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The Order Upper Bound on Parity Embedding of a Graph

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A graph Γ is *parity embedded* in a surface if a closed path in the graph is orientation preserving or reversing according to whether its length is even or odd. The *parity demigenus* of Γ is the minimum of $2-\chi(S)$ (where χ is the Euler characteristic) over all surfaces S in which Γ can be parity embedded. We calculate the maximum parity demigenus over all graphs, simple or not, of order n.

Let us try to embed a graph Γ, not necessarily simple, in a surface so that every odd polygon (the graph of a simple closed path of odd length), regarded as a path in the surface, reverses orientation while every even polygon preserves it. What is the smallest surface in which this is possible? That is, what is the minimum *demigenus* $d(S)=2-\chi(S)$ over all embedding surfaces S? We call this kind of embedding *parity embedding*[1] and the smallest $d(S)$ the *parity demigenus* of Γ, written $d(-\Gamma)$. There is in general no exact formula but there is a simple lower bound based on Euler's polyhedral formula and the obvious fact that a face boundary must (with trivial exceptions) have length at least 4:

$$d(-\Gamma)\geqslant\left\lceil\frac{m}{2}\right\rceil-n+2 \tag{1}$$

if Γ is connected and has no multiple edges and $m\geqslant 2$, where $n=|V|$, the order of Γ, and $m=|E|$, the number of edges. There is also an obvious upper bound in terms of the order, namely $d(-K_n^\circ)$ where K_n° is the complete graph with a loop at every vertex (since multiple edges do not affect parity embeddability, but loops do). Here we establish the value of this upper bound by proving that, except when $n\leqslant 5$, $d(-K_n^\circ)$ equals the lower bound given by (1).

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[1] As far as I know, the concept and name of parity embedding originated with Lins [1].

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THEOREM. *For any graph Γ of order n,*

$$d(-\Gamma) \leqslant d(-K_n^\circ) = \begin{cases} n, & \text{if } n \leqslant 4, \\ 6, & \text{if } n = 5, \\ \lceil \frac{1}{4}n(n-3) \rceil + 2, & \text{if } n \geqslant 6. \end{cases} \tag{2}$$

We sketch another interpretation of the theorem. Let $\tilde{\Gamma}$ be a connected bipartite graph with bipartition $V = V_1 \cup V_2$, embedded in T_g (the sphere with g handles) so that an involutory autohomeomorphism τ of T_g whose quotient is U_{g+1} (the sphere with $g+1$ crosscaps) carries $\tilde{\Gamma}$ to itself while exchanging the two independent vertex sets V_1 and V_2. The minimum possible g is $d(-\Gamma) - 1$, where Γ is the quotient of $\tilde{\Gamma}$ by τ. In particular, if $\tilde{\Gamma}$ is $K_{n,n}$ with all edges of the form $v\tau(v)$ doubled (a technical necessity), the smallest g is $d(-K_n^\circ) - 1$, which exceeds by approximately $n/4$ the minimum when the embedding is unrestricted, which is $\lceil \frac{1}{4}(n-2)^2 \rceil$. (Let us call this graph $K_{n,n}(\tau)$. The difference in the minimum g is largely due not to antipodality but merely to the fact that, if $n \geqslant 2$, an antipodal embedding of $K_{n,n}(\tau)$ has no digonal faces. This simple property raises the Eulerian lower bound on g to $\lceil \frac{1}{4}n(n-3) \rceil + 1$, which turns out to be the true minimum number of handles needed for an embedding of $K_{n,n}(\tau)$ without digonal faces when $n \geqslant 2$. We omit the details.)

A reader familiar with the usual, unsigned graph embedding may wonder why we allow loops. In unsigned embedding, loops and multiple edges have no effect on the surfaces in which a graph embeds. In parity embedding, multiple edges have no effect, but loops, in contrast, can alter the minimal surface. Indeed, the largest parity demigenus of a simple graph, which is obviously $d(-K_n)$, equals $\lceil \frac{1}{4}n(n-5) \rceil + 2$ for $n \geqslant 6$ (a result that will appear separately). This is smaller by about $\frac{1}{2}n$ than the overall upper bound $d(-K_n^\circ)$.

