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SIGNED GRAPHS*

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Dedicated to Professor Fred Supnick of the City College of the City University of New York

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A signed graph is a graph with a sign attached to each arc. This article introduces the matroids of signed graphs, which generalize both the polygon matroids and the even-circle (or unoriented cycle) matroids of ordinary graphs. The concepts of balance, switching, restriction and contraction, double covering graphs, and linear representation of signed graphs are treated in terms of the matroid, and a matrix-tree theorem for signed graphs is proved. The examples treated include the all-positive and all-negative graphs (whose matroids are the polygon and even-circle matroids), sign-symmetric graphs (related to the classical root systems), and signed complete graphs (equivalent to two-graphs).

Replacing the sign group by an arbitrary group leads to voltage graphs. Most of our results on signed graphs extend to all voltage graphs.

0. Introduction

A signed graph is a graph whose arcs are labelled by signs. This article develops the matroid theory of signed graphs and of voltage graphs (where the sign group is replaced by an arbitrary group), generalizing the theory of the ordinary graphic or polygon matroid.

I began this work in an attempt to deduce by purely combinatorial means the number of chambers of a classical root system R in $\mathbb{R}^n$ – that is, to obtain from the characteristic polynomial of the matroid of R the number of regions (n-dimensional cells) of the arrangement of dual hyperplanes R^*. (One evaluates the polynomial at -1 and takes the absolute value. Cf. [20], Theorem A.) The hyperplanes in these arrangements have equations of the forms $x_i = x_j$, $x_i + x_j = 0$ (where $i \neq j$), and $x_i = 0$. Taking all such hyperplanes gives the largest classical root system arrangement in $\mathbb{R}^n$, known as B_n^*; taking only those of the first type gives A_{n-1}^*. (The root systems B_n and A_{n-1} consist of those dual vectors having null and unit entries.)

It is well known in matroid theory that the subsets S of A_{n-1} correspond in essence to the simple graphs Γ on n nodes in such a way that the linear dependence matroid of S equals the graphic matroid $G(\Gamma)$. Hence the number of

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regions of S^* can be calculated from $G(\Gamma)$. The combinatorial problem I faced was to extend this treatment to B_n and its subarrangements. Although one can reduce to ordinary graph theory the problem of counting the regions of "sign-symmetric" subarrangements and thus handle all the irreducible classical root systems (see [22]), it is not possible to so treat all subarrangements. To adequately represent them combinatorially appears to require signed graphs. Thus I was led to define the matroid $G(\Sigma)$ of a signed graph Σ in such a way that the natural correspondence of Σ with a subset of B_n would be a matroid isomorphism.

In this paper I treat the matroid of a signed graph and the fundamental related ideas of balance, switching, restriction and contraction, the double covering graph, and linear representation, including the natural correspondence of graphs to vector sets. The principal results are the existence and description of the matroid $G(\Sigma)$ (Theorem 5.1), its relationship to the matroid of the double covering graph of Σ (Theorem 6.5 and its lemmas), and the representability of $G(\Sigma)$ by vectors over fields of various characteristic (see especially Theorem 8B.1). There is also a matrix-tree theorem for signed graphs (Theorem 8A.4). Most of these ideas and results extend to voltage graphs (Section 9).

To solve the original geometrical problem one needs also a way to compute the characteristic polynomial of $G(\Sigma)$. That is most neatly accomplished by means of a coloring theory of signed (and voltage) graphs, directly generalizing ordinary graph coloring (cf. [25] and [26]). One can also strengthen the correspondence between signed graphs and arrangements of hyperplanes by proving that the regions of the arrangement of Σ are in one-to-one correspondence with the acyclic orientations of Σ, thus generalizing an observation of Curtis Greene's about ordinary graphs (see [24]).

Several previously known matroids are signed graphic. Besides the graphic matroids, corresponding to the all-positive signed graphs, there are the even-circle or unoriented cycle matroids, corresponding to the all-negative graphs, and the Dowling lattices of the sign group, corresponding to the full signed expansions of complete graphs. There are also the previously unrecognized matroids of two-graphs. (These examples are discussed in Section 7.) Generalizing to voltage graphs, one has as examples of voltage-graphic matroids all the Dowling lattices (see Section 9) and the bicircular matroid (described in [27]). The further generalization of signed graphs to biased graphs, in which groups are no longer required, will appear elsewhere (cf. [21]).

1. Review of graphs

Our definition of an unsigned graph, to suit the needs of signed graph theory, has to be more general than the usual one. A *graph* (or *unsigned graph*) Γ consists of a set of nodes, $N(\Gamma)$, and a set of arcs, $E(\Gamma)$. We sometimes write $\Gamma=(N, E)$ to mean that the node set is N and the arc set is E. Loops and multiple arcs are allowed as well as the half arcs and free loops to be defined shortly. We use the term *ordinary graph* to mean there are no half arcs or free loops. There are *no finiteness restrictions* except when explicitly stated.

Arcs are of four types. A *link*, written $e\colon vw$, has two distinct endpoints, v and w. A *loop*, $e\colon vv$, has two coincident endpoints (we say e is incident to v twice). A *half arc*, written $e\colon v$, has one endpoint, v, the other end trailing off into space (it may be thought of as half of a loop—a description which is precise in the vectorial representation). A *free loop*, denoted $e\colon\emptyset$, has no endpoints.

A *circle* is a simple closed path in Γ: a loop, a pair of parallel links, a triangle, etc. (The terms "circuit" and "cycle" we reserve for other meanings.) The class of circles of Γ is denoted $\mathscr{C}(\Gamma)$.

Let $X \subseteq N$. By $\Gamma{:}X$, the subgraph *induced* by X, we mean $(X, E{:}X)$, where $E{:}X$ consists of the arcs whose endpoints are in Γ, but not the free loops. A *node component* of Γ is $\Gamma\colon W$ where W is a maximal node set (not void) connected by the arcs in Γ. A node component is *balanced* if it has no half arcs. We write

$$\pi(\Gamma)=\{W\subseteq N\colon \Gamma\colon W \text{ is a node component}\}$$

(a partition of N) and

$$\pi_b(\Gamma)=\{W\subseteq N\colon \Gamma\colon W \text{ is a balanced node component}\}$$

(a partial partition of N). The number of node components of Γ is

$$c(\Gamma)=\#(\pi(\Gamma)).$$

Let $S\subseteq E$. By $N(S)$ we mean the nodes which belong to arcs in S. The *restriction* of Γ to S is the spanning subgraph (N,S); we usually write it $\Gamma|S$ or simply S.

The *contraction* of Γ by S, written Γ/S, has for its node set $\pi_b(\Gamma|S)$. Its arcs are the members of $E\setminus S$ but with new endpoints: the endpoint v is changed to V such that $v\in V\in\pi_b(S)$ or is deleted if no such V exists. By this definition the contraction of an ordinary graph is that usual in the literature.

The same holds for $G(\Gamma)$, the *matroid* ("graphic geometry", "polygon matroid", "cycle matroid") of Γ. (See Welsh [19], Section 1.10. For matroids in general see Welsh or Crapo-Rota [3].) The matroid of a general unsigned graph Γ can be defined as that of the ordinary graph Γ_0 obtained by connecting every half arc to an extra node v_0. We mention here the notations Lat Γ for the lattice of closed sets (or flats) of $G(\Gamma)$, $\mathrm{Lat}^b\,\Gamma$ for the semilattice of flats which are balanced (free of half arcs), and rk_Γ for the rank function. For an ordinary graph, $\mathrm{Lat}^b\,\Gamma$ coincides with Lat Γ.

2. Signed graphs and balance

A signed graph Σ consists of an unsigned graph, denoted $|\Sigma|$, and a partial mapping $\sigma\colon E(|\Sigma|)\to\{\pm\}$, the *arc labelling*, which is defined on all arcs except half arcs and is required to be positive on free loops. (The constraints on σ have the same technical justifications as do half arcs and free loops: principally to make contraction work properly.) We write $N(\Sigma)$ for the node set, $E(\Sigma)$ for the arcs. We may also write $\Sigma=(\Gamma,\sigma)$ if we want to say that the underlying graph is Γ, or $\Sigma=(N,E,\sigma)$ to show that the node set is N and the arc set is E. Signed graphs were introduced by Harary in [9].

