QUASIGROUP ASSOCIATIVITY AND BIASED EXPANSION GRAPHS

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Abstract. We present new criteria for a multary (or polyadic) quasigroup to be isotopic to an iterated group operation. The criteria are consequences of a structural analysis of biased expansion graphs. We mention applications to transversal designs and generalized Dowling geometries.

1. Associativity in multary quasigroups

A *multary quasigroup* is a set with an n-ary operation for some finite $n \geq 2$, say $f : Q^n \to Q$, such that the equation $f(x_1, x_2, \ldots, x_n) = x_0$ is uniquely solvable for any one variable given the values of the other n variables. An (associative) *factorization* is an expression

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_i, h(x_{i+1}, \ldots, x_j), \ldots, x_n) \tag{1}$$

where g and h are multary quasigroup operations. For instance, if f is constructed by iterating a group operation,

$$f(x_1, \ldots, x_n) = x_1 \cdot x_2 \cdot \cdots \cdot x_n,$$

then it has every possible factorization. We study the degree to which an arbitrary multary quasigroup with some known factorizations is an iterated group. We employ a new method, the structural analysis of biased expansion graphs.

An operation may be disguised by isotopy, which means relabelling each variable separately; or by conjugation, which means permuting the variables. Precisely, we call operations f and f' *isotopic* if there exist bijections $\alpha_i : Q \to Q$ such that

$$f'(x_1, \ldots, x_n)^{\alpha_0} = f(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n});$$

we call them *circularly conjugate* if

$$x_0 = f'(x_1, \ldots, x_n) \iff x_i = f(x_{i+1}, \ldots, x_n, x_0, x_1, \ldots, x_{i-1})$$

or

$$x_0 = f'(x_1, \ldots, x_n) \iff x_i = f(x_{i-1}, \ldots, x_1, x_0, x_n, \ldots, x_{i+1})$$

for some $i = 0, 1, \ldots, n$. Neither isotopy nor circular conjugation affects the existence of factorizations. The exact factorization formulas may change under circular conjugation, but the factorizations of f and f' correspond.

If a ternary quasigroup factors in both possible ways,

$$f(x_1, x_2, x_3) = g_1(h_1(x_1, x_2), x_3) = g_2(x_1, h_2(x_2, x_3)) \tag{2}$$

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(this is known as *generalized associativity*), then g_1, g_2, h_1, h_2 are all isotopic to a single group multiplication, so f is isotopic to an iterated group operation [3, 8, 1]. To generalize this result to higher n, we create an undirected *factorization graph* $\Delta(f)$. The vertex set is $\{v_0, v_1, \ldots, v_n\}$ and the edge set contains $e_{01}, e_{12}, \ldots, e_{n-1,n}, e_{n0}$ (where e_{ij} denotes an edge whose endpoints are v_i and v_j) as well as an edge e_{ij} for every factorization (1). It follows easily from the theorem of generalized associativity that, if Δ is complete, then f is an iterated group isotope (and the converse is obvious). According to Dudek [6], V.D. Belousov, who introduced the notion of n-ary quasigroup in a paper with Sandik [4], conjectured that the same conclusion follows if Δ is any 3-connected graph. (I have not been able to locate this conjecture anywhere.) I can prove the conjecture.

Theorem 1. *An n-ary quasigroup operation f such that $\Delta(f)$ is 3-connected is isotopic to an iterated group operation.*

An immediate corollary (mentioned by a referee) is a characterization of iterated group isotopes among all n-ary operations of whatever kind.

Corollary 2. *An n-ary operation is isotopic to an iterated group operation if and only if it is an n-ary quasigroup operation whose factorization graph is 3-connected.*

Theorem 1 is the best possible result. A factorization graph can have a 2-separation. Indeed we can explicitly describe all possible factorization graphs. *Edge amalgamation* of two disjoint graphs means identifying one edge in the first graph with one in the second graph.