Before proceeding to the proof let us see how parity embedding fits into the more general scheme of orientation embedding of signed graphs. A graph with signed edges is said to be *orientation embedded* in a surface if it is embedded so that a closed path preserves orientation if and only if its sign product is positive. Parity embedding is therefore the same as orientation embedding of $-\Gamma$, the all-negative signing of Γ. Let us call the *demigenus* $d(\Sigma)$ of a signed graph Σ the smallest demigenus of any surface in which it orientation embeds. I believe that $d(-K_n^\circ)$ maximizes not only $d(-\Gamma)$ but also $d(\Sigma)$ for all signed graphs of order n that have no parallel edges. Equivalently, and a bit more simply,

Conjecture. $d(-K_n^\circ)$ is the maximum demigenus of any signed K_n°.

The Proof of the Theorem

Let δ_n denote the right-hand side of (2).

Proof that $d(-K_n^\circ) \geqslant \delta_n$. Taking account of (1) and the fact that the n loops already require n crosscaps, we deduce that

$$d(-K_n^\circ) \geqslant \max\left(n, \left\lceil \frac{n^2-3n}{4} \right\rceil + 2\right). \tag{3}$$

This takes care of all cases except $n=5$, where, as it happens, both quantities on the right-hand side of (3) equal 5. We need to prove that there is no parity embedding of K_5° in U_5.

Suppose there were such an embedding. Being minimal it would be *cellular*; every face would be an open 2-cell. Let f_i denote the number of faces whose boundaries have length i (briefly, *i-faces*). Then $f_i = 0$ unless $i = 4, 6, 8, \ldots$ because a complete walk around a face boundary must be orientation preserving. Consequently $2m = 4f_4 + 6f_6 + \cdots = 4f + 2(f_6 + 2f_8 + \cdots)$; since by Euler's formula $f = m - n + \chi(U_5) = 7$, we have $f_6 = 1$ and $f_4 = 6$. We show that these face numbers are unable to carry the five orientation-reversing loops.

We shall think of the boundary ∂F of a face F as a walk in the graph. If an edge appears twice on a face boundary we therefore count it as two edges in ∂F.

We now prove some lemmas about face boundaries of any cellularly parity-embedded K_n°.

LEMMA 1. *Two appearances of a vertex* v *on a face boundary* ∂F *cannot be separated along* ∂F *by exactly one vertex not equal to* v.

Proof. Suppose ∂F contained the vertex sequence v, w, v. Then the edges vw and wv are consecutive in ∂F. But that can happen only if w is monovalent, which cannot occur in K_n°. ∎

LEMMA 2. *A* 4-*face can have only one loop on its boundary. A* 6-*face can have as many as four boundary loops only if* $n = 2$. (*We count a repeated loop as two.*)

Proof. If the same loop appears twice on ∂F, Lemma 1 is violated unless F is a 6-face and the repeated loop appears consecutively. Thus for ∂F to contain two loops (if a 4-face) or four loops (if a 6-face), the edges of ∂F must be $wv, vv, (vv), vw, ww, (ww)$. Let us see what this entails at v. The loop vv splits a neighborhood of v into two sides. Tracing ∂F from w to v, then along vv, we are on the opposite side of v from wv. Therefore ∂F cannot immediately go back along vw. This disposes of the 4-face case. In the 6-face case ∂F must be able to continue along vv; this means no edges meet v on the side opposite wv. After tracing vv a second time, ∂F must return along vw, so there can be no other edge incident to v on the wv-side of v. Hence v has only one neighbor, from which it follows that $n=2$. (If $n=2$ it is clear that there is just one face and its boundary length is indeed 6.) ∎

Now, in a parity embedding of K_5° in U_5, the six 4-face boundaries and the one 6-face boundary can account for at most $6(1)+(3)=9$ loop appearances, which is too few for five loops. Therefore the supposed embedding does not exist. We have proved $d(-K_5^\circ)>5$. ∎

Proof that $d(-K_n^\circ)\leqslant\delta_n$. The constructions diagrammed in Figs. 1 to 8 demonstrate the existence of a parity embedding of K_n° in U_{δ_n} for all n. We begin with an explanation of how to read the diagrams.

