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The Signed Chromatic Number of the Projective Plane and Klein Bottle and Antipodal Graph Coloring

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A graph with signed edges (a *signed graph*) is *k-colorable* if its vertices can be colored using only the colors 0, $\pm 1, ..., \pm k$ so that the colors of the endpoints of a positive edge are unequal while those of a negative edge are not negatives of each other. Consider the signed graphs without positive loops that embed in the Klein bottle so that a closed walk preserves orientation iff its sign product is positive. All of them are 2-colorable but not all are 1-colorable, not even if one restricts to the signed graphs that embed in the projective plane. If the color 0 is excluded, all are 3-colorable but, even restricting to the projective plane, not necessarily 2-colorable. © 1995 Academic Press, Inc.

A signed graph Σ (a graph with signed edges) is said to be (*properly*) *colored in k colors* if its vertices are labelled by the "colors" 0, ± 1, ± 2, ..., $\pm k$ so that the endpoints of a positive edge have different colors, while those of a negative edge have colors which are not negatives of each other. The *chromatic number* $\chi(\Sigma)$ is the smallest value of k for which there exists a proper k-coloring; the *zero-free chromatic number* $\chi^*(\Sigma)$ is defined similarly for colorings that do not use the color 0. For example, let $\pm K_n^\circ$ denote the *complete signed graph* of order n, which consists of n vertices with all possible positive and negative links (non-loop edges) and negative loops, and let $\pm K_n$ denote the *complete signed link graph*, that is, $\pm K_n^\circ$ without the loops. Then $\chi(\pm K_n^\circ) = n$, since any two vertices must be colored with different nonzero absolute values, and $\chi(\pm K_n) = n - 1$, since one vertex can be colored 0.

Suppose we embed signed graphs in a surface S according to the rule of *orientation embedding*: a closed walk reverses orientation if and only if its sign product is negative. (This is the only embedding rule we shall use here. The surfaces we are mainly interested in are the unorientable surfaces U_h for $h \geqslant 1$, consisting of the sphere with h crosscaps. The *demigenus* $d(\Sigma)$ is

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the smallest demigenus ($=2-$ Euler characteristic) of any surface, oriented or not, in which Σ embeds.) The largest chromatic number of any signed graph without positive loops that orientation embeds in S is called its *signed chromatic number*, written $\chi_{\pm}(S)$; the largest zero-free chromatic number is called the *zero-free signed chromatic number* of S and is written $\chi^*_{\pm}(S)$. I propose that, with the exception of $\chi_{\pm}(U_1)$, $\chi_{\pm}(U_h)$ equals the largest order of a complete signed graph that embeds in U_h and $\chi^*_{\pm}(U_h)$ equals the largest order of a complete signed link graph that so embeds. Here we take a first step towards a proof by evaluating the numbers for the projective plane U_1 and the Klein bottle U_2.

THEOREM 1. $\chi_{\pm}(U_1)=2$ *and* $\chi^*_{\pm}(U_1)=3$.

THEOREM 2. $\chi_{\pm}(U_2)=2$ *and* $\chi^*_{\pm}(U_2)=3$.

These results agree with the conjecture above. Their demonstration is based on the method of double contraction employed to prove the planar five-color theorem in [3, p. 32, Second Proof] and earlier, in a less simple dual form, in [1, Appendix 1 to Chapter V]. (I do not know the origin of this ingenious technique, which was also used in [2, p. 72] to evaluate the unsigned chromatic number of the Klein bottle.)

Our results determine chromatic numbers for antipodal coloring of antipodally embedded planar and toroidal graphs. In the g-fold torus T_g there is an involutory self-homeomorphism α (not unique, but we arbitrarily choose one such mapping) whose quotient space is U_{g+1}. A graph Γ is *antipodally embedded* in T_g if it is embedded so that α restricted to the embedded Γ is an automorphism of Γ. An *antipodal coloring* of Γ is an ordinary coloring using integer colors in which the colors of antipodal vertices (that is, a vertex v and its image $\alpha(v)$) are negative to each other. The *antipodal chromatic number* of T_g is the smallest number of colors chosen from the integers that suffice to color antipodally all antipodally embedded loopless graphs in T_g. The antipodal interpretation of the signed chromatic numbers of U_h is the following statement.