Theorem 3. *A finite, simple graph with at least 3 vertices is a factorization graph of a multary quasigroup if and only if it has a Hamiltonian circuit and is obtained by edge amalgamation of circuits and complete graphs.*

If $|Q|$ is very small, f may be obliged to factor in every way. According to Dudek [6], Belousov and collaborator(s) proved that $\Delta(f)$ is complete when $|Q| = 2$, and also when $|Q| = 3$ although this proof was too long to publish. On the other hand, one can construct a multary quasigroup of order $|Q| = 4$ whose factorization graph is any graph satisfying Theorem 3 because there exist irreducible n-ary quasigroups with $|Q| = 4$ for all $n \geq 3$, by [4, Section 5] and [7]; see [2]. One can deduce the results for $|Q| \leq 3$ from a second general criterion for group isotopy. A *residual* multary quasigroup of a multary quasigroup is obtained by fixing the values of some of the independent variables.

Theorem 4. *If f has arity at least three and each residual ternary quasigroup is an iterated group isotope (not necessarily of the same group), then f is isotopic to an iterated group operation.*

Corollary 5. *If $|Q| \leq 3$, then f is isotopic to an iterated group operation.*

2. Biased expansion graphs

The approach we take to proving these results is that of biased graphs, and more specifically, biased expansions of a graph. Intuitively, a biased expansion of a graph Δ is a kind of branched covering of Δ, whose branch points are the vertices. The precise definition is somewhat complicated.

First we define a *biased graph* [11, Part I]. It is a pair $\Omega = (\Gamma, \mathcal{B})$ where Γ is a graph (multiple edges being allowed) and $\mathcal{B}$ is a *linear subclass* of the class of all circuits: this means that, whenever B_1, B_2 are circuits in $\mathcal{B}$ whose union $B_1 \cup B_2$ consists of three simple paths that are internally disjoint and have the same endpoints, then the third circuit in $B_1 \cup B_2$ also belongs to $\mathcal{B}$. Circuits in $\mathcal{B}$ are called *balanced*.

The prototype of a biased graph is a gain graph. Let us assign to each oriented edge $\tilde{e}$ of Γ a value $\varphi(\tilde{e})$ in some fixed group $\mathfrak{G}$, in such a way that the same edge with the opposite orientation, which we denote by $\tilde{e}^{-1}$, has value $\varphi(\tilde{e}^{-1}) = \varphi(\tilde{e})^{-1}$. Then (Γ, φ) is called a *gain graph* and $\varphi(\tilde{e})$ is the *gain* of $\tilde{e}$. We obtain a biased graph by taking as balanced circuits all those circuits $\tilde{C} = \tilde{e}_1\tilde{e}_2 \cdots \tilde{e}_l$ such that, after orienting the edges in the indicated direction around the circuit, the gain product $\varphi(\tilde{C}) = \varphi(\tilde{e}_1)\varphi(\tilde{e}_2)\cdots\varphi(\tilde{e}_l) = 1$, the group identity. It is easy to see that, although the actual value of $\varphi(\tilde{C})$ may depend on the chosen orientation and starting point, the class $\mathcal{B}$ of balanced circuits is independent of the choices. (Gain graphs are called "voltage graphs" in topological graph theory; however, our problems and methods are quite different, having their origin in matroid theory.)

A *biased expansion* of Δ [11, Example III.3.8 and Part V], written $\Omega \downarrow \Delta$, consists of Δ (called the *base graph*), a biased graph Ω with the same vertex set as Δ, and a *projection* mapping $p : \Omega \to \Delta$ which maps vertices to vertices and edges to edges (and preserves incidence of vertices and edges), is the identity on vertices, is surjective on edges, maps no balanced digon onto a single edge, and has a property we call the *circle lifting property*. This is the property that, whenever C is a circuit in Δ, e is an edge in C, and $\tilde{P}$ is a path in Ω that projects bijectively onto $C \backslash e$ (that is, $p|_{\tilde{P}} : \tilde{P} \to C \backslash e$ is a graph isomorphism), then there exists exactly one edge $\tilde{e} \in p^{-1}(e)$ for which the circuit $\tilde{P} \cup \{\tilde{e}\}$ is balanced.

The prototype of a biased expansion is a *group expansion* of Δ by a group $\mathfrak{G}$ [11, Example I.6.7]. Here Ω is a gain graph with vertex set $V(\Delta)$ and edge set $\mathfrak{G} \times E(\Delta)$; the projection takes $(g, e) \in E(\Omega)$ to $e \in E(\Delta)$. To define the gain function φ we fix an arbitrary orientation of Δ and carry it over to Ω, orienting (g, e) the same way as e. An edge $\tilde{e} = (g, e)$ of Ω has gain $\varphi(\tilde{e}) = g$ if $\tilde{e}$ is directed as in the fixed orientation and g^{-1} if not. The general rule for gain graphs makes Ω a biased graph, which one can verify is a biased expansion of Δ.