A lens shape with a number (say, i) inside denotes a crosscap in the form of a hole on whose rim opposite points are identified. The sharp ends are vertex i and the sides are the loop at i. The two points representing the vertex are labelled $+$ and $-$ (only the $+$ being shown). We call these figures *loop lenses*.

A crosscap which is not a loop lens is drawn as a circle or oval with a tilde inside. Again this denotes a hole whose opposite boundary points are identified. In Fig. 2 the outer circular boundary also denotes a crosscap: that is, opposite points on it are identified.

A handle is drawn as a pair of circular holes (called its *ends*) whose boundaries are identified with each other in opposite senses or in the same sense. In the former case the handle preserves orientation with respect to the plane of the figure, which means that a closed path passing through that handle and no other handle or crosscap is orientation preserving. Such a handle is called a *prohandle*. In the latter case the handle reverses orientation; it is called an *antihandle*. (In some cases only one of the two ends is depicted—this is the case for the "outside handles" to be defined later.)

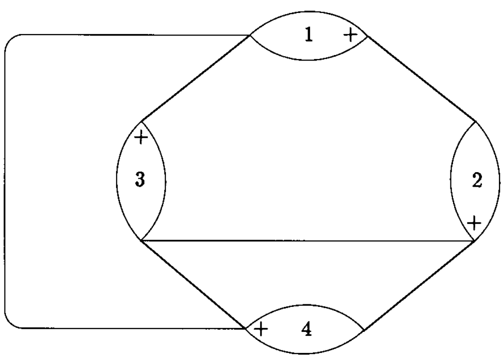

FIG. 1. A parity embedding of K_4° in U_4. The whole figure lies in the plane. By deleting vertices (and incident edges and lenses) we get minimal parity embeddings of K_3°, K_2°, and K_1°.

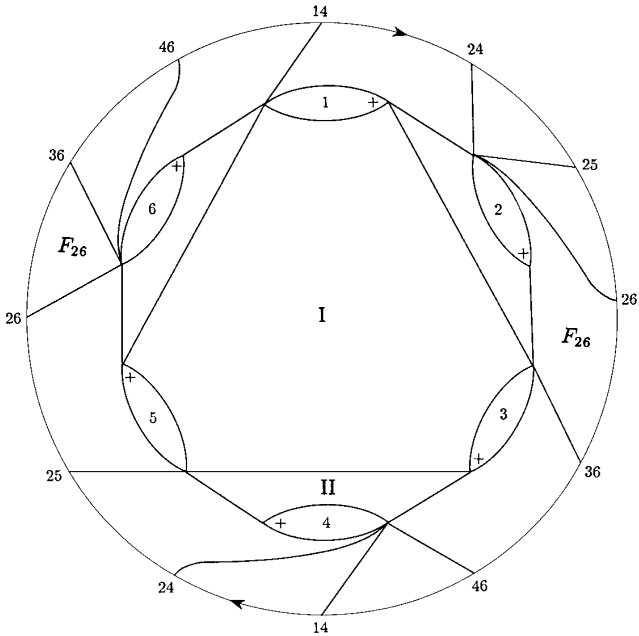

FIG. 2. A parity embedding of K_6° in U_7, presented as six loop lenses in the projective plane. By deleting one vertex we obtain a parity embedding of K_5° in U_6.

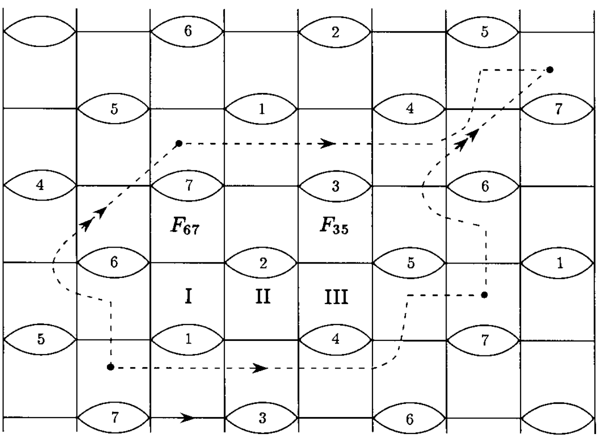

FIG. 3. A parity embedding of K_7° in U_9, drawn as a torus with seven loop lenses. The torus is presented as a planar tessellation. The + end of each lens is at the right. A fundamental domain is outlined in dashes.