PROPOSITION 1. *The antipodal chromatic number of* T_g *equals the lesser of* $2\chi_{\pm}(U_{g+1})+1$ *and* $2\chi^*_{\pm}(U_{g+1})$.

Thus by Theorems 1 and 2 the antipodal chromatic number of the sphere is 5, greater than the ordinary chromatic number, while the antipodal chromatic number of the torus, also 5, is less than the ordinary chromatic number of 7. (The easy proof of Proposition 1 will appear elsewhere.)

Now, some technicalities. Since, in orientation embedding a signed graph, it clearly suffices to constrain only the signs of *polygons* (graphs of simple closed walks), what really matters is the polygon signs. *Switching* a signed graph Σ means reversing the signs of all edges having one endpoint in a certain subset X of the vertex set and the other endpoint in the complementary subset. Two signings of the same graph are related by switching if and only if they have the same polygon signs. (This is well known in signed graph theory. For an easy proof see, e.g., [6, Proposition 3.2].) When switching a colored signed graph we also switch the colors by negating those of vertices in the switched set X; thus switching preserves propriety of a coloring (that is, the property of being proper). Since by switching we can always reduce the color set to $\{0, +1, +2, ..., +k\}$, it makes sense to regard $+i$ and $-i$ as essentially one color. So, we shall call each of 0, $+1$, -1, $+2$, ... a *semicolor*.

To *contract* edge e in a signed graph Σ, if it is a link, we switch so that it is positive, then identify its endpoints and delete the edge. The resulting signed graph Σ/e is determined up to switching. Topologically, if Σ is embedded in a surface, contraction means shrinking the edge to a point, so if Σ embeds in S, Σ/e does as well. Note that we do not contract loops.

Given a signed graph Σ, we write $V = V(\Sigma)$ and $E = E(\Sigma)$. Σ_+ is the (unsigned) subgraph on vertex set V whose edges are the positive edges of Σ. The *neighborhood* in Σ of a vertex v is $N(v) = \{w \in V$: w is adjacent but not equal to $v\}$; the *complete neighborhood* is $N'(v) = N(v) \cup \{v\}$. Each of these sets induces a subgraph of Σ, denoted respectively by $\Sigma(v)$ and $\Sigma'(v)$. The graph of *double adjacency* of Σ is $\Sigma_{(2)} = (V, E_{(2)})$, where $E_{(2)}$ is the set of vertex pairs vw such that v and w are both positively and negatively adjacent in Σ.

A basic fact about U_h is that it contains no $h+1$ pairwise disjoint, orientation-reversing, simple closed curves. (This is well known and easy to prove. Suppose the contrary. Cut along the $h+1$ curves; that leaves a closed surface with boundary equal to $h+1$ disjoint circles, whose Euler characteristic is still $2-h$ because, the curves being one-sided, the cut surface remains connected. Attach $h+1$ disks to the circles to form a closed surface. Its characteristic is $(2-h)+(h+1)=3$, which is impossible.)

To begin the proof of Theorems 1 and 2, note that if Σ has a k-coloring, it has a zero-free $(k+1)$-coloring. Thus $\chi^*(\Sigma)$ equals $\chi(\Sigma)$ or $\chi(\Sigma)+1$. An instance of the latter case is $\pm K_3$. Clearly, $\chi(\pm K_3)=2$ while $\chi^*(\pm K_3)=3$. Since $\pm K_3$ embeds in U_1, $\chi^*_\pm(U_h) \geqslant 3$ and $\chi_\pm(U_h) \geqslant 2$ for $h \geqslant 1$. Furthermore, $\chi_\pm(U_h)=2$ implies $\chi^*_\pm(U_h)=3$.