Balance. Any path $P=e_1e_2\cdots e_k$ not containing a half arc has a *value*, obtained by multiplying the signs of its constituent arcs:

$$\sigma(P)=\sigma(e_1)\sigma(e_2)\cdots\sigma(e_k).$$

A circle whose value is + is called *balanced*. An arc set is called balanced when it contains no half arcs and every circle in it is balanced. (A free loop is therefore balanced.) The class of balanced circles of Σ is written $\mathscr{B}(\Sigma)$. The balanced circle classes of signed graphs are characterized in [23].

The notion of balance was introduced by Harary in [9]; it plays a central role in the matroid theory of signed graphs. After setting forth a few notations, we will state some characterizations of balanced graphs.

Notation. Most of the notation employed for ordinary graphs carries over to signed graphs. If for instance $\Sigma=(\Gamma,\sigma)=(N,E,\sigma)$ is a signed graph and $X\subseteq N$, then the subgraph *induced* by X is the signed graph $\Sigma{:}X=(X,E{:}X,\sigma|(E{:}X))$. If $S\subseteq E$, the *restriction* $\Sigma|S$ is the signed graph $(N,S,\sigma|S)$; it may be written simply S when no confusion can arise between the arc set and the signed graph. Contractions will be discussed in Section 4.

The node components of an arc set S determine, besides the partition $\pi(S)$ of the node set N, also a partial partition:

$$\pi_b(S)=\{X\in\pi(S)\colon S{:}X \text{ is balanced}\}.$$

Thus for example if v is an isolated node of $\Sigma|S$ or supports only positive loops, then $\{v\}\in\pi_b(S)$. We write

$$b(S)=b(\Sigma|S)=\#(\pi_b(S)),$$

the number of balanced node components of S in Σ. We also write

$$c(S)=c(\Sigma|S)=\#(\pi(S)),$$

the total number of node components, and

$$N_u(S)=N\setminus[\textstyle\bigcup\pi_b(S)],$$

the set of nodes which lie in unbalanced components of S. In particular Σ has $c(\Sigma)$ node components, $b(\Sigma)$ balanced ones (as a reminder: free loops do not count but isolated nodes do).

If S is an arc set, let

$$S_+=S\cap\sigma^{-1}(+)\quad\text{and}\quad S_-=S\cap\sigma^{-1}(-).$$

For instance $E_+(\Sigma)$ means $\sigma^{-1}(+)$.

Characterization of balanced graphs. Harary found the criteria which we present in (i) and (ii) of the next theorem. (For proofs see [9], or [10], Theorem 13.2, for instance. Most of this can be found in König [13], Theorems I.8 and X.11, stated without signs.) Part (iii) is a useful variant of (ii); note that we regard an isolated node as a degenerate bipartite graph. In (iv) we think of an arc set as a vector in $GF(2)^E$; a set $\mathscr{T}$ of circles *spans* if it generates every circle by summation. Part (iv) follows from [23], Theorem 2; for the spanning set a fundamental system of circles it was proved by König (Theorem X.13).

Proposition 2.1. *Let Σ be a signed graph. An arc set S is balanced if and only if it contains no half arcs and satisfies any of the following conditions.*

(i) *Two paths in S with the same endpoints have the same sign.*

(ii) *$N(\Sigma)$ can be partitioned into two sets, X and Y (possibly void), such that every negative arc in S stretches between X and Y, while every positive arc is within X or Y or is a free loop.*

(iii) *The contraction S/S_+ is bipartite.*

(iv) *There is a spanning set of circles of which all are balanced.* □

For still more characterizations of balance see Corollary 3.3, Corollary 5.5, and Propositions 6.2 and 8A.5.

3. Switching

Suppose $\Sigma=(\Gamma,\sigma)$ is a signed graph and $\nu\colon N(\Sigma)\to\{\pm\}$ is any sign function. *Switching Σ by ν* means forming the switched graph $\Sigma^\nu=(\Gamma,\sigma^\nu)$, whose underlying graph is the same but whose sign function is defined on an arc $e{:}vw$ by

$$\sigma^\nu(e)=\nu(v)\sigma(e)\nu(w).$$

Observe that $\sigma^\nu(e)\neq\sigma(e)$ only if e extends between $X=\nu^{-1}(-)$ and $Y=\nu^{-1}(+)$. Thus we may also speak of *switching Σ by X*. Since $(\sigma^\nu)^\mu=\sigma^{\nu\mu}$, any switching is the product of switchings by single nodes.

Switching is important because it leaves the signed-graphic matroid invariant (Corollary 5.4) and because it is needed to define contractions of signed graphs. It is also useful in proofs, especially in regard to coloring (see [25] and [26]). Switching was first described by Abelson and Rosenberg [1]; we obtained it as the signed-graphic version of the graph switching originated by van Lint and Seidel [14]. Adapting Seidel's terminology (cf. [15]) we call Σ_1 and Σ_2 *switching equivalent*, written $\Sigma_1\sim\Sigma_2$, if there is a switching function ν such that $\Sigma_2=\Sigma_1^\nu$.

Lemma 3.1. *Let Γ be a graph and T a maximal forest. Each switching equivalence class of signed graphs on Γ has a unique representative which is + on T. Indeed given any prescribed sign function $\sigma_T\colon T\to\{\pm\}$, each switching class has a single representative which agrees with σ_T on T.*

Proof. This follows easily once one roots the forest T. □

The lemma tells us that the switching classes on Γ have a canonical form with respect to T by which they are in natural one-to-one correspondence with the signed graphs on $\Gamma \setminus T$. A frequent example in the literature is the case of a simple graph Γ having a node v_0 adjacent to every other; T is taken to be all links at v_0. Then switching classes on Γ correspond to signed graphs on $\Gamma:\{v_0\}^c$.

The next result follows at once from Lemma 1.

Proposition 3.2. *Two signed graphs on the same underlying graph are switching equivalent if and only if they have the same list of balanced circles.* □

As a particular case we have another characterization of balanced graphs to add to Proposition 2.1. The case of finitely many nodes is essentially König [13], Theorem X.10.

Corollary 3.3. *A signed graph is balanced if and only if it is switching equivalent to an all-positive graph without half arcs.*

An arc set $S \subseteq E(\Sigma)$ is balanced if and only if it contains no half arcs and Σ can be switched to make S positive. □

4. Contractions and minors

Let us consider a signed graph $\Sigma = (\Gamma, \sigma) = (N, E, \sigma)$. We have already defined the *restriction* to an arc set S to be $\Sigma | S = (N, S, \sigma | S)$. We now have to define the contractions of Σ. Thus will be defined the *minors* of Σ: the results of any number of contractions and restrictions.

The *contraction* of Σ by S, written Σ/S, has node set $\pi_b(S)$, arcs $E \setminus S$ but with endpoints reconnected, and sign function switched from σ. To construct Σ/S, first Σ must be switched to Σ^v so that all arcs in balanced components of S have positive sign. Then S may be discarded and $E \setminus S$ reconnected by the following rule: an endpoint $v \in N$ is replaced by the node set $V \in \pi_b(S)$ which contains it, but if there is no such V (i.e., if $v \in N_u(S)$) then the arc loses that endpoint. It is thus possible for an arc to lose both endpoints (becoming a free loop) or one (becoming a half arc if it had two distinct endpoints), or none; this is why we require free loops and half arcs. Finally $\sigma_{\Sigma/S}$ is set equal to $\sigma^v | (E \setminus S)$.

The sign function $\sigma_{\Sigma/S}$ is ambiguous, since there are many ways to switch so S becomes positive. So there are several contractions Σ/S. This ambiguity is unavoidable but harmless. All we ever need is that the switching class of Σ/S be well defined, and indeed it is true that *contraction and restriction are well defined operations on switching classes:* for Σ and Σ^v have the same contractions and, if Σ_1 is one contraction Σ/S, then Σ_2 is another if and only if $\Sigma_2 \sim \Sigma_1$. It would be possible to write signed graph theory in terms of switching classes rather than signed graphs; but at the cost of readability. Thus we abjure switching classes and speak of "the contraction Σ/S"; that will mean any arbitrary representative of the contraction switching class.

Some technically important properties of minors are listed in the next two propositions.