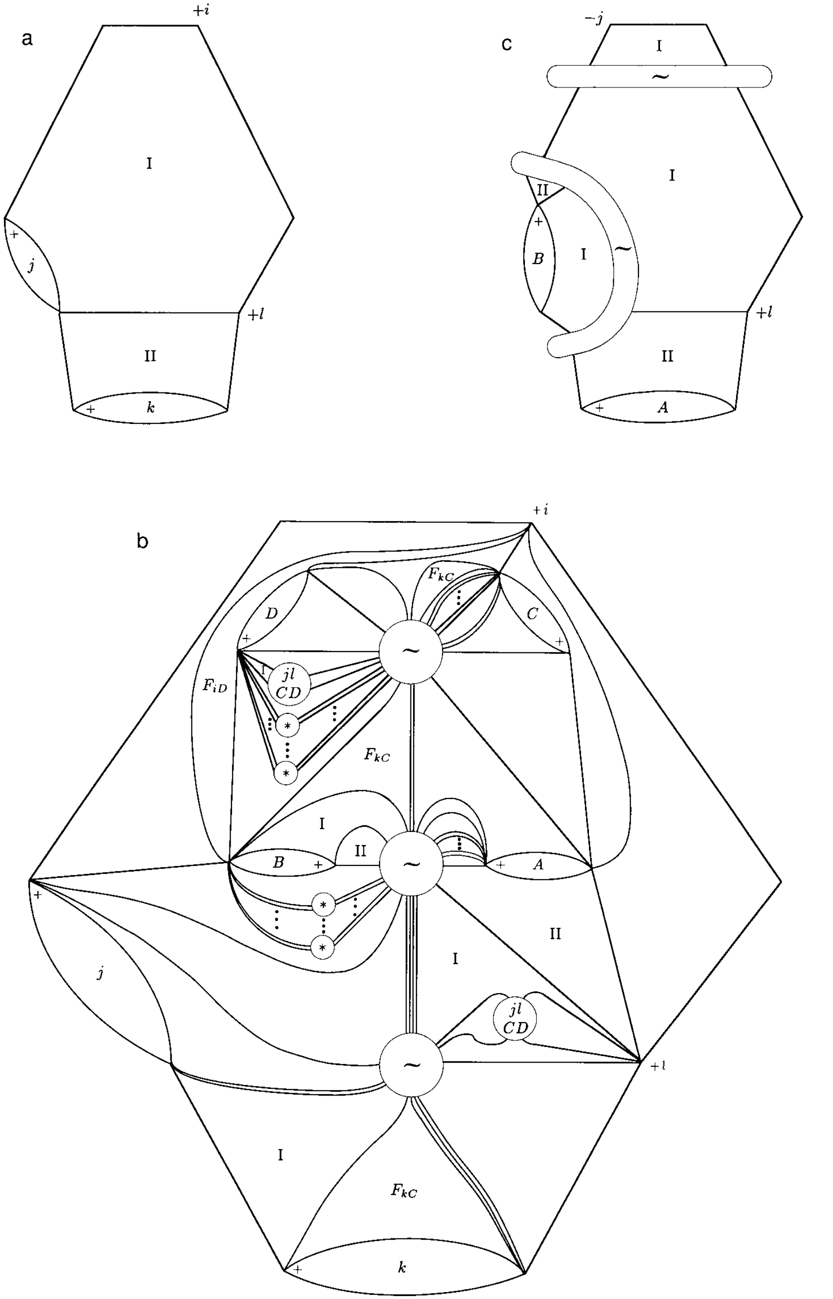

FIG. 4. The inductive scheme for parity embedding of K°_{6+4s}: (a) the environment before gadget insertion, (b) the gadget in the environment, and (c) the new environment, from (b).