For the proof that $\chi_\pm(U_1)=\chi_\pm(U_2)=2$, let Σ be embedded in U_2 and *simply signed* (that is, without positive loops or parallel edges of the same sign; this in particular rules out multiple negative loops at a vertex), and suppose that Σ is not 2-colorable but every simply signed, U_2-embeddable graph having fewer vertices is 2-colorable. (Hence Σ is connected.) Our task is to prove that Σ is 2-colorable.

Standard reasoning, as for instance in the proof of [4, Lemma 4.1], enables us to bound the average degree. (The degree is the number of edge ends at the vertex. A loop counts 2.) We briefly sketch the argument. A graph embedding is *cellular* if every face is an open 2-cell. Suppose Σ is embedded in S, whose Euler characteristic is ε. We can cut out any non-cellular face and replace it by one or more cells to make the embedding cellular. (Youngs calls this operation *capping* [5]. Abstractly, it can be done with rotation systems; see [7, Section 6] for definition and references.) Call the resulting surface S_0; it has Euler characteristic $\varepsilon_0 \geqslant \varepsilon$. Σ is orientation embedded in S_0 because a neighborhood of Σ is preserved during the capping. Now let f be the number of faces of the capped embedding in S_0. Because Σ is simply signed, all face boundaries have length at least 3; therefore $3f \leqslant$ sum of face boundary lengths $= 2|E|$. Cellularity of the capped embedding implies Euler's formula, $|V| - |E| + f = \varepsilon_0$. Thus the average degree is at most $6(1 - \varepsilon_0/n)$. In our situation $S = U_1$ or U_2, so $\varepsilon_0 \geqslant \varepsilon \geqslant 0$, whence the average degree is at most 6. Moreover, if it equals 6 then $\varepsilon_0 = \varepsilon = 0$ so $S_0 = S$ (see [5, p. 309, (3) and (4)]) and the original embedding is a triangulation of U_2. Consequently, Σ has a vertex of degree at most 6.

Case 1. If Σ had a vertex of degree four or less, or one of degree five supporting a negative loop, its 2-colorability would be obvious by induction. So Σ has no such vertex.

Case 2. Suppose there is a vertex v of degree five, not supporting a negative loop. If v has distinct neighbors v_1 and v_2 joined to v by positive edges e_1 and e_2 but not positively adjacent to each other, then we contract e_1 and e_2. Thus v_1 and v_2 are merged into v. No positive loop is created; hence by induction the contracted graph Σ' has a 2-coloring. We apply this 2-coloring to $\Sigma \backslash v$, giving v_1 and v_2 the color of v in Σ'. That leaves v with at least one of the five semicolors available, so Σ is 2-colorable.

If v is linked by edges e_1 and e_2 to distinct vertices v_1 and v_2 and there is no edge which forms with e_1 and e_2 a positive triangle, then by switching we may make e_1 and e_2 positive and thence conclude that Σ is 2-colorable.

Now let us switch so that each neighbor of v is positively adjacent to it. We see that Σ is 2-colorable unless every neighbor is positively adjacent to every other. If Σ has four or more neighbors, we therefore have $+K_5$ (an all-positive K_5) embedded in U_2. But an all-positive graph embedded in U_1 or U_2 is planar (that is, can be embedded in the plane, although the actual embedding in U_2 need not contractible. See, e.g., [7, Lemma 3.3] and, for proofs, the references cited therein). Since K_5 is nonplanar, we have a contradiction. It follows that (because Σ is not 2-colorable) v can have no more than three neighbors and that, for any edges e_1 and e_2 from v to distinct neighbors v_1 and v_2, there is an edge e_{12} such that $e_1 e_{12} e_2$ is a positive triangle. We conclude that $\Sigma'(v)$ is $\pm K_4$ minus an edge at v, possibly with negative loops added at some of the three neighbors of v.

If Σ happens to be projective planar our proof is done. For Σ necessarily has a vertex of degree less than 6, but, because U_1 has no two disjoint essential curves, Σ can have no two vertex-disjoint negative polygons. Hence by contradiction we have Theorem 1.

