch Σ to Σ^ζ so every balanced component of S is all positive. Then $\sigma_{\Sigma/S}(e) := \sigma^\zeta(e)$.

Lemma 2.9. (a) *Given Σ a signed graph and $S \subseteq E(\Sigma)$, all contractions Σ/S (by different choices of switching Σ) are switching equivalent. Any switching of one contraction Σ/S is another contraction and any contraction Σ^ζ/S of a switching of Σ is a contraction of Σ.*

(b) *If $|\Sigma_1| = |\Sigma_2|$, $S \subseteq E$ is balanced in both Σ_1 and Σ_2, and Σ_1/S and Σ_2/S are switching equivalent, then Σ_1 and Σ_2 are switching equivalent.*

(c) *For $e \in E$, $[\Sigma/e]$ (defined in Section 2.2.1) and $[\Sigma/\{e\}]$ are essentially the same switching class.*

Part (a) means that the switching class $[\Sigma/S]$ is uniquely defined, even though the signed graph Σ/S is not unique.

2.2.3. *Minors.*

A *minor* of Σ is any contraction of any subgraph.

Theorem 2.10. *Given a signed graph Σ, the result of any sequence of deletions and contractions of edge and vertex sets of Σ is a minor of Σ. In other words, a minor of a minor is a minor.*

Proof. See [4, Proposition 4.2]. □

2.3. **Frame circuits.**

A *frame circuit* of Σ is a subgraph, or edge set, that is either a positive circle or a loose edge, or a pair of negative circles that intersect in precisely one vertex and no edges (this is a *tight handcuff circuit*), or a pair of disjoint negative circles together with a minimal path that connects them (this is a *loose handcuff circuit*). We regard a tight handcuff circuit as having a connecting path of length 0 (it is the common vertex of two the circles).

Proposition 2.11. Σ *contains a loose handcuff circuit if and only if there is a component of Σ that contains two disjoint negative circles.*

Proposition 2.12. *Let $e \in E$. If Σ contains a handcuff circuit C such that $e \in C$, then e is in an unbalanced component Σ' of Σ and e is not a partial balancing edge. If e is in an unbalanced component Σ' of Σ and e is not a partial balancing edge, then Σ contains a frame circuit C such that $e \in C$.*

Proof. If there exists C as stated in the proposition, then the component Σ' that contains e also contains C so it is unbalanced. As $C \setminus e$ is unbalanced, $\Sigma' \setminus e$ has no balanced component (this requires checking cases), so e is not a partial balancing edge (this requires checking definitions).

Conversely, suppose e is not a partial balancing edge and it is in an unbalanced component Σ'. Since $\Sigma' \setminus e$ is unbalanced, it has a negative circle C_1. If e is an unbalanced edge at v, there is a path P in Σ' from v to C_1; then $C = C_1 \cup P \cup e$.

If e is a balanced edge, it is a link with endpoints v, w. If it is an isthmus, then $\Sigma' \setminus e$ has two components, both unbalanced (by Proposition 2.1), so C is a negative circle in each of those components, together with a connecting path (which must contain e). If e is not an isthmus, it lies in a circle C'. If C' is positive, let $C = C'$. But suppose C' is negative; then there are three subcases, depending on how many points of intersection C' has with C_1. If there are no such points, take a minimal path P connecting C' to C_1 and let $C = C_1 \cup P \cup C'$. If there is just one such point, $C = C_1 \cup C'$. If there are two or more such points, take P to be a maximal path in C' that contains e and is internally disjoint from C_1. Then $P \cup C_1$ is a theta graph in which C_1 is negative; hence one of the two circles containing P is positive, and this is the circuit C. □

Theorem 2.13 ([B, Slilaty (2007a)]). *Σ has no two vertex-disjoint negative circles if and only if one or more of the following is true:*

(1) *Σ is balanced,*
(2) *Σ has a balancing vertex,*
(3) *Σ embeds in the projective plane, or*
(4) *Σ is one of a few exceptional cases.*

We will not discuss projective planarity, which is a large topic in itself (see [B, Zaslavsky (1993a)], [B, Archdeacon and Debowsky (2005a)]).

2.4. **Closure and closed sets.**

The *balance-closure* of an edge set R is

$$\operatorname{bcl}(R) := R \cup \{e \in R^c : \exists \text{ a positive circle } C \subseteq R \cup e \text{ such that } e \in C\} \cup E_0(\Sigma).$$

The *closure* of an edge set S is

$$\operatorname{clos}(S) := \big(E{:}V_0(S)\big) \cup \Big(\bigcup_{i=1}^{k} \operatorname{bcl}(S_i)\Big) \cup E_0(\Sigma),$$

where $S_1, \dots, S_k$ are the balanced components of S.