Any path has a sign which is calculated according to the following rule. An endpoint which is a vertex is signed (as mentioned earlier); any other endpoint is treated as positive. To calculate the path sign we multiply the endpoint signs and negate once for each crosscap or antihandle through which the path passes. By this rule a closed path has positive sign if and only if it preserves orientation. Also, the sign of a concatenation of two paths (say from P to Q and Q to R) equals the product of the signs of the paths. In particular, the sign of a polygon of length l in the graph is $(-1)^l$ if every edge is negative. Thus we have a parity embedding if we make every edge negative.

Now we describe the minimal parity embeddings. Embeddings for $n \leqslant 7$ are shown in Figs. 1 to 3. We construct embeddings for higher order inductively. In outline, starting from embeddings of K_6° and K_7° we repeatedly add a four-vertex "tetradic gadget" (different for $n \equiv 6$ and 7 modulo 4) to get embeddings for all larger $n \equiv 6$ and 7 (mod 4). We solve $n \equiv 8$, 9 (mod 4) by adding to an embedded K_{6+4s}° a "dyadic" or "triadic" gadget of two or three vertices. In every embedding the parity property is assured by making every edge negative according to the rule stated earlier.

In greater detail: A gadget added to an embedded K_n° is inserted in a suitable environment, which is a portion of the embedding having a certain shape (which is constant within each residue class of n modulo 4). The environment is a union of closed faces of the embedded graph; its boundary therefore consists of some edges and (signed) vertices of K_n° and its interior consists of open faces and edges (but no vertices, as it happens). The environments are shown in Figs. 4a and 7a. Note that some vertices are unlabelled: they play no part in adding a gadget, so they can be anything. On the other vertices, different labels signify that the vertices are actually different. The vertex signs are not in themselves important; their role is to tell us (by the usual rule) the sign of a path drawn through the environment between two labelled boundary vertices. A different presentation of the environment in which parts of it pass through crosscaps or antihandles might have different vertex signs, but it is the same environment so long as the path signs are the same. In Figs. 4 and 7, for example, (c) is the same environment as (a) although drawn differently.

In (a) the faces are labeled I and II. In (b, c) the new environment is similarly labeled. The vertices i, j, k, l of the old environment correspond to j, B, A, l in the new. The new vertex pairs are iD and kC; their attachment faces are labeled accordingly. In (b) the two holes of the internal handle, which is an antihandle, are labelled $jlCD$. Outside handles are starred; only the ends within the replacement surface are depicted.

An initial environment (that is, for $n=6$) with $i=1$, $j=5$, $k=4$, and $l=3$ is shown in Fig. 2. The initial pairing of outside vertices is 26.