In general, however, Case 2 requires more work and there is a third case.

Case 2, *resumed.* Let $N(v) = \{v_1, v_2, v'\}$, the missing edge of the $\pm K_4$ being $-vv'$. (Note that the sign on the missing edge is arbitrary; it can be reversed by switching v or v'.)

LEMMA 1. *If Σ has a second vertex of degree* 5, *it is v'.*

Proof. If there is a second vertex w of degree 5, let $N(w) = \{w_1, w_2, w'\}$ and (without loss of generality) $\Sigma'(w)$ be $\pm K_4 \setminus (-ww')$, perhaps with negative loops at some of the vertices in $N(w)$. Let us assume that w is non-adjacent to v, whence v, v', w are distinct.

Were w_1 not adjacent to v, then Σ would contain three vertex-disjoint negative polygons: two digons in $\Sigma'(v)$ and one digon on ww_1. But U_2 cannot contain such a configuration. So w_1, $w_2 \in N(v)$. But now we have a different problem: the subgraph induced on $\{v, v', w, w', v_1, v_2\}$ has average degree greater than 6 (whether or not w' is adjacent to v), which is too high for a simply signed graph embedded in the Klein bottle. The only possible conclusion is that w is a neighbor of v, whence $w = v'$. ∎

LEMMA 2. *Σ has no separating vertex.*

Proof. Suppose to the contrary that there is a separating vertex p; that is, $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 \cap \Sigma_2 = \{p\}$, Σ_1 and Σ_2 have order smaller than Σ, and $\Sigma_1 \supseteq \Sigma'(v)$. The formula of [7, Theorem 8.1] states that $d(\Sigma) = d(\Sigma_1) + d(\Sigma_2) - \delta$, where $\delta = 0$ usually, but $\delta = 1$ if, for both $i = 1$ and $i = 2$, adding a negative loop to p in Σ_i does not raise the demigenus. Since $2 \geqslant d(\Sigma) \geqslant d(\Sigma_1) \geqslant d(\Sigma'(v)) \geqslant 2$, we deduce that $d(\Sigma_2) = \delta \leqslant 1$.

In case $\delta = 0$, Σ_2 is planar. Make it all positive by switching. We can 2-color Σ by 2-coloring Σ_1 (in the signed sense) and then 5-coloring the unsigned graph $|\Sigma_2|$ (in the ordinary sense) with the colors $\{0, \pm 1, \pm 2\}$ so that p has the same color as in Σ_1. In case $\delta = 1$, let l_p be a loop at p; then both graphs $\Sigma_i \cup \{-l_p\}$ embed in U_2 and have smaller order than Σ, so they are 2-colorable. The loop prevents p from being colored 0, so we can arrange the colorings of Σ_1 and Σ_2 to agree on p, thus 2-coloring Σ. ∎

Let m_1, m_2, and m' be the amounts by which the degrees of v_1, v_2, and v' exceed 6, 6, and 5, respectively. Lemma 1 implies that the sum of all degrees is at least $6|V| + m_1 + m_2 + (m' - 1) - 1$. Since the average degree is at most 6 we can conclude that $m_1+m_2+m' \leqslant 2$. Thus at least one of v_1 and v_2 has no loop; therefore $\Sigma'(v)$ is 2-colorable. It follows that Σ has more than the four vertices of $\Sigma'(v)$, so $\Sigma'(v)$ is attached by one or two edges to a graph Σ_2; this entails that $\Sigma'(v)$ has no loops. By Lemma 2 the number of attaching edges is two, they are vertex disjoint, and Σ_2 is connected.