An edge set is *closed* if it equals its own closure: $\operatorname{clos} S = S$. We write

$$\operatorname{Lat}\Sigma := \{S \subseteq E : S \text{ is closed}\}.$$

When partially ordered by set inclusion, $\operatorname{Lat}\Sigma$ is a lattice.

Note that a half edge and a negative loop are equivalent in everything that concerns closure or circuits.

The usual closure operator in a graph Γ is the same as closure in $+\Gamma$. In that case, since $+\Gamma$ is balanced, a frame circuit is simply a positive circle (or a loose edge).

Lemma 2.14. $\operatorname{bcl}(R)$ *is balanced if and only if R is balanced. Furthermore,* $\operatorname{bcl}(\operatorname{bcl} R) = \operatorname{bcl}(R) = \operatorname{clos}(R)$.

Lemma 2.15. *For an edge set S, $\pi_{\mathrm{b}}(\operatorname{clos} S) = \pi_{\mathrm{b}}(\operatorname{bcl} S) = \pi_{\mathrm{b}}(S)$ and $V_0(\operatorname{clos} S) = V_0(S)$.*

Let E be any set; its *power set* $\mathcal{P}(E)$ is the class of all subsets of E. A function $J : \mathcal{P}(E) \to \mathcal{P}(E)$ is an *(abstract) closure operator on E* if it has the three properties

(C1) $J(S) \supseteq S$ for every $S \subseteq E$ (increase).
(C2) $R \subseteq S \implies J(R) \subseteq J(S)$ (isotonicity).
(C3) $J(J(S)) = J(S)$ (idempotence).

Theorem 2.16. *The operator* clos *on subsets of $E(\Sigma)$ is an abstract closure operator.*

Proof. The definition makes clear that clos is increasing and isotonic. What remains to be proved is that $\operatorname{clos}(\operatorname{clos}(S)) = \operatorname{clos}(S)$.

Let $\pi_{\mathrm{b}}(S) = \{B_1, \ldots, B_k\}$; thus, $S{:}B_i$ is balanced. Using Lemma 2.15,

$$\begin{aligned}\operatorname{clos}(\operatorname{clos} S) &= \big(E{:}V_0(\operatorname{clos} S)\big) \cup \bigcup_{i=1}^{k} \operatorname{bcl}\big((\operatorname{clos} S){:}B_i\big) = \big(E{:}V_0(S)\big) \cup \bigcup_{i=1}^{k} \operatorname{bcl}\big((\operatorname{bcl} S){:}B_i\big) \\ &= \big(E{:}V_0(S)\big) \cup \bigcup_{i=1}^{k} \operatorname{bcl}(S{:}B_i) = \operatorname{clos} S. \qquad \square\end{aligned}$$

Theorem 2.17. *For $S \subseteq E$,*

$$\operatorname{clos} S = S \cup \{e \notin S : \exists \text{ a frame circuit } C \text{ such that } e \in C \subseteq S \cup e\}.$$

Proof. We treat a half edge as if it were a negative loop.
Necessity. We want to prove that if $e \in \operatorname{clos} S$, then a frame circuit C exists. There are three cases, depending on where the endpoints of e are located.

Case 0. A trivial case is where e is a loose edge. Then $e \in \operatorname{clos} S$ and $C = \{e\}$.

Case 1. Suppose e has its endpoints within one component, S'. Then there is a circle C' in $S' \cup e$ that contains e. If C' is positive it is our circuit C. (This includes the case of a positive loop e, where $C = \{e\}$.) If S' is balanced, then $e \in \operatorname{bcl} S'$ so there exists a positive circle C'. If S' is unbalanced and e is not a partial balancing edge, then C exists by Proposition 2.11.

Suppose S' is unbalanced and e is a partial balancing edge. Then Proposition 2.1 tells us that e cannot be in $\operatorname{clos} S$ (this requires checking the three cases of that proposition).

Case 2. Suppose e has endpoints in two different components, S' and S''. For e to be in the closure, it must be in $E{:}V_0$. Hence, S' and S'' are unbalanced. Each of them contains a negative circle, C' and C'' respectively, and there is a connecting path P in $S \cup e$ which contains e. Then $C' \cup P \cup C''$ is the desired circuit.