Lemma 4.1. *Suppose S is balanced in Σ and $A \supseteq S$. Then A is balanced in $\Sigma \Leftrightarrow A \setminus S$ is balanced in Σ/S.*

Let us assume (by adequate switching) that S is positive in Σ. Then after contraction every arc in S^c retains the same sign. Again by appeal to switching, the property of balance of A and $A \setminus S$ can be replaced by positivity. Then the lemma is obvious. □

Proposition 4.2. *Let Σ be a signed graph, $T \subseteq S \subseteq E(\Sigma)$, $R \subseteq E(\Sigma) \setminus S$. Then*

$$(\Sigma|S)|T = \Sigma|T, \quad (\Sigma|S)/T = (\Sigma/T)|(S/T), \quad \text{and} \quad (\Sigma/S)/R = \Sigma/(S \cup R).$$

The first two equations are obvious. The third depends on Lemma 1. We must formalize the map $v \overset{\bullet}{\to} V$ as a partial function,

$$\varphi_S : N(\Sigma) \to N(\Sigma/S) = \pi_b(S).$$

The maps

$$\varphi_{R \cup S} : N(\Sigma) \to N(\Sigma/(R \cup S)) = \pi_b(R \cup S),$$

$$\Phi_R : N(\Sigma/S) \to N((\Sigma/S)/R) = \pi_b((\Sigma/S)|R)$$

are similar. The codomains of the latter two correspond naturally; if $W \in \pi_b((\Sigma/S)|R)$, then $f(W) = \{V \in \pi_b(S) : V \subseteq W\}$ is a block in $\pi_b(R \cup S)$. This follows from Lemma 1. Now it is clear that $\varphi_{R \cup S} = f \circ \Phi_R \circ \varphi_S$, which is the hard part of proving the desired equality of contractions. □

Corollary 4.3. *Any minor of Σ is the restriction of a contraction and the contraction of a restriction.* □

5. Matroids

The central observation of this paper is the existence of the signed-graphic matroid.

An *unbalanced figure* of Σ is any unbalanced circle or half arc. A *circuit* of Σ is an arc set of any of the following types:

(1) a balanced circle;

(2) the union of two unbalanced figures which meet at a single node;

(3) the union of two node-disjoint unbalanced figures and a simple path which meets one figure at each end and is otherwise disjoint from them.

An *improving set* is a set of arcs whose removal increases the number of balanced node components. It is somewhat analogous to a cut set of an ordinary graph. The *balanced closure* of an arc set S is

$$\mathrm{bcl}(S) = S \cup \{e \notin S\colon e \in C, \text{ a balanced circle in } S \cup e\}.$$

S is *balance-closed* if $S = \mathrm{bcl}(S)$. Thus if S is balance-closed in Σ it contains all the free loops.

Theorem 5.1. *Let Σ be a signed graph. There is a matroid $G(\Sigma)$ whose points are the arcs of Σ, defined by any of the following equivalent statements* (a)–(j). *This matroid is finitary.*

(a) *The closure of an arc set S is*

$$\mathrm{clos}_\Sigma(S) = \mathrm{bcl}(S) \cup E(\Sigma){:}N_u(S).$$

(b) *An arc set A is closed $\Leftrightarrow$ it is the union of $E(\Sigma){:}X$ (where X is some set of nodes, possibly $\emptyset$), the free loops, and a balanced, balance-closed subset of $E(\Sigma){:}X^c$.*

(c) *An arc set is independent $\Leftrightarrow$ each of its components contains at most one circle (and that one unbalanced) or one half arc, but not both.*

(d) *An arc set S is dependent $\Leftrightarrow$ it contains a balanced circle or else two unbalanced figures connected within S.*

(e) *An arc set is a circuit of $G(\Sigma)$ $\Leftrightarrow$ it is a circuit of Σ (as defined above).*

(f) *An arc set spans $\Leftrightarrow$ in each node component T of Σ, it is connected and reaches every node of T; and in each unbalanced component it also contains an unbalanced figure. (It need not contain the free loops.)*

(g) *An arc set S is a basis $\Leftrightarrow$ for each $B \in \pi_b(\Sigma)$, $S{:}B$ is a spanning tree; while for each $B \in \pi(\Sigma) \setminus \pi_b(\Sigma)$, $S{:}B$ is a spanning tree plus either a half arc or an arc forming an unbalanced circle in $S{:}B$.*

(h) *An arc set H is a copoint $\Leftrightarrow$*

$$H = A \cup E(\Sigma){:}N(A)^c \cup \{\text{free loops of } \Sigma\},$$

where $N(A) \neq \emptyset$, A is connected, and (letting $\Sigma{:}Y$ be the component of Σ which contains A) either $\Sigma{:}Y$ is unbalanced and every component of $H{:}Y$ except A is unbalanced while A is a balance-closed subset of $E(\Sigma){:}Y$, or else $\Sigma{:}Y$ is balanced, $A = E(\Sigma){:}N(A)$, and $E(\Sigma){:}[Y \setminus N(A)]$ is connected.

(i) *An arc set is a bond of $G(\Sigma)$ $\Leftrightarrow$ it is a minimal improving set.*

(j) *The rank of an arc set S is*

$$\mathrm{rk}_\Sigma(S) = \# N_u(S) + \sum_{B \in \pi_b(S)} (\# B - 1),$$

which may be infinite; if $n = \# N(\Sigma)$ is finite, then

$$\mathrm{rk}_\Sigma(S) = n - b(S).$$

Proof. We begin by proving that the function $\mathrm{rk} = \mathrm{rk}_\Sigma$ in (j) is a Whitney ran function (cf. [3], Section 5, or [19], Section 1.2). The necessary properties are:

(1) $\mathrm{rk}\,\emptyset = 0$, which is obvious;

(2) $\mathrm{rk}\,S \le \mathrm{rk}\,T$ when $S \subseteq T$;

(3) unit increase: $\mathrm{rk}(S \cup \{e\}) \le \mathrm{rk}\,S + 1$ for an edge $e \in S$; and

(4) submodularity, for which it is sufficient to have "elemental" submodularit

$$\mathrm{rk}(S) + \mathrm{rk}(S \cup \{e, f\}) \le \mathrm{rk}(S \cup \{e\}) + \mathrm{rk}(S \cup \{f\}).$$

Proof of (2). We may assume $\mathrm{rk}\,T < \infty$. Clearly: $N_u(S) \subseteq N_u(T)$; the blocks o $\pi_b(S)$ in a block C of $\pi_b(T)$ form a partition π_C of C; and the blocks of $\pi_b(S)$ i $N_u(T)$ form a partition π_0 of $N_u(T) \setminus N_u(S)$. What must be proved is that

$$\sum_{B \in \pi_0} (\# B - 1) \le \sum_{B \in \pi_0} (\# B) + \sum_{C \in \pi_b(T)} \{(\# C - 1) - \sum_{B \in \pi_C} (\# B - 1)\}.$$

This is obvious.

Proof of (3). We use the notation of (2) with $T = S \cup \{e\}$, now assuming $\mathrm{rk}\,S < \infty$. What we have to prove is that

$$(*) \qquad \sum_{B \in \pi_0} (\# B - 1) + 1 \ge \sum_{B \in \pi_0} (\# B) + \sum_{C \in \pi_b(T)} \{(\# C - 1) - \sum_{B \in \pi_C} (\# B - 1)\}.$$

The three possibilities are:

(α) $\pi_b(S) = \pi_b(T)$. This case arises when e has its endpoints in $N_u(S)$ or when its ends are in a block $B_1 \in \pi_b(S)$ and $(S{:}B_1) \cup \{e\}$ is balanced.

(β) Just one block $B_1 \in \pi_b(S)$ is not in $\pi_b(T)$. This case arises when e joins B_1 to $N_u(S)$ or when e has its endpoints in B_1 but $(S{:}B_1) \cup \{e\}$ is unbalanced.

(γ) Just two blocks B_1, $B_2 \in \pi_b(S)$ are not in $\pi_b(T)$. This case occurs when e has one end in B_1 and the other in B_2. Then $B_1 \cup B_2 \in \pi_b(T)$.

It is easy to see that $(*)$ holds with equality in cases (β) and (γ), with inequality in case (α).

Proof of (4). We may assume $\mathrm{rk}\,S < \infty$. We apply (2) and (3) with $T = S \cup \{e\}$, $S' = S \cup \{f\}$, and $T' = S' \cup \{e\}$. We must prove

$$(\dagger) \qquad \mathrm{rk}\,(T') - \mathrm{rk}\,(S') \le \mathrm{rk}\,(T) - \mathrm{rk}\,(S).$$

This is certainly true (by unit increase) unless $\mathrm{rk}\,T - \mathrm{rk}\,S = 0$. Then case ($\alpha$) holds (in regard to S and T).