Say $p, p' \in N'(v)$ are adjacent respectively to $q, q' \in V(\Sigma_2)$. By switching we ensure that the edges pq and $p'q'$ are positive. Let Σ_2' be Σ_2 with an edge $+qq'$. (If Σ_2 contains that edge, Σ_2' is Σ_2.) Since there is a positive path from q to q' through $N'(v)$, Σ_2' embeds in U_2, hence is 2-colorable. Choose a 2-coloring. One at least of q and q' is not colored 0; say the color $c(q)$ is not 0. We now 2-color $\Sigma'(v)$ so that $c(p) \neq c(q)$ and $c(p') \neq c(q')$, which is possible because Σ_2 can only prevent our using zero at the one vertex p' of $\Sigma'(v)$, while $\Sigma'(v)$ lets us use zero at either v_1 or v_2. Therefore Σ is 2-colorable, contrary to hypothesis. We conclude that Case 2 cannot occur.

Case 3. Suppose Σ is 6-regular. Here we need not only double but triple contraction and more.

We focus on a vertex v. Bear in mind that, under our general hypotheses on Σ, a loop is always negative and a double adjacency, or digon, consists of a positive and a negative link.

LEMMA 3. *If v is doubly adjacent to x_1 and x_2, then x_1 and x_2 are doubly adjacent to each other.*

Proof. If they are not, we may suppose after adequate switching that $+x_1x_2$ is absent from Σ. Contracting $+vx_1$ and $+vx_2$ gives a graph Σ_1 that by induction is 2-colorable. Coloring all vertices except v as in Σ_1, we see that $c(x_1)=c(x_2)$ so there is a semicolor available for v. But this contradicts the hypotheses on Σ. ∎

LEMMA 4. *At most two vertices are doubly adjacent to v.*

Proof. If v had three double neighbors, by Lemma 3 $\Sigma'(v)$ would be $\pm K_4$. But $d(\pm K_4) > 2$ by Proposition 2 in the Appendix. ∎

LEMMA 5. $|N(v)| \neq 6$.

Proof. Suppose to the contrary that v had six distinct neighbors, say $x_1, \ldots, x_6$ in cyclic order around v as embedded. (We take subscripts on the x_i modulo 6.) We prove that Σ is 2-colorable.

First we switch so all edges vx_i are positive. Recall that Σ triangulates U_2. Since each vx_{i-1} and vx_i are boundary edges of a face F_i, there is an edge $+x_{i-1}x_i$ completing the boundary. (It is positive because a face boundary is positive.) The union of the closed faces $\bar{F}_i$ is a disk D with all-positive boundary polygon $x_0x_1 \cdots x_6$.

Suppose x_1, x_3, x_5 mutually nonadjacent in Σ_+. Contract them all into v, 2-color the resulting graph, and carry the coloring back to $\Sigma \backslash v$. Since $N(v)$ rules out at most four semicolors, Σ is 2-colorable.

We may therefore suppose that in the set $\{x_1, x_3, x_5\}$ there is a positive edge e, and similarly that in $\{x_2, x_4, x_6\}$ there is a positive edge f. If e and f crossed with respect to the hexagon $x_0x_1 \cdots x_6$, $\Sigma(v)_+$ would contain a subdivided K_4, so in $\Sigma'(v)_+$ there would be a subdivided K_5. But as we noted earlier, an all-positive subdivided K_5 cannot embed in U_2; thus e and f do not cross. For the same reason there is at most one more positive edge in $\Sigma(v)$ and, without loss of generality, we may assume that $\Sigma(v)_+$ consists of the hexagon $x_0x_1 \cdots x_6$ with diagonals $+x_1x_3$ and $+x_4x_6$ and possibly $+x_1x_4$.

This information allows us to 2-color Σ. In the embedded Σ, D contains v and six radial edges to the x_i. Replace this by edges $+x_1x_5$ and $+x_2x_4$ in D, contract them, 2-color the result, and pull back to a 2-coloring of $\Sigma \backslash v$ in which $c(x_1) = c(x_5)$ and $c(x_2) = c(x_4)$. Now $N(v)$ has at most four semicolors, so v can be colored, as claimed. ∎

The preceding three lemmas imply that the double-adjacency graph $\Sigma_{(2)}$ is the disjoint union of K_2's and K_3's. Furthermore, because U_2 contains no three mutually disjoint orientation-reversing curves, $\Sigma_{(2)}$ has at most two components and, if it has two components, at most one vertex in each edge component and none in each triangle component can support a negative loop of Σ.