Sufficiency. Assuming a circuit C exists, we want to prove that $e \in \operatorname{clos} S$. Again there are three cases, this time depending on C and its relationship with e.

Case 0. C is balanced. Then $e \in \operatorname{bcl} S \subseteq \operatorname{clos} S$.

Case 1. C is unbalanced and e is not in the connecting path. Let C_1, C_2 be the two negative circles and P the connecting path of C, and assume $e \in C_1$. Since $C \setminus e$ is connected, it lies in one component S' of S. Thus, $C_2 \subseteq S'$, whence S' is unbalanced. It follows that $e \in E{:}V_0 \subseteq \operatorname{clos} S$.

Case 2. C is unbalanced and e is in the connecting path. With notation as in Case 1, now $C \setminus e$ has two components, one containing C_1 and the other containing C_2. The components of S that contain C_1 and C_2 are unbalanced. (There may be one such component or two, depending on whether C_1 and C_2 are connected by a path in S.) Therefore, e has both endpoints in V_0, so again, $e \in E{:}V_0 \subseteq \operatorname{clos} S$. $\square$

3. Geometry

In this section we write the vertex set as $V = \{v_1, v_2, \ldots, v_n\}$. $\mathbf{F}$ denotes any field. The most important field will be the real numbers $\mathbb{R}$. Other important fields are $\mathbb{F}_2$, the 2-element field of arithmetic modulo 2, and $\mathbb{F}_3$, the 3-element field of arithmetic modulo 3.

3.1. Vectors for edges.

We have a signed graph Σ of order n. For each edge e there is a vector $x(e) \in \mathbf{F}^n$, whose definition is, for the four types of edge:

$$
\begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \\ \mp\sigma(e) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\qquad
\begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \mp \sigma(e) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\qquad
\begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\qquad
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

a link $e{:}v_iv_j$, a loop e at v_i, a half edge e at v_i, a loose edge.

These vectors are well defined only up to sign, i.e., the negative of $x(e)$ is another possible choice of $x(e)$. We make an arbitrary choice $x(e)$ for each edge e. The choice does not affect the linear dependence properties.

For a set $S \subseteq E$, define $x(S) := \{x(e) : e \in S\}$.

Theorem 3.1. *Let S be an edge set in Σ and consider the corresponding vector set $x(S)$ in the vector space $\mathbf{F}^n$ over a field $\mathbf{F}$.*

(1) When $\operatorname{char}\mathbf{F} \neq 2$, $x(S)$ is linearly dependent if and only if S contains a frame circuit.
(2) When $\operatorname{char}\mathbf{F} = 2$, $x(S)$ is linearly dependent if and only if S contains a circle or a loose edge.

The proof is not short.

Corollary 3.2. *The minimal linearly dependent subsets of $x(E)$ are the sets $x(C)$ where C is a frame circuit in Σ.*

The proofs of the next results are short. Define a set $S \subseteq E(\Sigma)$ to be *independent* if $x(S)$ is linearly independent.

Corollary 3.3. *A set $S \subseteq E(\Sigma)$ is independent if and only if it does not contain a frame circuit.*

The vector subspace generated by a set $X \subseteq \mathbf{F}^n$ is denoted by $\langle X \rangle$. We write

$$\mathcal{L}_{\mathbf{F}}(\Sigma) := \{\langle X \rangle : X \subseteq x(E)\}.$$

When partially ordered by set inclusion, $\mathcal{L}_{\mathbf{F}}(\Sigma)$ is a lattice.

Corollary 3.4. *For $S \subseteq E(\Sigma)$, $x(E) \cap \langle x(S) \rangle = x(\operatorname{clos} S)$. Thus, $\mathcal{L}_{\mathbb{R}}(\Sigma) \cong \operatorname{Lat} \Sigma$.*

The *rank* of $S \subseteq E$ is defined to be

$$\operatorname{rk} S := n - b(S).$$

Theorem 3.5. *Let $S \subseteq E$. Then $\dim\langle x(S) \rangle = \operatorname{rk} S$.*

Proof. The proof is simplest when expressed in terms of the frame matroid (Section 3.4), so I omit it; see [4, Theorem 8B.1 and following remarks]. The essence of the proof is using Corollary 3.3 to compare the minimum number of edges required to generate S by closure in Σ to the minimum number of vectors $x(e)$ required to generate $\langle x(S) \rangle$. □

3.2. The incidence matrix.

The *incidence matrix* $\mathrm{H}(\Sigma)$ (read 'Eta of Sigma') is a $V \times E$ matrix (thus, it has n rows and m columns where $m := |E|$) in which the column corresponding to edge e is the column vector $x(e)$.