If e has both ends in $N_u(S)$, then since $N_u(S) \subseteq N_u(S')$ we have (α') (that is, (α) for S' and T') and hence ($\dagger$).

The other possibility is that e has both ends in $B_1 \in \pi_b(S)$ and $T{:}B_1 = (S{:}B_1) \cup \{e\}$ is balanced. If $B_1 \subseteq N_u(S')$, we have (α') and hence ($\dagger$). If $B_1 \cup B_2 \in \pi_b(S')$, then f joins B_1 and B_2; since any circle of $T'{:}(B_1 \cup B_2)$ is contained in $T{:}B_1$ or $S{:}B_2$ and both these are balanced graphs, then $T'{:}(B_1 \cup B_2)$ is balanced; thus (α') holds true and hence ($\dagger$) is valid. In the remaining case $B_1 \in \pi_b(S')$. If f has its endpoints in B_1, we apply Harary's path value criterion for balance (Proposition 2.1(i)) as follows. Let e have endpoints v and w, and let the value of a path $P: v \to w$ in $S:B_1$ be ε. Since $T:B_1$ is balanced, $\varepsilon = \sigma(e)$. Since $S':B_1$ is balanced, any path $P': v \to w$ in $S':B_1$ has also the same value ε. Therefore any path $Q': v \to w$ in $T':B_1$ has the same value ε. We conclude that $T':B_1$ is balanced, (α') holds, and (†) is valid, provided f has its endpoints in B_1. But if it does not, then $T':B_1 = T:B_1$, which is balanced, and again we have (α'), hence (†). This proves (4).

Thus rk_Σ is a Whitney rank function. Henceforth we denote its matroid by $G(\Sigma)$.

There is no great difficulty in verifying (a), (b), (c), and (d) in that order, nor (f), (g), (h), and (i). To prove (e), first we observe that every set containing a circuit of Σ is among the dependent sets determined by (d). Conversely let D be a minimal dependent set as defined by (d); we must show D is a circuit of Σ. The only case that needs attention is where D is unbalanced and contains no half arc. Then it contains just two independent circles and is either a circuit of Σ or a theta graph. But in a signed graph, every theta graph contains a balanced circle. Thus D must not have been a theta graph. (This step is the only one in the proof that uses the fact that the voltage group has order 2.) □

$G(\Sigma)$ is called the *signed graphic matroid* of Σ. Some more notation: A closed set is also called a *flat*. The *lattice of flats* of Σ, or of $G(\Sigma)$, is

$$\mathrm{Lat}\ \Sigma = \{A : A \text{ is a closed arc set}\};$$

the *semilattice of balanced flats* is

$$\mathrm{Lat}^{\mathrm{b}}\Sigma = \{A : A \text{ is closed and balanced}\}.$$

The lattice of flats is, of course, a geometric lattice.

Theorem 5.2. *Let Σ be a signed graph and $S \subseteq E(\Sigma)$. Then $G(\Sigma | S) = G(\Sigma) | S$ and $G(\Sigma/S) = G(\Sigma)/S$.*

The first equation is obvious. The second, by the definition of matroid contraction in terms of rank and the finitarity of signed graphic matroids, is equivalent to the assertion that $\mathrm{rk}_{\Sigma/S}(R) = \mathrm{rk}_\Sigma(R \cup S) - \mathrm{rk}_\Sigma(S)$. By Theorem 1(j), this is equivalent to $b((\Sigma/S)|(R \cup S))$, a consequence of the bijectivity of the correspondence f in the proof of Theorem 4.2. □

Problem 5.1. *How narrowly does $G(\Sigma)$ determine Σ? Are two signed graphs with the same matroid related in any simple way?*

This problem is certainly more complicated than the corresponding one for graphs. Note for example that $G(\Sigma)$ may be graphic even though Σ is not balanced. For instance all four signed graphs in Fig. 1 have matroids isomorphic to $G(K_4)$. There also exist two quite different signed graphs with matroid $G(K_5)^\perp$ (this will appear elsewhere).

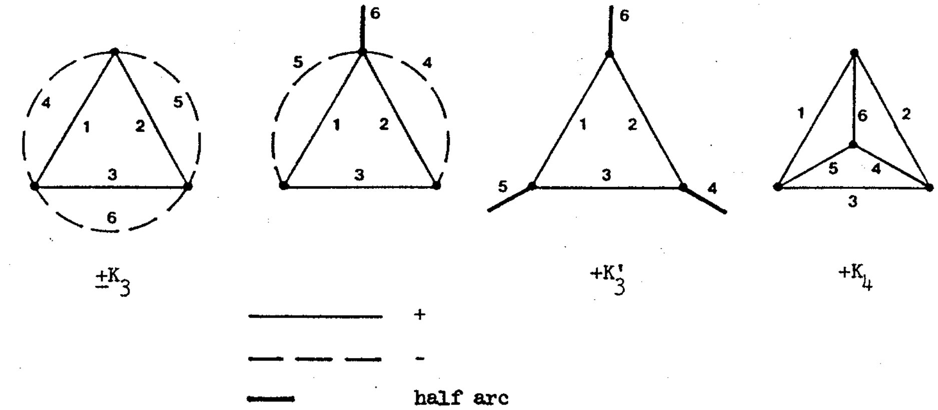

Fig. 1. Four signed graphs with isomorphic matroids. Isomorphisms are indicated by the arc numberings.

Problem 5.2. Theorem 2 implies that signed graphic matroids can be characterized by excluded minors. *What are the minimal exclusions?* Aside from what Corollary 3 and Corollary 8B.4 say, this is an open question.

Corollary 5.3. *No signed-graphic matroid contains as a minor a five-point line.*

Proof. By Theorem 2 it suffices to prove that no $G(\Sigma)$ can be a five-point line. That can be done by inspection of cases. □

It is apparent from Theorem 1(e) that switching, which preserves balance of circles, leaves the signed graphic matroid invariant. In fact since, for a given underlying graph Γ, $\mathscr{B}(\Sigma)$ and $G(\Sigma)$ determine each other, we have matroidal versions of Proposition 3.2 and Corollary 3.3.

Corollary 5.4. *Two signed graphs on the same underlying graph are switching equivalent if and only if they have the same matroid.* □

Corollary 5.5. *A signed graph $\Sigma = (\Gamma, \sigma)$ is balanced if and only if Γ has no half arcs and $G(\Sigma) = G(\Gamma)$. Then also* $\mathrm{Lat}\ \Sigma = \mathrm{Lat}^{\mathrm{b}}\Sigma = \mathrm{Lat}\ \Gamma$.

An arc set S is balanced if and only if it has no half arcs and $G(\Sigma | S) = G(\Gamma | S)$. □

Balanced flats can be characterized in various ways. If X and Y are disjoint node sets, by $E{:}\langle X, Y\rangle$ we mean the set of arcs in E with one end in X and the other in Y (thus half arcs are excluded). If $X_1, Y_1, \ldots, X_k, Y_k$ are disjoint node sets, we call

$$(*) \qquad E{:}\langle X_1, Y_1\rangle \cup \cdots \cup E{:}\langle X_k, Y_k\rangle$$

a *bipartitionally induced arc set*. We do not exclude empty sets from among X_i and Y_i, but of course $E{:}\langle \emptyset, Y\rangle = \emptyset$. The next two propositions are easy consequences of Proposition 2.1(ii), Theorem 1(a), and suitable switching.

Proposition 5.6. *Let Σ be a signed graph. An arc set S is a balanced flat if and only if it satisfies any one of the following equivalent conditions.*

(i) *S is balanced and balance-closed in Σ.*

(ii) *$S=S_+\cup S_-$, S_- is a bipartitionally induced subset of $E_-(\Sigma)$ of the form $(*)$, and $S_+=E_+(\Sigma):\{X_1, Y_1, \ldots, X_k, Y_k\}$.*

(iii) *Σ can be switched to Σ^v so that $S=E_+(\Sigma^v):\pi$ for some partition π of N.*

(iv) *S has no half arcs, S_+ is closed, S_-/S_+ is bipartitionally induced in Σ/S_+, and for every component of S_-/S_+ with left (or right) set X (say), $E_+(\Sigma/S_+):X$ is void.* □

Corollary 5.7. *Let $A=\mathrm{clos}_\Sigma S$. Then A is balanced $\Leftrightarrow$ S is balanced.* □

For the next result we need the notation

$$(\mathrm{Lat}\ \Sigma)/S=\{A\setminus S: A\in \mathrm{Lat}\ \Sigma \text{ and } A\supseteq S\}$$

and $(\mathrm{Lat}^b\Sigma)/S$ defined similarly.