Now we show that in every case Σ is 2-colorable. The case $|V| \leqslant 2$ is trivial. When $|V| = 3$, Σ has a loopless vertex which can be colored 0 while the other vertices are colored $+1$ and $+2$. Thus if $\Sigma_{(2)}$ is connected we are done. Otherwise, if one component is an edge with loopless vertex x, then x has five neighbors in Σ; yet $|V| \leqslant 5$, a contradiction. If $\Sigma_{(2)}$ has two triangle components, say with vertex sets $X = \{x_0, x_1, x_2\}$ and $Y = \{y_0, y_1, y_2\}$, the simple edges of Σ form a 2-regular bipartite graph between X and Y, hence a hexagon. Supposing X and Y labelled so that $x_i \in X$ and $y_i \in Y$ are nonadjacent in Σ for $i = 0, 1, 2$, we color $c(x_i) = c(y_i) = +i$. That concludes the proof of Theorem 2. ∎

Appendix: The Demigenus of $\pm K_4$

Proposition 2. $d(\pm K_4) = 3$.

Proof. Figure 1 shows $\pm K_4 \backslash (-v_2v_3)$ embedded in U_2. To embed $\pm K_4$ in U_3, place a crosscap next to edge $+v_2v_3$ and run $-v_2v_3$ through it.

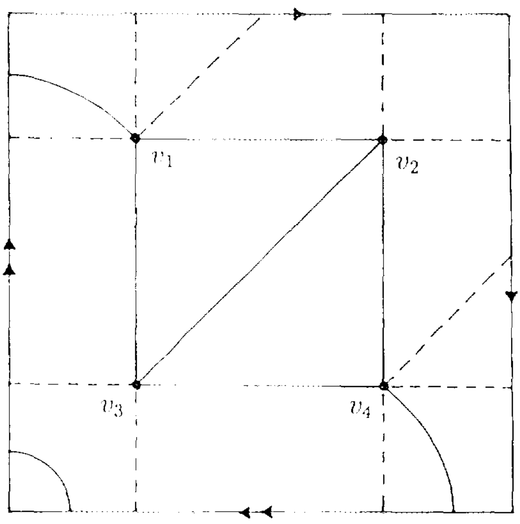

FIG. 1. $\pm K_4 \setminus (-v_2v_3)$ embedded in the Klein bottle, represented as a square with sides suitably identified. Positive edges are solid; negative ones are dashed.

The proof that $\pm K_4$ cannot embed in U_2 is based on three lemmas that apply rather generally. For them we need definitions and terminology.

Let Σ denote any loopless, simply signed graph embedded in a surface S. We think of each vertex as oriented in an arbitrary but fixed way. The *rotation* $\mathbf{R}(v)$ at a vertex v of degree k is the cyclic sequence $\langle e_1, e_2, ..., e_k \rangle$ of signed edges at v, read counterclockwise as viewed from above in the local orientation at v. We write $e_i = \varepsilon_i v v_i$, where ε_i is the sign. Each consecutive pair (e_{i-1}, e_i) (subscripts taken modulo k) is separated by a face F_i. If F_i is a triangle (that is, its boundary has length three; the boundary vertices and edges need not be distinct), then the third bounding edge is $(\varepsilon_{i-1}\varepsilon_i) v_{i-1} v_i$, because a face boundary walk must be positive. Thus one can deduce a portion of $\mathbf{R}(v_i)$ from $\mathbf{R}(v)$ if it is known that F_i is triangular.

LEMMA 6. *If Σ is a loopless signed graph, then each triangular face has distinct vertices.*

Proof. Suppose a triangular face F had two coincident vertices. One would need a loop at that vertex in the boundary of F. But we assumed Σ is loop free. ▮

It follows that a triangular face has distinct edges; thus its closure is a closed 2-cell.