Theorem 3.6. *Let $S \subseteq E$. The rank of $\mathrm{H}(\Sigma|S)$ is $\operatorname{rk} S$.*

Proof. The column rank is the dimension of the span of the columns corresponding to S, which is the span of $x(S)$. Apply Theorem 3.5. □

3.3. Arrangements of hyperplanes.

An *arrangement of hyperplanes* in $\mathbb{R}^n$, $\mathcal{H} = \{h_1, h_2, \dots, h_m\}$, is a finite set of hyperplanes. The *complement* is $\mathbb{R}^n \setminus \big(\bigcup_{k=1}^m h_k\big)$. A *region* of $\mathcal{H}$ is a connected component of the complement. We write $r(\mathcal{H}) :=$ the number of regions. The *intersection lattice* is the family $\mathcal{L}(\mathcal{H})$ of all subspaces that are intersections of hyperplanes in $\mathcal{H}$, partially ordered by $s \le t \iff t \subseteq s$ (reverse inclusion). The *characteristic polynomial* of $\mathcal{H}$ is

$$p_{\mathcal{H}}(\lambda) := \sum_{\mathcal{S} \subseteq \mathcal{H}} (-1)^{|\mathcal{S}|} \lambda^{\dim \mathcal{S}}, \tag{3.1}$$

where $\dim \mathcal{S} := \dim \big(\bigcap_{h_k \in \mathcal{S}} h_k\big)$.

Theorem 3.7 ([3, Theorem A])**.** *We have $r(\mathcal{H}) = (-1)^n p_{\mathcal{H}}(-1)$.*

A signed graph Σ, with edge set $\{e_1, e_2, \dots, e_m\}$, gives rise to a hyperplane arrangement

$$\mathcal{H}[\Sigma] := \{h_1, h_2, \dots, h_m\}$$

where

$$h_k \text{ has the equation } \begin{cases} x_j = \sigma(e_k) x_i, & \text{if } e_k \text{ is a link or loop with endpoints } v_i, v_j, \\ x_i = 0, & \text{if } e_k \text{ is a half edge or a negative loop at } v_i, \\ 0 = 0, & \text{if } e_k \text{ is a loose edge or a positive loop.} \end{cases}$$

(The last equation has the solution set $\mathbb{R}_n$, so it is not truly a hyperplane, but I allow it under the name 'degenerate hyperplane'.) The hyperplane h_k is the solution set of the equation $x(e_k) \cdot x = 0$; i.e.,

$$h_k = \{x \in \mathbb{R}^n : x(e_k) \cdot x = 0\}.$$

($\cdot$ is the usual inner product or 'dot product'.)

Lemma 3.8. *Let $\mathcal{S} = \{h_{i_1}, \dots, h_{i_l}\}$ be the subset of $\mathcal{H}[\Sigma]$ that corresponds to the edge set $S = \{e_{i_1}, \dots, e_{i_l}\}$. Then $\dim \mathcal{S} = b(S)$.*

Proof. Apply vector space duality to Theorem 3.5. □

Theorem 3.9. *$\mathcal{L}(\mathcal{H}[\Sigma])$ is isomorphic to $\mathcal{L}_{\mathbb{R}}(\Sigma)$ and* $\operatorname{Lat}\Sigma$.

Proof. The isomorphism between $\mathcal{L}(\mathcal{H}[\Sigma])$ and $\mathcal{L}_{\mathbb{R}}(\Sigma)$ is standard vector-space duality. The isomorphism $\mathcal{L}_{\mathbb{R}}(\Sigma) \cong \operatorname{Lat}\Sigma$ is in Corollary 3.4. □

3.4. **Matroid.**

The *frame matroid* $G(\Sigma)$ is an abstract way of describing all the previous characteristics of a signed graph: linearly dependent edge sets, minimal dependencies, rank, closure, and closed sets. See [4, Section 5] for more information. For matroid theory, consult [Oxley].

4. COLORING

We color a signed graph from a color set

$$\Lambda_k := \{\pm 1, \pm 2, \dots, \pm k\} \cup \{0\}$$

or a zero-free color set

$$\Lambda_k^* := \Lambda_k \backslash \{0\} = \{\pm 1, \pm 2, \dots, \pm k\}.$$

A *k-coloration* (or *k-coloring*) of Σ is a function $\gamma : V \to \Lambda_k$. A coloration is *zero free* if it does not use the color 0. Coloring comes from [5] and [B, Zaslavsky (1982c)].