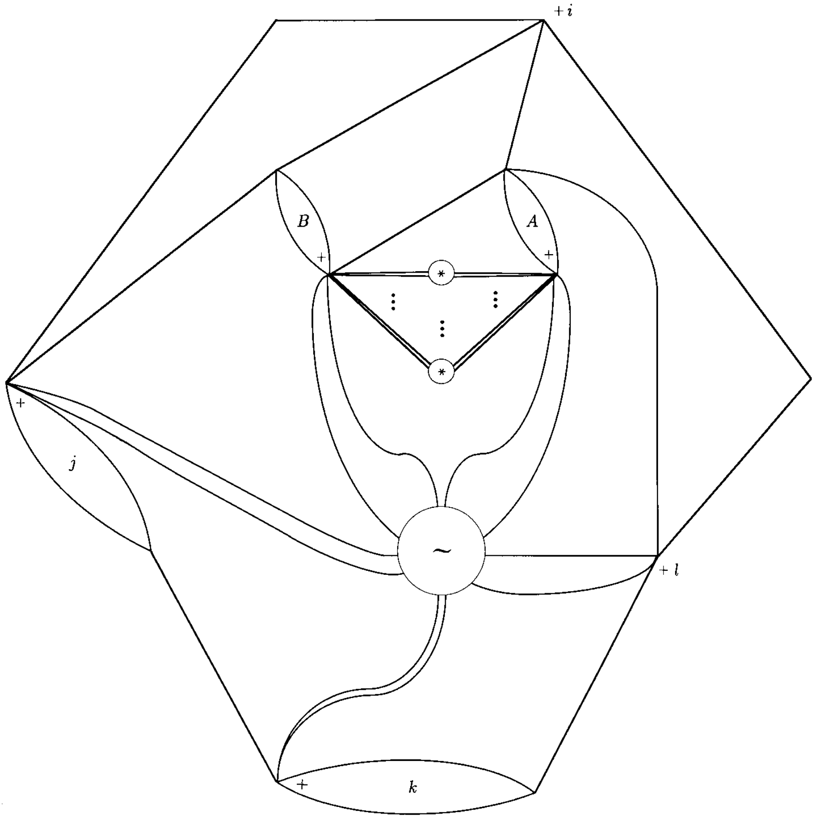

FIG. 5. A parity embedding of K°_{8+4s}, shown as a dyadic gadget inserted in the environment of K°_{6+4s}.

Gadget insertion can be viewed as a two-step process. First, the interior of the environment is removed and replaced by a surface which contains a parity-embedded K°_{4} (or K°_{2} or K°_{3}) and edges joining the new vertices to some of those on the boundary of the environment. (We call these latter vertices *direct*. There are four of them if $n = 6 + 4s$, three if $n = 7 + 4s$. The remaining old vertices are called *indirect*.) This *replacement surface* contains, besides the loop lenses, several crosscaps and handles to permit all the connecting edges to be drawn with negative sign and no crossings.

In the second step one adds *outside handles* to carry edges from the new vertices to the indirect old ones. Each such handle carries edges to two indirect vertices from two or (in the triadic case) three new vertices. The two indirect vertices, say p and q, must therefore be on a common face F_{pq}, called their *attachment face*, which the handle reaches from a suitable face of the replacement surface. To make all this possible one wants in advance,

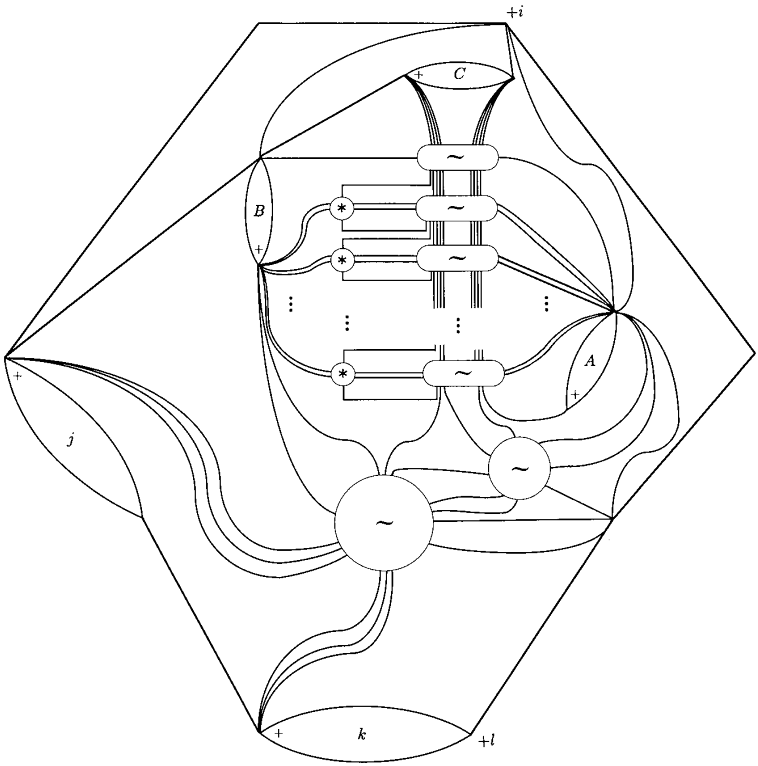

FIG. 6. A parity embedding of K_{9+4s}°, shown as a triadic gadget inserted in the environment of K_{6+4s}°.