Corollary 5.8. *Let Σ be a signed graph and $S\subseteq E(\Sigma)$. Then* $(\mathrm{Lat}\ \Sigma)/S=\mathrm{Lat}(\Sigma/S)$ *and* $(\mathrm{Lat}^b\Sigma)/S=\mathrm{Lat}^b(\Sigma/S)$.

Proof. The first equation is another way of saying $G(\Sigma)/S=G(\Sigma/S)$. The second follows from the first and Lemma 4.1. □

Finally we describe the separators (or summands) of $G(\Sigma)$. A *block* of a graph is a maximal arc set S which has no cut nodes. (For instance any loop or free loop is a block.) A block which is unbalanced or which contains an arc lying on some simple path between unbalanced blocks is an *inner block;* the other blocks are *outer.* A *circle of balanced blocks* is a union $\Sigma_1\cup\Sigma_2\cup\cdots\cup\Sigma_r$ where each Σ_i is 2-connected (or a link) and balanced and

$$\Sigma_i\cap\Sigma_j=\begin{cases}\emptyset & \text{if } j\neq i, i\pm 1 \pmod r,\\ \text{one node} & \text{if } j=i\pm 1 \text{ and } r\geq 3,\\ \text{two nodes} & \text{if } j\neq i \text{ and } r=2.\end{cases}$$

Theorem 5.9. *The irreducible separators of $G(\Sigma)$ are: each outer block of $|\Sigma|$, and each component of the union of all inner blocks; except that if such a component is an unbalanced circle of balanced blocks $\Sigma_1\cup\Sigma_2\cup\cdots\cup\Sigma_r$, then each Σ_i is an irreducible separator of $G(\Sigma)$.*

Proof. We begin the proof with a few observations.

(1) A connected component of Σ is a separator. Thus we can restrict attention to connected graphs.

(2) Call *equivalence* of arcs the transitive closure of the relation: e and f belong to a common balanced circle. Then equivalent arcs are inseparable from each other. In particular a balanced block of Σ is inseparable.

(3) Since an outer block cannot meet an unbalanced circuit, it must be an irreducible separator. Thus we can assume there are no outer blocks.

Case 1. Σ has more than one block. Then any block that is an end of the block-cut tree of Σ is unbalanced. Consider an arc f in an interior block B. Say B lies between end blocks B_1 and B_2. Then there are unbalanced figures in B_1 and B_2, and a simple path between them and including f, whose union is a circuit of Σ. So f is inseparable from some arc in some end block.

Now consider an end block B and an unbalanced figure C in B. Let e be any arc in another end block B'. If there is an unbalanced figure C' including e, then there is an unbalanced circuit of Σ containing $C\cup C'$, whence e is inseparable from C. On the other hand suppose no unbalanced figure contains e. Let C' be an unbalanced circle in B' and let $e'\in C'$. Then e' is inseparable from C. And since there is a circle containing e and e', which must be balanced, e is inseparable from e'. By the transitivity of inseparability, e is inseparable from C. We have thus shown that any arc in an end block $B'\neq B$ is inseparable from C.

It follows easily that any two arcs in end blocks and consequently any two arcs of Σ are inseparable from each other.

Case 2. Σ consists of one unbalanced block having more than one arc equivalence class. Suppose Σ is an unbalanced circle of balanced blocks. Then the balanced blocks are the arc equivalence classes, there are no unbalanced circuits, and consequently each balanced block is an irreducible separator. Notice that a circle is balanced if and only if it is contained in one of the balanced blocks of the circle.

Conversely suppose Σ has a nontrivial separator S. We can assume without loss of generality that S is connected. Note that it is a union of arc equivalence classes.

First we show S is balanced. If not, it contains an unbalanced circle C. Let T be a component of the complement S^c; it has at least two nodes of attachment v and w. Let Q be a simple path in T with endpoints v and w, and let $P\subseteq S$ be a minimal arc set containing C and connecting it to both v and w. Then $P\cup Q$ is either a circuit of Σ or a theta graph in which no circle is balanced. But the former contradicts the separability of S, while the latter is impossible in a signed graph. So S must have been balanced after all.

Next we show that S^c is connected. If it has components T_1 and T_2, let v_i and w_i be points of attachment of T_i to S and let Q_i be a path between them in T_i. Let P be a tree in S with end nodes v_1, v_2, w_1, w_2. Then $P\cup Q_1\cup Q_2$ is either a circuit of Σ or a completely unbalanced theta graph, which is the same *absurdum* as before.

Third we prove that S^c has only two nodes of attachment to S. If it has more, say u, v, w, then there are subgraphs $P\subseteq S$ and $Q\subseteq S^c$, each of which is either a tree with end nodes u, v, w or a path meeting all three of u, v, w and ending at two of them. Then $P\cup Q$ is another impossible graph of the type encountered before.

Now call v, w the nodes of attachment of S to S^c. Let C be an unbalanced circle in Σ. If C contains no arc of some block B of S, then B is an outer block of Σ; but by assumption there are none. So C meets every block of S, say in the order $B_1, B_2, \ldots, B_s$ going from v to w, and every block of S^c, say $B_{s+1}, \ldots, B_r$ going from w to v. (Clearly C cannot revisit a block of S or S^c.) It is easy to see that Σ consists of the unbalanced circle $B_1 \cup B_2 \cup \cdots \cup B_r$ of the balanced blocks $B_1, B_2, \ldots, B_r$, as we wished to prove.

Cases 1 and 2 cover all possibilities except that in which Σ is a block having one arc equivalence class, but this is easy. □

6. Covering graphs

Signed graphs and switching classes are essentially equivalent to signed coverings and double coverings of unsigned graphs. In fact there is a quite close relationship between the matroid of a signed graph and that of its double covering (see Theorem 6.5 and its lemmas).

Let $\Gamma=(N,E)$ be a graph. If $\tilde{\Gamma}=(\tilde{N},\tilde{E})$ is another and $p:\tilde{\Gamma}\to\Gamma$ is a graph homomorphism (i.e., an incidence-preserving map $\tilde{N}\cup\tilde{E}\to N\cup E$ mapping $\tilde{N}\to N$ and $\tilde{E}\to E$) such that

p is onto,
p is 2-to-1,
p is a local isomorphism (equivalently, the restriction of p to any one node of $\tilde{\Gamma}$ and all its incident arcs is an isomorphism),

then $\tilde{\Gamma}$ is a *double covering graph of* Γ with p as *projection map*. A *fiber* is a set $p^{-1}(x)$ where $x\in N\cup E$. A *fibered isomorphism* of coverings of the same graph, $p_1:\tilde{\Gamma}_1\to\Gamma$ and $p_2:\tilde{\Gamma}_2\to\Gamma$, is a graph isomorphism $\varphi:\tilde{\Gamma}_1\to\tilde{\Gamma}_2$ such that $p_1=p_2\circ\varphi$. (It is sometimes convenient to allow "branched" double coverings, in which a half arc $e:v$ may be covered by one arc $\tilde{e}:+v,-v$ and a free loop also may be simply covered, but as that complicates the theory slightly we do not do so here).

A *signed covering graph of* Γ is a double covering $p:\tilde{\Gamma}\to\Gamma$ together with a sign mapping $\varrho:\tilde{N}\to\{\pm\}$ which is a bijection on $p^{-1}(v)$ for each node $v\in N$. We write it $\tilde{\Sigma}=(\tilde{\Gamma},\varrho)$. A *signed covering graph of* $\Sigma=(\Gamma,\sigma)$ is a signed covering $\tilde{\Sigma}$ of Γ with the property

$$\sigma(p(\tilde{e}))=\varrho(\tilde{v})\varrho(\tilde{w})$$

for every arc $\tilde{e}:\tilde{v}\tilde{w}$ in $\tilde{E}$ which does not project to a half arc. Clearly a $\tilde{\Sigma}$ which covers Γ determines σ. A converse is also true. We say $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are *fibered-isomorphic* when $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are isomorphic coverings by an isomorphism φ which satisfies $\varrho_1=\varrho_2\circ\varphi$. According to the following lemma, one could base the theory of signed graphs on signed coverings.