A coloration γ is *proper* if it satisfies all the properties

$$\begin{cases} \gamma(v_j) \neq \sigma(e)\gamma(v_i) & \text{for a link or loop } e \text{ with endpoints } v_i, v_j, \\ \gamma(v_i) \neq 0 & \text{for a half edge } e \text{ with endpoint } v_i, \end{cases}$$

and there are no loose edges. (These are the negations of the equations of the hyperplanes h_k.)

4.1. **Chromatic polynomials.**

There are two chromatic polynomials of a signed graph. For aninteger $k \geq 0$, define

$$\chi_\Sigma(2k+1) := \text{ the number of proper } k\text{-colorations},$$

and

$$\chi^*_\Sigma(2k) := \text{ the number of proper zero-free } k\text{-colorations}.$$

Theorem 4.1. *The chromatic polynomials have the properties of*

(i) *Unitarity:* $\chi_\emptyset(2k+1) = 1 = \chi^*_\emptyset(2k)$ *for all* $k \geq 0$.
(ii) *Switching Invariance: If* $\Sigma \sim \Sigma'$, *then* $\chi_\Sigma(2k+1) = \chi_{\Sigma'}(2k+1)$ *and* $\chi^*_\Sigma(2k) = \chi^*_{\Sigma'}(2k)$.
(iii) *Multiplicativity: If Σ is the disjoint union of Σ_1 and Σ_2, then*

$$\chi_\Sigma(2k+1) = \chi_{\Sigma_1}(2k+1)\chi_{\Sigma_2}(2k+1) \quad \textit{and} \quad \chi^*_\Sigma(2k) = \chi^*_{\Sigma_1}(2k)\chi^*_{\Sigma_2}(2k).$$

(iv) *Deletion-Contraction: If e is a link, a positive loop, or a loose edge,*

$$\chi_\Sigma(2k+1) = \chi_{\Sigma\backslash e}(2k+1) - \chi_{\Sigma/e}(2k+1) \quad \textit{and} \quad \chi^*_\Sigma(2k) = \chi^*_{\Sigma\backslash e}(2k) - \chi^*_{\Sigma/e}(2k).$$

Outline of Proof. The hard part is the deletion-contraction property. The proof is similar to the usual proof for ordinary graphs: count proper colorations of $\Sigma \backslash e$. If e is a link, switch so it is positive. Then a proper coloration of $\Sigma \backslash e$ give unequal colors to the endpoints of e, and is a proper coloration of Σ, or it gives the same color to the endpoints, and it corresponds to a proper coloration of Σ/e. If e is a half edge or a negative loop, the two cases are when the endpoint gets a nonzero color or is colored 0. □

Theorem 4.2. *$\chi_\Sigma(\lambda)$ is a polynomial function of $\lambda = 2k+1 > 0$; specifically,*

$$\chi_\Sigma(\lambda) = \sum_{S \subseteq E} (-1)^{|S|} \lambda^{b(S)}. \tag{4.1}$$

Also, $\chi^*_\Sigma(\lambda)$ is a polynomial function of $\lambda = 2k \geq 0$. Specifically,

$$\chi^*_\Sigma(\lambda) = \sum_{S \subseteq E:\, balanced} (-1)^{|S|} \lambda^{b(S)}. \tag{4.2}$$

Proof. Apply Theorem 4.1 and induction on n. □

Therefore, we can extend the range of λ to all of $\mathbb{R}$. In particular, we can evaluate $\chi_\Sigma(-1)$. This lets us draw an important connection between the geometry and coloring of a signed graph.

Theorem 4.3. $\chi_\Sigma(\lambda) = p_{\mathcal{H}[\Sigma]}(\lambda)$.

Proof. Compare the summation expressions, (4.1) and (3.1), for the two polynomials, and note that by Lemma 3.8 $b(S) = \dim \mathcal{S}$ if $\mathcal{S} \subseteq \mathcal{H}[\Sigma]$ corresponds to the edge set S. □

Corollary 4.4. *The number of regions of $\mathcal{H}[\Sigma]$ equals $(-1)^n \chi_\Sigma(-1)$.*

To compute the chromatic polynomial it is often easiest to get the zero-free polynomial first and use

Theorem 4.5 (Zero-Free Expansion Identity)**.** *The chromatic and zero-free chromatic polynomials are related by*

$$\chi_\Sigma(\lambda) = \sum_{W \subseteq V:\ stable} \chi^*_{\Sigma \setminus W}(\lambda - 1).$$