3. Expansions and quasigroups

Biased expansions of a circuit C_{n+1} of length $n+1$ are equivalent to equivalence classes of n-ary quasigroups under isotopy and circular conjugation. To show this, first we construct an n-ary quasigroup (Q, f) from a biased expansion $\Omega \downarrow C_{n+1}$. It is not hard to show that every edge fiber $p^{-1}(e)$ has the same cardinality. Suppose $C_{n+1} = e_{01}e_{12}\cdots e_{n-1,n}e_{0n}$ on vertex set $\{v_0, v_1, \ldots, v_n\}$. (This involves a choice of base edge e_{01} and direction around C.) Choose a set Q and bijections $\beta_i : Q \to p^{-1}(e_{i-1,i})$ and $\beta_0 : Q \to p^{-1}(e_{0n})$. For $x_1, \ldots, x_n \in Q$ define

$$f(x_1, \ldots, x_n) = \beta_0^{-1}(\tilde{e}_{0n})$$

where $\tilde{e}_{0n}$ is the unique edge in $p^{-1}(e_{0n})$ that makes the circuit $\beta_1(x_1)\beta_2(x_2)\cdots\beta_n(x_n)\tilde{e}_{0n}$ balanced in Ω. It is easy to verify that f defines an n-ary quasigroup. The quasigroup is well defined only up to isotopy, because of the arbitrariness of the bijections β_i, and circular conjugacy, because of the arbitrariness of the base edge and direction.

Conversely, given an n-ary quasigroup (Q, f) it is easy to construct $\Omega \downarrow C_{n+1}$ that corresponds to (Q, f) in the previous manner. Let $Q_i = Q \times \{i\}$ for $i = 0, 1, \ldots, n$. Label the edges of C_{n+1} as before. Define Ω to have vertex set $V(C_{n+1})$ and edge set $Q_0 \cup Q_1 \cup \cdots \cup Q_n$, and let the endpoints of $(x, i) \in Q_i$ be v_{i-1} and v_i with subscripts modulo $n+1$. The projection of an edge is $p(x, i) = e_{i-1,i}$. Define a circle $(x_0, 0)(x_1, 1)\cdots(x_n, n)$ to be balanced if $x_0 = f(x_1, \ldots, x_n)$.

The connection between factorization and expansions is through extensions. Formally, an *extension* of $\Omega \downarrow \Delta$ (assuming Δ is simple, which is the only case of importance) is a biased expansion $\Omega' \downarrow \Delta'$ such that $\Delta \subseteq \Delta'$, $V(\Delta') = V(\Delta)$, Δ' is simple, $\Omega \subseteq \Omega'$, $p = p'|_\Omega$, and $(p')^{-1}(\Delta) = p^{-1}(\Delta) = \Omega$. Informally, an extension of $\Omega \downarrow \Delta$ is a biased expansion $\Omega' \downarrow \Delta'$ such that Ω' contains Ω and covers Δ in the same way as does Ω (this is what the conditions on p and p' mean), but Ω' may also cover additional edges not in Δ, namely, those in Δ' that are not in Δ. A biased expansion is *maximal* if it has no proper extensions. For instance, it is maximal if Δ is a complete graph. A central point of this work is that there are maximal extensions where Δ is incomplete.

The usefulness of extensions depends on the next result.

Theorem 6. *Every biased expansion of a 2-connected graph has a unique maximal extension (up to isomorphism).*

The connection between maximal extensions and factorizations of a multary quasigroup operation is this:

Theorem 7. *If $\Omega \downarrow C_{n+1}$ is the biased expansion corresponding to an n-ary quasigroup (Q, f), then the maximal extension of $\Omega \downarrow C_{n+1}$ has for base graph the factorization graph $\Delta(f)$.*

In order to prove Theorems 1 and 3 we need to know what a maximal extension of a biased expansion looks like. Call a graph *theta-complete* if any two vertices that are joined by three internally disjoint paths are adjacent. (The graph formed by the three paths is called a *theta graph.*) Our most difficult result is

Theorem 8. *If $\Omega \downarrow \Delta$ is maximal, then Δ is theta-complete. Equivalently (if Δ is 2-connected), Δ is obtained by edge amalgamation from complete graphs and circuits.*

To suggest the course of the proof we state the three principal lemmas. By saying that $\Omega \downarrow \Delta$ extends to e, we mean that e is an edge on the vertex set of Δ and there is an extension $\Omega' \downarrow \Delta'$ of $\Omega \downarrow \Delta$ for which $\Delta' = \Delta \cup \{e\}$. It is convenient to allow the trivial case in which e is in Δ.