besides the environment, a pairing of the indirect vertices such that all the attachment faces are distinct. In order to guarantee negative edge signs we choose the F_{pq} so that both vertices appear on ∂F_{pq} with the same sign. (Actually, one vertex appears twice, with both signs; we use the copy whose sign equals that of the other vertex.) We place in F_{pq} one end of each handle carrying edges to p and q (see Fig. 8); from there the handle edges can be distributed to p and q without passing through any crosscaps or handles, so the sign contribution to an edge at this end of the handle will be the same for p and for q. At the other end of the handle we arrange things so that the sign calculated there for every edge in the handle is the same. (For example, one edge may start at vertex $+x$ and pass through two crosscaps on its way to the handle end, while another starts at $-y$ and goes through one crosscap; thus both edges have $+$ signs on that side of the handle.) Now all we need do is choose a pro- or antihandle, whichever makes the handle edges negative. Since that is always possible, in the

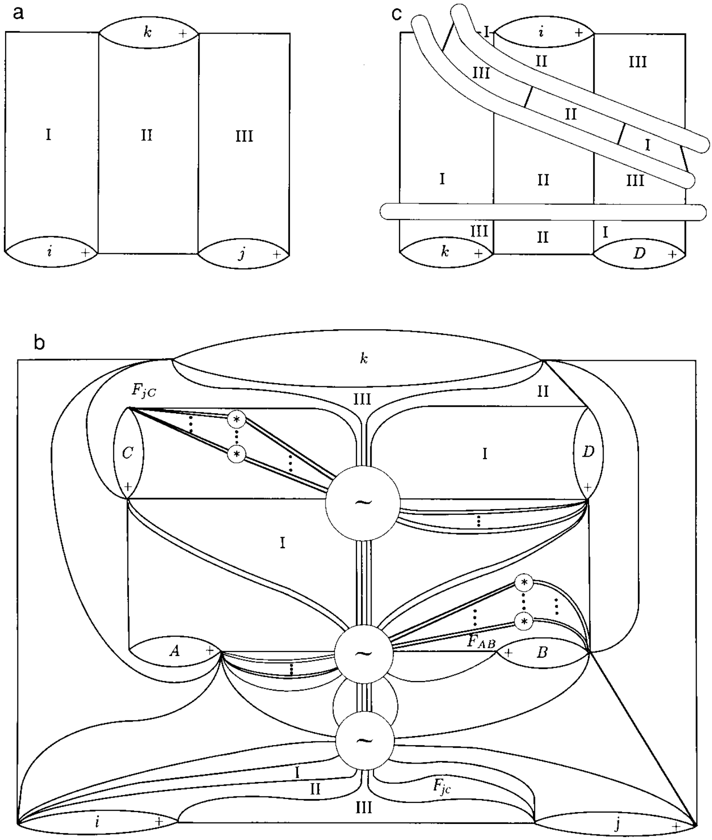

FIG. 7. The inductive scheme for parity embedding of K°_{7+4s}. (a, b, c) are as in Fig. 4. In (a) the faces are labeled I, II, III. In (b, c) the new environment is labeled similarly. The vertices i, j, k of the old environment correspond to D, k, i in the new one. The new vertex pairs are jC and AB.

An initial environment (for $n = 7$) with $i = 1$, $j = 4$, $k = 2$ and initial pairing 35, 67 is shown in Fig. 3.

drawings we can safely ignore the exact orientation type of the outside handles.

When we add the dyadic or triadic gadget, only one outside handle goes to each attachment face. In the triadic gadget, this handle carries three edges to each indirect vertex; those going to each such vertex must be grouped together in the handle, and the gadget is constructed to do this.