Proof. Let $\lambda = 2k+1$. For each proper k-coloration γ there is a set $W := \{v \in V : \gamma(v) = 0\}$, which must be stable. The restricted coloration $\gamma|_{V \setminus W}$ is a zero-free, proper k-coloration of $\Sigma \setminus W$. This construction is reversible. □

4.2. **Chromatic numbers.**

The *chromatic number* of Σ is

$$\chi(\Sigma) := \min\{k : \exists \text{ a proper } k\text{-coloration}\},$$

and the *zero-free chromatic number* is

$$\chi^*(\Sigma) := \min\{k : \exists \text{ a zero-free proper } k\text{-coloration}\}.$$

Thus, $\chi(\Sigma) = \min\{k \geq 0 : \chi_\Sigma(2k+1) \neq 0\}$ and $\chi^*(\Sigma) = \min\{k \geq 0 : \chi^*_\Sigma(2k) \neq 0\}$.

Almost any question about the behavior of chromatic numbers of signed graphs is open. What I know is in [B, Zaslavsky (1984a)], where I studied complete signed graphs with very large or very small zero-free chromatic number.

5. Examples

The standard basis vectors of $\mathbb{R}^n$ are $b_1 = (1, 0, \dots, 0), b_2, \dots, b_n$.

5.1. **Full signed graphs.** In this example Σ is a signed graph with no half or loose edges or negative loops, $\Sigma^\bullet$ is Σ with a half edge at every vertex, and Σ° is Σ with a negative loop at every vertex. Whether a half edge or negative loop is added makes little difference, because each is an unbalanced edge. Write f_i for the unbalanced edge added to v_i.

- *Graph theory*: The balanced subgraphs in $\Sigma^\bullet$ are the same as those of Σ.
- *Closed sets*: An edge set in $\Sigma^\bullet$ is closed if and only if it consists of the induced edge set $E(\Sigma^\bullet){:}W$ together with a balanced, closed subset of $E(\Sigma){:}W^c$, for some vertex set $W \subseteq V$. Σ° is similar.
- *Vectors*: $x(E(\Sigma^\bullet))$ is $x(E(\Sigma))$ together with the unit basis vectors b_i of $\mathbb{R}^n$. $x(E(\Sigma^\circ))$ is $x(E(\Sigma))$ together with the vectors $2b_i$.
- *Hyperplane arrangement*: $\mathcal{H}[\Sigma^\bullet] = \mathcal{H}[\Sigma^\circ]$, and they equal $\mathcal{H}[\Sigma]$ together with all the coordinate hyperplanes $x_i = 0$.
- *Chromatic polynomials*: $\chi_{\Sigma^\bullet}(\lambda) = \chi_{\Sigma^\circ}(\lambda)$, and both $= \chi_{\Sigma^\bullet}(\lambda) = \chi^*_\Sigma(\lambda - 1)$ by Theorem 4.5, since the only stable set is $W = \varnothing$.
  $\chi^*_{\Sigma^\bullet}(\lambda) = \chi^*_{\Sigma^\circ}(\lambda) = \chi^*_\Sigma(\lambda)$.
- *Chromatic numbers*: $\chi(\Sigma^\bullet) = \chi(\Sigma^\circ) = \chi^*(\Sigma)$ since the unbalanced edges prevent the use of color 0.

5.2. **All-positive signed graphs.** Assume Γ is a graph with no unbalanced edges and no loose edges. $+\Gamma$ has almost exactly the same properties as its underlying graph.

- *Graph theory*: Every subgraph is balanced; $b(S) = c(S)$ for all $S \subseteq E$.
- *Closed sets*: S is closed $\iff$ every edge with endpoints connected by S is in S. Closure in $+\Gamma$ is identical to the usual closure in Γ, and the closed sets in $+\Gamma$ are the same as in Γ.
- *Vectors*: If e has endpoints v_i, v_j, then $x(e) = \pm(b_j - b_i)$. All $x(e) \in$ the subspace $x_1 + \cdots + x_n = 0$.
  If $\Gamma = K_n$ and one takes both signs, the set of vectors is the classical root system A_{n-1}. Thus, $x(E)$ for any graph is a subset of A_{n-1}.
- *Incidence matrix*: $M(+\Gamma)$ is the 'oriented incidence matrix' of Γ.
- *Hyperplane arrangement*: If e_k has endpoints v_i, v_j, then h_k has equation $x_i = x_j$. All $h_k \supseteq$ the line $x_1 = \cdots = x_n$.
  Take $\Gamma = K_n$; then $\mathcal{H}[+K_n] = \mathcal{A}_{n-1}$, the hyperplane arrangement dual to A_{n-1}.
- *Chromatic polynomials*: $\chi_{+\Gamma}(\lambda) = \chi^*_{+\Gamma}(\lambda) = \chi_\Gamma(\lambda)$, the chromatic polynomial of Γ.
- *Chromatic numbers*: $\chi(+\Gamma) = \lfloor \chi(\Gamma)/2 \rfloor$ and $\chi^*(+\Gamma) = \lceil \chi(\Gamma)/2 \rceil$.