Lemma 9 (Common Extension)**.** *If $\Omega \downarrow \Delta$ extends to e_1 and to e_2, then it extends to $\Omega' \downarrow (\Delta \cup \{e_1, e_2\})$.*

Lemma 10 (Theta Extension)**.** *Any biased expansion of a theta graph with trivalent vertices v and w extends to the edge e_{vw}.*

Lemma 11 (Chordal Extension)**.** *Suppose Ω is a biased expansion of a 2-connected graph Δ and $e \notin E(\Delta)$. For any circuit $C \subseteq \Delta$ of which e is a chord, Ω extends to e if and only if the restricted expansion $p^{-1}(C) \downarrow C$ extends to e.*

A *chord* of C is an edge whose endpoints are vertices of C that are not adjacent in C. Lemma 9 is the core of the proof of Theorem 6. The other two lemmas are the key to Theorem 8. Lemma 10 shows that a theta subgraph $\Theta \subseteq \Delta$ admits an extension of the restricted biased expansion $p^{-1}(\Theta) \downarrow \Theta$ to an edge e_{vw} that joins the trivalent vertices of Θ. By Lemma 11, this extendibility applies to the whole biased expansion $\Omega \downarrow \Delta$.

Corollary 12. *If $\Omega \downarrow \Delta$ is maximal and Δ is 3-connected on 4 or more vertices, then Δ is a complete graph and Ω is a group expansion.*

The last part, that $\Omega \downarrow K_n$ implies Ω is a group expansion when $n \geq 4$, is essentially due to [9, pp. 490–492] and is also implicit in the theorem of generalized associativity. We have a new proof that is particularly suited to biased expansion graphs.

As for graphs with 2-separations, we have a construction.

Theorem 13. *Let Δ_1 and Δ_2 be finite base graphs of maximal biased expansions, assumed disjoint; let $e_i \in E(\Delta_i)$; and form Δ by amalgamating Δ_1 and Δ_2 along e_1 and e_2. There exist finite expansions $\Omega_1 \downarrow \Delta_1$ and $\Omega_1 \downarrow \Delta_2$, with $|p_1^{-1}(e_1)| = |p_2^{-1}(e_2)|$, which can be amalgamated along $p_1^{-1}(e_1)$ and $p_2^{-1}(e_2)$ so as to form a maximal biased expansion of Δ.*

Belousov's conjecture (Theorem 1) is a special case of Corollary 12, by Theorem 7. Theorem 3 is a consequence of Theorems 7 and 13, Corollary 12, and Tutte's 3-decomposition of graphs (see [10, Chapter IV]). Theorem 4 follows from a general property of biased and group expansions. A *minor* is a contraction of a subgraph.

Theorem 14. *A 2-connected biased expansion graph of order at least four, such that each minor with four vertices is a group expansion, is itself a group expansion.*

Full proofs are in [12].

4. Transversal designs and generalized Dowling geometries

We mention two other ways of interpreting our results.

An n-ary quasigroup is equivalent to a transversal t-design with $n+1$ point classes of size $|Q|$, strength $t = n$, and index $\lambda = 1$. Our results can be interpreted as indicating how such a design can be decomposed into smaller ones and, in some cases, into designs constructed from groups. Details are in [12].

To each biased graph is associated a matroid called the *bias matroid* [11, Section II.2]. The well-known Dowling geometries of a group [5] are bias matroids associated with group expansions of complete graphs. The matroids of biased expansions, and especially of those that are maximal, are therefore a generalization of Dowling geometries. In that connection, in [11, Example III.3.8] I stated that it was not known which (simple) graphs have a biased expansion that is not a group expansion. This question can now be answered: the graphs are those that have a block of order three, or a 2-separable block of order at least four, or more than one block of order at least four. The proof is the same as that of Theorem 3.

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