Lemma 6.1. *All signed coverings of a given signed graph are fibered-isomorphic.*

Proof. In fact one can construct $\tilde{\Sigma}$ from Σ (cf. Biggs [2], Definition 19.1, stated for voltage graphs over any group but not mentioning ϱ or half arcs). The node set $\tilde{N}$ is $\pm N$ (technically, $\{\pm\}\times N$); $\varrho(\tilde{v})$ is the sign part of $\tilde{v}=\varepsilon v$. For completeness we describe $\tilde{E}$ in the following Table 1. □

Table 1

Σ		$\tilde{\Sigma}$
Arc	$e:vw\in\sigma^{-1}(\varepsilon)$	2 arcs $\tilde{e}_1:+v,+\varepsilon w$ and $\tilde{e}_2:-v,-\varepsilon w$
Half arc	$e:v$	2 half arcs $\tilde{e}_1:+v$ and $\tilde{e}_2:-v$
Free loop	$e:\emptyset$	2 free loops

Covering and switching. A signed covering graph $\tilde{\Sigma}\to\Sigma$ can be switched by $v: N\to\{\pm\}$; the switched graph is $\tilde{\Sigma}^v=(\tilde{\Gamma},v\varrho)$. If we think of $\tilde{\Sigma}$ as lying above Σ with its + nodes at one height and its − nodes at another, switching $v\in N$ corresponds to reversing the heights of the nodes $\tilde{v}_1,\tilde{v}_2\in p^{-1}(\tilde{v})$. (See the nice pictures in [15], p. 488.) Clearly $\tilde{\Sigma}^v$ is a signed covering of Σ^v with projection p.

Now we get still another criterion for balance. The key observation is that, if $\tilde{\Sigma}$ covers each component of Γ with two components, then $\tilde{\Sigma}$ can be switched so only nodes with the same sign are connected.

Proposition 6.2. *Let $\tilde{\Sigma}$ be a signed covering graph (without half arcs) of Σ. Then Σ is balanced if and only if each node component of Σ is covered by two of $\tilde{\Sigma}$.* □

Suppose $\tilde{\Sigma}_1\to\Sigma_1$ and $\tilde{\Sigma}_2\to\Sigma_2$ are signed coverings of the same unsigned graph $\Gamma=|\Sigma_1|=|\Sigma_2|$. Our results on switching and uniqueness of signed coverings imply that, if $\Sigma_2\sim\Sigma_1$ (switching equivalence), then the underlying double coverings $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are fibered-isomorphic. On the other hand if $\tilde{\Gamma}_1\cong\tilde{\Gamma}_2$ then $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ can differ only in the choice of + node in each fiber $p^{-1}(v)$, in other words by switching, so $\Sigma_1\sim\Sigma_2$. Thus we have the next theorem (known in essence to Higman; cf. [15], Section 4), which says that the theory of switching classes of signed graphs could be based on double coverings.

Theorem 6.3. *Let Γ be an unsigned graph. There is a one-to-one correspondence between isomorphism types of double coverings of Γ and switching classes of signed graphs on Γ. To construct the class $[\Sigma]$ corresponding to a type $[\tilde{\Gamma}]$, choose any node labelling $\varrho:\tilde{N}\to\{\pm\}$ which makes $(\tilde{\Gamma},\varrho)$ a signed covering and let Σ be the signed graph so determined. To construct the covering type $[\tilde{\Gamma}]$ corresponding to a class $[\Sigma]$, construct a signed covering $\tilde{\Sigma}\to\Sigma$ and let $\tilde{\Gamma}$ be the graph underlying $\tilde{\Sigma}$.* □

It is not hard to see how $\tilde{\Gamma}$ is affected by contraction of Σ. Bear in mind that a node of $\tilde{\Gamma}/T$ is a maximal T-connected set of nodes of $\tilde{\Gamma}$.

Proposition 6.4. *Let Σ be a signed graph, $p:\tilde{\Gamma}\to\Sigma$ be its double covering, and $S\subseteq E(\Sigma)$. Then the double covering of Σ/S is obtained from $\tilde{\Gamma}/p^{-1}(S)$ by removing any node V which contains the full inverse image $p^{-1}(v)$ of a node of Σ. (If S is balanced, no nodes need be removed).* □

The covered matroid. Theorem 3 informs us that a double covering $\tilde{\Gamma}\to\Gamma$ determines the matroid G of the corresponding signed graphs. The manner of determination is remarkably simple and implies that projection is a strong map and G is the weakest (i.e., least dependent) matroid on $E(\Gamma)$ consistent with that fact.

Theorem 6.5. *Let $p:\tilde{\Gamma}\to\Gamma$ be a double covering and let G be the matroid of a signed graph corresponding to $\tilde{\Gamma}$. The matroid G is determined by any of the rules:*
(i) *S is closed in G $\Leftrightarrow$ $p^{-1}(S)$ is closed in $G(\tilde{\Gamma})$.*
(ii) $p^{-1}(\mathrm{clos}_G S)=\mathrm{clos}_{G(\tilde{\Gamma})}p^{-1}(S)$.
(iii) *S spans in G $\Leftrightarrow$ $p^{-1}(S)$ spans in $G(\tilde{\Gamma})$.*
(iv) *S is independent in G $\Leftrightarrow$ $p^{-1}(S)$ is independent or contains just one circle.*

Proof. For the proof of (i) let Σ be a signed graph corresponding to $\tilde{\Gamma}$. It is obvious that, if $S\in\mathrm{Lat}\,\Sigma$, then $p^{-1}(S)\in\mathrm{Lat}\,\tilde{\Gamma}$.

Conversely let S be such that $p^{-1}(S)$ is closed. Assume (without loss of generality) that S is connected and let $X=N(S)$. Then $N(p^{-1}(S))=p^{-1}(X)$. If $p^{-1}(S)$ is connected, it equals $\tilde{E}:p^{-1}(X)$ because it is closed; but then $S=pp^{-1}(S)=E:X$, which is closed in Σ.

But suppose $p^{-1}(S)$ is not connected. By the remark before Proposition 2, there is a node labelling ϱ for $\tilde{\Gamma}$ such that the components of $p^{-1}(S)$ are $p^{-1}(S):\varrho^{-1}(+)$ and $p^{-1}(S):\varrho^{-1}(-)$. Because each of these is closed, S itself is closed in Σ. That proves the converse.

Part (ii) is easy from (i) and (iii) from (ii), while (iv) follows from two lemmas which show how the circuits of Σ can be partly characterized by the double covering. □

Lemma 6.6. *Let Σ be a signed graph and $p:\tilde{\Gamma}\to\Sigma$ its corresponding double covering. Any circuit of Σ is the image of a circuit in $\tilde{\Gamma}$.*

Proof. Suppose C is a balanced circle in Σ. Then $p^{-1}(C)$ consists of two components, each isomorphic to C. So that case is easy.

Suppose C is a circuit consisting of two unbalanced figures, C' and C'', and a simple path P meeting $C^{(i)}$ at $v^{(i)}$ (possibly P has zero length). The set $p^{-1}(P)$ consists of two copies of P, namely P_j between v_j' and v_j'' (for $j=1,2$), which do not touch. The inverse image of C' contains a path C_1' between v_1' and v_2' (in fact also another if C' is a circle). Similarly in $p^{-1}(C'')$ there is a path C_1'' between v_1'' and v_2''. Now $C_1'\cup P_1\cup C_1''\cup P_2$ is a circle. That finishes the proof. □

Lemma 6.7. *The projection of a circle of $\tilde{\Gamma}$ is dependent in Σ.*

Proof. Let $\tilde{C}$ be a circle of $\tilde{\Gamma}$. Each pendant arc of $p(\tilde{C})$ must be a half arc. Hence if $p(\tilde{C})$ contains a balanced circle, or no circle, or no balanced circle and is not equal to a circle, it is dependent by Theorem 5.1(d).

In the remaining case $p(\tilde{C})$ is an unbalanced circle C. Thus $p^{-1}(C)$ is a single circle. But the circle $\tilde{C}\subseteq p^{-1}(C)$, hence equality holds and the lemma is proven. □

7. A sample kit of signed graphs

We examine some special kinds of signed graphs, among them four varieties obtained from ordinary graphs through different modes of construction.