The full graph $+\Gamma^\bullet$ is very much like $\Gamma + v_0 := \Gamma$ with an extra vertex v_0 which is adjacent to all of V by edges e_{0i} between v_0 and v_i. Define $\alpha : E(+\Gamma^\bullet) \to E(\Gamma + v_0)$ by $\alpha(e) := e$ if $e \in E(\Gamma)$ and $\alpha(f_i) := e_{0i}$.

- *Graph theory*: S is balanced if and only if it does not contain an unbalanced edge f_i.
- *Closed sets*: S is closed $\iff$ $\alpha(S)$ is closed in $\Gamma + v_0$.
- *Chromatic polynomials*: $\chi_{+\Gamma}(\lambda) = \chi^*_{+\Gamma}(\lambda - 1) = \chi_\Gamma(\lambda - 1)$.
- *Chromatic numbers*: $\chi(+\Gamma^\bullet) = \chi^*(+\Gamma^\bullet) = \lceil \chi(\Gamma)/2 \rceil$.

5.3. **All-negative signed graphs.** Assume Γ is a graph with no unbalanced edges. $-\Gamma$ is very interesting.

- *Graph theory*: A subgraph is balanced $\iff$ it is bipartite. $b_{-\Gamma}(S) =$ the number of bipartite components of S (including isolated vertices).
- *Closed sets*: S is closed if the union of its non-bipartite components is an induced subgraph.
- *Vectors*: If e has endpoints v_i, v_j, then $x(e) = b_i + b_j$ (or its negative).
- *Incidence matrix*: $M(-\Gamma)$ is the 'unoriented incidence matrix' of Γ.
- *Hyperplane arrangement*: h_k has equation $x_i + x_j = 0$ if e_k has endpoints v_i, v_j. Also, $r(\mathcal{H}[-\Gamma]) = \sum_{F \in \operatorname{Lat}\Gamma} |\chi_{\Gamma/F}(-\frac{1}{2})|$.
- *Chromatic polynomials*: $\chi^*_{-\Gamma}(\lambda) = \sum_{F \in \operatorname{Lat}\Gamma} \chi_{\Gamma/F}(\frac{1}{2}\lambda)$ [B, Zaslavsky (1982c), Theorem 5.2]. $\chi_{-\Gamma}(\lambda)$ has not seemed interesting.
- *Chromatic numbers*: $\chi^*(-\Gamma) =$ the largest size of a matching in the complement of a contraction of Γ [B, Zaslavsky (1982c), page 299]. $\chi(-\Gamma)$ has not seemed interesting.

5.4. **Complete signed graphs.** The signed expansions $\pm K_n$, called the *complete signed link graph*, and $\pm K_n^\bullet$, called the *complete signed graph*, have very simple properties.

- *Closed sets*: $\operatorname{Lat}(\pm K_n^\bullet) \cong$ the lattice of signed partial partitions of V [B, Dowling (1973b)].
- *Vectors*: $x(E(\pm K_n)) = \{\pm(b_j - b_i), \pm(b_j + b_i) : i \neq j\}$ where we take either $+$ or $-$ for each vector. $x(E(\pm K_n^\bullet)) = \{\pm(b_j - b_i), \pm(b_j + b_i) : i \neq j\} \cup \{\pm b_i\}$ (if f_i is a half edge; but $\pm 2b_i$ if f_i is a negative loop) where we take either $+$ or $-$ for each vector.