Now we come to the last crucial point. In order to make induction possible, adding the tetradic gadget must reproduce the environment: the new embedding must contain a group of closed faces whose union is similar to the original environment. Furthermore, it must admit a suitable pairing

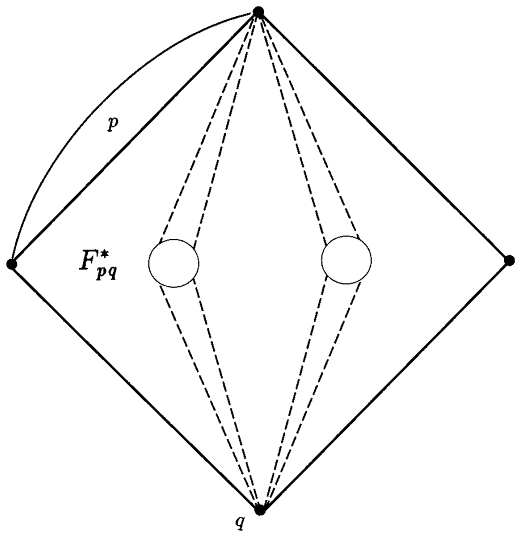

FIG. 8. An attachment face F_{pq}, showing the indirect vertices p and q, the ends in F_{pq} of the two outside handles, the four edges (dashed) that run through the handles to each of p and q, and the new attachment face F^*_{pq}.

of vertices which are outside the new environment. To explain how we meet these requirements we note that the new surface consists of two parts, the replacement surface and the remnant of the old surface, both modified by the addition of outside handles. The boundary between the two parts we call the *border*. Bear in mind that each outside handle consists of two circular holes with identified boundaries, one hole in the remnant and one in the replacement surface; thus it is topologically a circle. The border therefore consists of the original environment's boundary and the outside handles. Note that the border is transverse to the embedded $-K^{\circ}_{n+4}$: they intersect in a finite set of points, none a vertex.

The border may cut a face F of the new embedding into components, each of which is clearly a topological disk. A component is a *pseudopod* if its boundary consists of a path in the border and a path in ∂F and contains exactly one vertex. One can check in Figs. 4b, 7b, and 8 that every face cut by the border has exactly one component that is not a pseudopod.

Now examine Figs. 4 and 7. The diagrams 4c and 7c show the new environment, which, as the diagrams 4b and 7b demonstrate, lies within the replacement surface. The new pairing is chosen to be the old one together with two new pairs tu whose attachment faces F_{tu} are in the replacement surface (except for pseudopods). The old attachment faces F_{pq} are broken up by the outside handles and their edges but a new attachment face F^*_{pq} can be found by taking the part of F_{pq}, as split up, which abuts the loop pp. (See Fig. 8.) The new face F^*_{pq} is outside the replacement surface (again, aside from pseudopods). It follows that the new attachment faces are all distinct.

The final verification of all requirements is by the reader's inspection of Figs. 4–7.

It remains to show that the surface resulting from addition of the gadget to a minimal parity embedding of K_n°, $n \geqslant 6$, has the right demigenus. A tetradic gadget contains 4 loop lenses and 3 other crosscaps. If $n \equiv 6 \pmod 4$ there is also one internal handle. The $\lceil (n-4)/2 \rceil$ indirect vertex pairs require $2\lceil (n-4)/2 \rceil$ handles. The total increment to $d(-K_n^\circ)$ is therefore $2n+1$. Assuming $d(-K_n^\circ) = \delta_n$, we have K_{n+4}° embedded in demigenus

$$\left\lceil \frac{n^2 - 3n}{4} \right\rceil + 2 + (2n+1) = \left\lceil \frac{n^2 + 5n + 4}{4} \right\rceil + 2,$$

which equals δ_{n+4}. The dyadic gadget has 2 loop lenses, one other crosscap, and $(n-4)/2$ outside handles for a total demigenus increment of $n-1$; and $\delta_n + (n-1) = \frac{1}{4}(n^2 + n - 2) + 2 = \delta_{n+2}$. The triadic gadget has 3 loop lenses and 3 other crosscaps plus one for each outside handle. There are $(n-4)/2$ outside handles. The total demigenus increment is $\frac{3}{2}n$, and $\delta_n + \frac{3}{2}n = \frac{1}{4}(n^2 + 3n + 2) + 2 = \delta_{n+3}$. Thus in every case the new graph is parity embedded in the desired demigenus. ∎

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