7A. *All-positive graphs*

If we label by + the arcs (except the half arcs) of an unsigned graph Γ, we get the *positive signed graph on* Γ, written $(\Gamma,+)$ or simply $+\Gamma$. This in most respects is the signed-graphic analog of Γ. For instance $G(+\Gamma)=G(\Gamma)$ (Proposition 1); the two graphs have the same adjacency and incidence matrices, and so forth. But we should not regard Γ and $+\Gamma$ as identical, for the signed graph which best mirrors the properties of Γ is not always the positive one (cf. some of our comments on all-negative and signed complete graphs).

In signed graph theory half arcs and free loops are unavoidable—for one reason, to represent root systems one needs to distinguish half arcs from negative loops—but not so for unsigned or positive graphs. To get rid of them in the latter cases we can let v_0 be a new node, not in N, and replace each half arc $e:v$ by a link $e:vv_0$, each free loop $e:\emptyset$ by a loop $e:v_0v_0$. Call the new graph Γ_0 (or $+\Gamma_0$, if signed positive). It is easy to verify by inspection that $G(\Gamma_0)=G(+\Gamma_0)=G(+\Gamma)$. Thus in ordinary graphic matroid theory there is no need for half arcs and free loops. Indeed (in view of our definition $G(\Gamma)=G(\Gamma_0)$ in Section 1) we have the matroid identity:

Proposition 7A.1. *For any unsigned graph Γ, $G(+\Gamma)=G(\Gamma)$.* □

7B. *Fullness*

A graph in which every node carries an unbalanced arc (a half arc or negative loop) is called *full.* An individual node is *full* if it supports an unbalanced arc; otherwise it is *empty.* Any signed graph may be made full by filling (adjoining unbalanced arcs to) its empty nodes. The resultant full graph is denoted Σ° if each empty node gains a loop, Σ' if a half arc, $\Sigma^{\bullet}$ if either one indifferently (each node gains only one arc, however).

Full graphs are distinguished by the relative simplicity of their chromatic polynomials (for which see [26], Theorem 1.1) and by being a class closed under formation of minors through deletion and contraction of links. These properties make it possible to prove facts about signed graphs inductively by treating full graphs first, then gradually removing unbalanced arcs, as we do in the treatment of chromatic polynomials in [22].

7C. Sign symmetry and signed expansions

A signed graph is *sign-symmetric* if, whenever it contains a link $e:vw$ of one sign, it also contains a link $f:vw$ of the opposite sign. The *signed expansion* of an unsigned graph Γ is the signed graph $\pm\Gamma$ obtained by replacing each link or loop of Γ (excluding free loops) by two copies of itself, one positive and one negative. Half arcs and free loops remain undoubled. The *full signed expansion* $(\pm\Gamma)^{\bullet}$, loosely written $\pm\Gamma^{\bullet}$, is the result of filling each empty node of $\pm\Gamma$.

Obviously every signed expansion is sign symmetric. Conversely a sign-symmetric graph Σ is equivalent to a signed expansion in a sense to be explained. Let Γ be the unsigned (simple) graph which has node set $N(\Sigma)$ and has a link vw whenever v and w are adjacent in Σ and a half arc at every full node of Σ. If we "simplify" Σ by eliminating balanced loops and multiplicities and replacing all the unbalanced arcs at each full node by a single half arc, the result is $\pm\Gamma$. Since balanced loops and multiplicity of arcs make no real difference to the closure properties of arcs in Σ nor to the balance of flats, we see that Lat Σ and Lat $(\pm\Gamma)$ are naturally isomorphic, as are $\mathrm{Lat}^b\Sigma$ and $\mathrm{Lat}^b(\pm\Gamma)$.

Full sign-symmetric graphs, and to a lesser degree all sign-symmetric graphs, are perhaps the nicest of signed graphs. For one thing, the chromatic invariants of $\pm\Gamma^{\bullet}$ are very simply related to those of Γ (without being identical, as are those of $+\Gamma$); those of $\pm\Gamma$ are nearly as well behaved. (See [22] for this.) Since chromatic invariants are unaffected by "simplification" of the graph, this property of signed expansions carries over to sign-symmetric graphs. Another nice property is that the balanced flats of a signed expansion $\pm\Gamma$ are neatly characterized by those of Γ. A third is that full sign-symmetric graphs, and in a more complicated way all sign-symmetric graphs, are classes closed under contractions. The latter facts are made precise in Propositions 1 and 2.

Proposition 7C.1. *Let Γ be an ordinary graph (no half arcs). Every balanced flat of $\pm\Gamma$ is obtained by choosing a flat $A\in\mathrm{Lat}\,\Gamma$ and labelling A in a balanced way. The number of such balanced labellings is $2^{\mathrm{rk}\,A}$.*

Proof. For the verification we note that a balanced flat F of $\pm\Gamma$ cannot contain a pair of parallel arcs with opposite signs, whence $F=(A,\sigma)$ for some $A\subseteq E(\Gamma)$ and sign function σ; and that the number of balanced labellings follows from Lemma 3.1. □

When Γ has half arcs Proposition 1 still remains valid if A is limited to being a balanced flat.

Let us write here Σ^U for Σ with an unbalanced arc attached to each empty node in $U\subseteq N(\Sigma)$.

Proposition 7C.2. *Let Γ be an unsigned graph and let $U\subseteq N(\Gamma)$ include all nodes which carry loops or half arcs. Let F be a balanced flat of $\pm\Gamma^U$ derived from the flat A of Γ as in Proposition 1, and let*

$$X=\{V\in\pi(A):\#(V)>1 \text{ or } V\subseteq U\}.$$

Then $(\pm\Gamma^U)/F$ and $\pm(\Gamma/A)^X$ simplify to the same signed graph. In particular $(\pm\Gamma^{\bullet})/F$ and $\pm(\Gamma/A)^{\bullet}$ simplify to the same graph.

Proof. The verifications are implicit in [22], Sections 6 (the full case) and 8. The particular result explains the primacy of the full case seen in [22], where results on chromatic invariants are proved first for full, then (by removing unbalanced arcs) for general sign-symmetric graphs. □

Proposition 7C.3. Lat $\pm K_n^{\bullet}$ *is the Dowling lattice $Q_n(\{\pm\})$ of the sign group.*

This fact, generalized to voltage graphs (Section 9), was known to Dowling. (The proof for vector-representable Dowling lattices, which include $Q_n(\{\pm\})$, appears in [5b], p. 109, and also in [5a], Section 5.3, p. 200). We indicate how the proof fits into our general theory.

Proof. Each flat F of $\pm K_n^{\bullet}$ is obtained by choosing a partial partition π of the node set N (say π partitions X^c) and labeling $A=E(K_n):\pi$ in a balanced way (by Theorem 5.1(b) and Proposition 1). Let $[\nu]$ be the class of switching functions ν on X^c such that A^{ν} is labeled all $+$; thus ν is determined up to (right) multiplication by a constant on each block of π. So $[\nu]$ is a partial $\{\pm\}$-partition of N as defined by Dowling [6].

This construction is reversible, hence the proposition. □

7D. All-negative graphs and the even-circle matroid

The *negative signed graph on Γ*, written $(\Gamma,-)$ or simply $-\Gamma$, is Γ with every arc (except half arcs) labelled $-$. Thus there can be no free loops. The matroid $G(-\Gamma)$ may be called the *even-circle matroid* of Γ, for it is determined by the fact that the balanced circles are the ones of even length. (It is also known as the *unoriented cycle matroid.*) The published treatments of this matroid have been Doob's studies [4, 5] of the eigenspace of -2 of a line-graphic adjacency matrix, the article [16] by Simões-Pereira on "matroidal families" of graphs, and a brief discussion of even-circle circuits at the end of Tutte's paper [18] on the dual chain group. Descriptions of, for example, the closed arc sets of $-\Gamma$, its bases, its rank function, or its chromatic polynomial are easy consequences of the general theory of signed graphs.

To begin with, Proposition 2.1 gives the interpretation of all-negative balance.

Corollary 7D.1. *Let Γ be an unsigned graph (with no free loops, so $-\Gamma$ is defined). An arc set S is balanced in $-\Gamma$ if and only if it is bipartite.* □

The interpretation of balanced flats from Proposition 5.6 is:

Corollary 7D.2. *An arc set S is a balanced flat in $-\Gamma$ if and only if it is bipartitionally induced* (defined in Section 5). □

For a description of the even-circle matroid $G(-\Gamma)$ we restate some of the parts of Theorem 5.1 for $-\Gamma$. They are labelled to correspond with Theorem 5.1. Here $N_0(S)$ is the set of nodes of non-bipartite components of S. The *even closure* of S is the set of all arcs in S or whose endpoints are connected by an odd simple path in S; we denote it $\mathrm{ecl}(S)$.