  If we take both signs, we get the classical root systems $D_n := \{\pm(b_j - b_i), \pm(b_j + b_i) : i \neq j\}$ (where we take both $+$ and $-$ signs) from $\pm K_n$, and $B_n := D_n \cup \{\pm b_i\}$ and $C_n := D_n \cup \{\pm 2b_i\}$ from $\pm K_n^\bullet$ (the former if all f_i are half edges, the latter if they are negative loops).
- *Hyperplane arrangement*: $\mathcal{H}[\pm K_n^\bullet] = \mathcal{B}_n = \mathcal{C}_n$ and $\mathcal{H}[\pm K_n] = \mathcal{D}_n$, the duals of B_n, C_n, and D_n. The numbers of regions are $2^n n!$ and $2^{n-1} n!$, respectively.
- *Chromatic polynomials*: $\chi_{\pm K_n^\bullet}(\lambda) = (\lambda - 1)(\lambda - 3)\cdots(\lambda - 2n + 1)$,
  $\chi_{\pm K_n}(\lambda) = (\lambda - 1)(\lambda - 3)\cdots(\lambda - 2n + 3)\cdot(\lambda - n + 1)$, and
  $\chi^*_{\pm K_n}(\lambda) = \chi^*_{\pm K_n^\bullet}(\lambda) = \lambda(\lambda - 2)\cdots(\lambda - 2n + 2)$.
- *Chromatic numbers*: $\chi(\pm K_n^\bullet) = \chi^*(\pm K_n^\bullet) = \chi^*(\pm K_n) = n$ and $\chi(\pm K_n) = n - 1$.

5.5. **Signed expansion graphs.** The properties of $\pm\Gamma$ and $\pm\Gamma^\bullet$ are closely related to those of Γ.

- *Graph theory*: Each balanced set $S \subseteq E(\Gamma)$ gives $2^{b(S)}$ balanced subsets of $E(\pm\Gamma)$.
- *Hyperplane arrangement*: $r(\mathcal{H}[\pm\Gamma^\bullet]) = 2^n(-1)^n\chi_\Gamma(-1) = 2^n|\chi_\Gamma(-1)|$ and
$$r(\mathcal{H}[\pm\Gamma]) = \sum_{W \subseteq V:\ \text{stable in } \Gamma} (-2)^{n-|W|}|\chi_{\Gamma\setminus W}(-1)|.$$
- *Chromatic polynomials*: $\chi_{\pm\Gamma^\bullet}(\lambda) = 2^n\chi_\Gamma(\frac{1}{2}(\lambda - 1))$, $\chi^*_{\pm\Gamma}(\lambda) = 2^n\chi_\Gamma(\frac{1}{2}\lambda)$, and
$$\chi_{\pm\Gamma}(\lambda) = \sum_{W \subseteq V:\ \text{stable in } \Gamma} 2^{n-|W|}\chi_{\Gamma\setminus W}(\tfrac{1}{2}(\lambda - 1)).$$
- *Chromatic numbers*: $\chi(\pm\Gamma^\bullet) = \chi^*(\pm\Gamma) = \chi(\Gamma)$, the chromatic number of Γ, and $\chi(\pm\Gamma) = \chi(\Gamma) - 1$.

Bibliography

References in the style [B, Name (yeara)] are to the author Name, item (yeara) in [B].

[B] Thomas Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas. *Electronic J. Combin.*, Dynamic Surveys in Combinatorics (1998), No. DS8 (electronic), vi + 151 pp. Current version (about 240 pp.) available at `http://www.math.binghamton.edu/zaslav/Bsg/`. MR 2000m:05001a. Zbl 898.05001.

[G] Thomas Zaslavsky, Glossary of signed and gain graphs and allied areas. *Electronic J. Combinatorics* (1998), Dynamic Surveys in Combinatorics, #DS9 (electronic), 41 pp. MR 2000m:05001b. Zbl 898.05002.

[1] F. Harary, On the notion of balance of a signed graph. *Michigan Math. J.* **2** (1953–54), 143–146 and addendum preceding p. 1. MR 16, 733h. Zbl 56, 421c (e: 056.42103).

[2] J.G. Oxley, *Matroid Theory.* Oxford Univ. Press, Oxford, 1992. MR 94d:05033. Zbl 784.05002.

[3] Thomas Zaslavsky, *Facing Up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes.* Mem. Amer. Math. Soc., No. 154 (= Vol. 1, Issue 1). American Mathematical Society, Providence, R.I., 1975. MR 50 #9603. Zbl 296.50010.

[4] Thomas Zaslavsky, Signed graphs. *Discrete Appl. Math.* **4** (1982), 47–74. Erratum. *Discrete Appl. Math.* **5** (1983), 248. MR 84e:05095. Zbl 503.05060.

[5] Thomas Zaslavsky, Signed graph coloring. *Discrete Math.* **39** (1982), 215–228. MR 84h:05050a. Zbl 487.05027.