LEMMA 7. *If Σ is a loopless, simply signed graph,*

$$\mathbf{R}(v) = \langle \varepsilon_1 vv_1, \varepsilon_2 vv_2, ..., \varepsilon_{i-1} vv_{i-1}, \varepsilon_i vv_i, ... \rangle,$$

where $4 \leqslant i \leqslant k$, and F_2 and F_i are triangular faces, then $(v_1, v_2) \neq (v_i, v_{i-1})$.

Proof. Suppose to the contrary that equality held. Since Σ is simply signed, $\varepsilon_i = -\varepsilon_1$ and $\varepsilon_{i-1} = -\varepsilon_2$. By switching we may assume that $\varepsilon_1 = \varepsilon_2 = +$. The boundary of F_2 in the clockwise direction (in the local orientation at v) is $\langle +vv_2, +v_2v_1, +v_1v \rangle$ while that of F_i is $\langle -vv_1, +v_1v_2, -v_2v \rangle$. The closed walk $(+vv_2, -v_2v)$ is negative, therefore orientation reversing. It is also homotopic to a path from v to v in S, to wit, through F_2 to the center of $+v_1v_2$ and on through F_i to v. This latter path is orientation preserving because it lies in an annulus formed by the union of the closures of F_1, F_2, and a small neighborhood of v. Thus we have a homotopy between an orientation-reversing and an orientation-preserving path, which is impossible. ∎

LEMMA 8. *Suppose that every vertex of Σ has degree at least equal to 5. If*

$$\mathbf{R}(v) = \langle \varepsilon_1 vv_1, \varepsilon_2 vv_2, \varepsilon_3 vv_3, ..., \varepsilon_{i-2} vv_{i-2}, \varepsilon_{i-1} vv_{i-1}, \varepsilon_i vv_i, ... \rangle,$$

where $6 \leqslant i \leqslant k$, and F_2, F_3, F_{i-1}, and F_i are triangles, then $(v_1, v_2, v_3) \neq (v_{i-2}, v_{i-1}, v_i)$.

Proof. Suppose to the contrary that there were equality. By suitable switching make $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$ and $\varepsilon_{i-2} = \varepsilon_{i-1} = \varepsilon_i = -$. Then the third edges bounding F_2 and F_3, and also F_{i-1} and F_i, are $+v_1v_2$ and $+v_2v_3$. We see that $\mathbf{R}(v_2)$ can only be $\langle +v_2v_3, +v_2v, +v_2v_1, -v_2v \rangle$ (or the inverse rotation). But then v_2 has degree 4, contrary to hypothesis. ∎

Now we apply the lemmas to $\pm K_4$. If it could embed in U_2, the embedding would be a triangulation. (This follows by standard arguments based on the Euler charcteristic.) Let us abbreviate the rotation at a vertex, say v_1, by writing not the edges $\varepsilon_i v_1 v_i$ but only the subscripts of the second vertices. Then $\mathbf{R}(v_1)$ is a cyclic permutation of the multiset $\{2, 2, 3, 3, 4, 4\}$ in which there is no adjacent pair ii (by Lemma 6), no reflected pair $ij \cdots ji$ (but $iji \cdots$ is permitted), and no repeated triple $ijk \cdots ijk \cdots$. One can easily verify that these conditions cannot be met. Hence, $\pm K_4$ cannot embed in U_2. ∎

A *forbidden link minor* for orientation embedding in a surface S is a signed graph which does not embed in S, but such that every proper subgraph and every contraction by a link do embed.

Corollary 1. *$\pm K_4$ is a forbidden link minor for orientation embedding in the Klein bottle.*

Proof. Proposition 2 and Fig. 1 leave to be proved only that $\pm K_4$/edge embeds in U_2. This contraction consists of $\pm K_3$ with a negative loop at a vertex v and all links at v doubled. The doubling of links can be ignored. Embedding $\pm K_3$ in U_1, one crosscap added in a face abutting v suffices to embed the loop as well. Thus $\pm K_4$/edge embeds in U_2. ∎

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