Corollary 7D.3. *Let $\Gamma=(N,E)$ be an unsigned graph without free loops. The even-circle matroid $G(-\Gamma)$ is the finitary matroid defined by any of the following equivalent statements.*

(a) *The closure of an arc set S is $\mathrm{ecl}(S)\cup E{:}N_0(S)$.*

(b) *An arc set A is closed $\Leftrightarrow$ it is the union of $E{:}X$ (where X is a node set, possibly $\emptyset$) and a bipartitionally induced arc set in $\Gamma{:}X^c$.*

(d) *An arc set S is dependent $\Leftrightarrow$ it contains an even circle, or else a pair of odd circles or half arcs (or one of each) connected within S.*

(e) *An arc set is a circuit of $G(-\Gamma)$ $\Leftrightarrow$ it is an even circle, or it consists of a pair of odd circles or half arcs (or one of each) connected either at one node or by a simple path which meets one of them at each endpoint.*

(g) *An arc set S is a basis $\Leftrightarrow$ in each bipartite component of Γ it is a spanning tree, and in each other component it is a spanning tree T plus either a half arc or an arc which forms an odd circle with T.*

(j) *The rank of an arc set S is $n-b(S)$ if $n=\#(N)$ is finite; here $b(S)$ denotes the number of bipartite node components of S (including isolated nodes).* □

Another fact about $G(-\Gamma)$ is that the closure of a bipartite arc set is bipartite (from Corollary 5.7).

A restriction of $-\Gamma$ is, of course, all-negative but contractions generally are not.

7E. Signed complete graphs and two-graphs

A signed complete graph (K_n,σ) is characterized by its negative subgraph; this it is

convenient to emphasize by the notation. The *signed complete graph with negative graph Γ* is

$$K_\Gamma=+\Gamma^c\cup-\Gamma.$$

In a signed complete graph K_Γ the class of balanced circles is completely determined by the class $\mathscr{B}^{(3)}(K_\Gamma)$ of unbalanced triangles. For the class $\mathscr{C}^{(3)}(K_n)$ of triangles spans $C(K_n)$; therefore by [23], Theorem 2, if we know which triangles are balanced we know $\mathscr{B}(K_\Gamma)$. This observation underlies the "two-graphs" introduced by G. Higman (see [15] and [17]).

A two-graph is usually taken to be a class of triples of nodes. But to apply signed graph theory it is better to regard it as a class of triangles. So we define a *two-graph* on n nodes to be a class $\mathscr{T}\subseteq\mathscr{C}^{(3)}(K_n)$ such that, among the four triangles on any quadruple of nodes, an even number belong to $\mathscr{T}$. (Equivalently: an even number are not in $\mathscr{T}$. Hence the complement $\mathscr{T}^c$ is also a two-graph). This condition amounts to saying (in the language of [23]) that $\mathscr{T}$ is 4-additive. Since $\mathscr{C}^{(3)}(K_n)$ has as a 4-generating basis the set of all triangles through a node, we have as special cases of [23], Corollaries 4 and 8:

Proposition 7E.1. (1) *If Σ is a signed complete graph, then $\mathscr{B}^{(3)}(\Sigma)$ is a two-graph. And any two-graph is a $\mathscr{B}^{(3)}(\Sigma)$ for some signed complete graph Σ.*

(2) *If Σ_1 and Σ_2 are signed complete graphs, then $\mathscr{B}^{(3)}(\Sigma_1)=\mathscr{B}^{(3)}(\Sigma_2)$ if and only if Σ_1 and Σ_2 are switching equivalent.*

(3) *There is a one-to-one correspondence between two-graphs and switching classes of signed complete graphs.* □

This proposition is the translation into signed-graphic language of basic properties of two-graphs summarized in [15], Theorem 4.2.

We call the set $\mathscr{B}^{(3)}(K_\Gamma)$ of unbalanced triangles the *two-graph of K_Γ*. (It is the same as Seidel's two-graph of Γ as defined in [15].) The correspondence of Proposition 1(3), combined with Corollary 5.4, shows that the *matroid of a two-graph $\mathscr{T}$* is well defined: it is $G(\mathscr{T})=G(K_\Gamma)$ where K_Γ is any signed graph whose two-graph is $\mathscr{T}$.

8. Linearity: matrices and representation

The signed graphic matroid $G(\Sigma)$ has linear representations as an oriented incidence matrix (hence a linear operator), as a set of vectors (over various fields and rings), and as a set of hyperplanes which partition real n-space. These representations, generalizations of those of ordinary graphic matroids, retain many interesting properties of the latter. Here we discuss the incidence and adjacency matrices of a signed graph, and examine representability by vectors in characteristics 2 and otherwise. Our results imply that $G(\Sigma)$ is isomorphic to the linear dependence matroid of the corresponding root system subarrangement and thus that the problem of counting regions raised in the introduction can be solved within signed graph theory. (The actual solution appears in [25] and [26].)

8A. *Matrices and determinants*

An incidence matrix has a row for each node and a column for each arc, the entry for node v and arc e being 0 if v and e are not incident. In ordinary graph theory a graph Γ has two kinds of incidence matrices, unoriented (entries 0,1) and oriented (entries $0, \pm 1$). In signed graph theory all incidence matrices are oriented; the two kinds belonging to Γ are viewed as the matrices respectively of $-\Gamma$ and $+\Gamma$.

The incidence matrix associated with Σ we denote $M(\Sigma)$. It is best described by way of its column vectors. Let $M_e = (m_{ve};\ v \in N)$ be the vector corresponding to the arc e. When v is an endpoint of e, the following rules define m_{ve}:

if $e: vw$ is a link:	$m_{ve} = \pm 1$ and $m_{we} = -\sigma(e) m_{ve}$;
if $e: vv$ is a loop:	$m_{ve} = 0$ if $\sigma(e) = +$, ± 2 if $\sigma(e) = -$;
if $e: v$ is a half arc:	$m_{ve} = \pm 1$;
if $e: \emptyset$ is a free loop:	$M_e = 0$.

The general rule behind these is: each incidence of e with a node counts ± 1; the two incidences of a link or loop e must have opposite signs if $\sigma(e) = +$, the same signs if $\sigma(e) = -$; for a loop at v the two incidences are summed.

There are thus in general two possibilities for each column M_e, one the negative of the other. The choice between them corresponds to a choice of orientation of e; hence $M(\Sigma)$ is properly called *an (oriented) incidence matrix*. However since the choice of orientation is immaterial here, we speak of "the" incidence matrix $M(\Sigma)$.

The fact that $M_e = 0$ for positive and for free loops is one reason a free loop should be regarded as positively labelled.

The connection between the matrix $M(\Sigma)$ and the matroid $G(\Sigma)$ is as with ordinary graphs: the kernel of $M(\Sigma)$ acting as a linear operator on $\mathbb{R}^{E(\Sigma)}$ is a chain group whose matroid is $G(\Sigma)$ (Corollary 8B.2).

The *adjacency matrix* of Σ is the matrix $A(\Sigma)$ whose off-diagonal entries are

$$a_{vw} = d_+(v, w) - d_-(v, w) \quad \text{for } v \neq w$$

and whose diagonal entries are

$$a_{vv} = 2d_+(v, v) - 2d_-(v, v),$$

where $d_\varepsilon(v, w)$ is the number of arcs between v and w labelled ε (the number of loops, if $v = w$). As with ordinary graphs, $-M(\Sigma)M(\Sigma)^t$ is a *modified adjacency matrix* $A'(\Sigma)$ with diagonal entries changed to $a'_{vv} = a_{vv} - d(v)$, where $d(v)$ is the

degree of v (the number of arc incidences at v; a loop at v counts as 2). For this matrix there is an analog of the Matrix-Tree Theorem (Theorem 4 below).

The determinantal properties of signed graphs are almost as nice as those of ordinary graphs. For instance $M(\Sigma)$ is the next thing to totally unimodular (see Lemma 3).

Let $X \subseteq N(\Sigma)$. The *restriction of* Σ *to* X, denoted $\Sigma|X$, is the signed graph obtained by deleting all nodes outside X but no arcs. (As a result a link $e: vw$, where $v \in X$ and $w \notin X$, becomes a half arc $e: v$, while an arc whose endpoints were in X^c becomes a free loop.)

Lemma 8A.1. *Let* $X \subseteq N(\Sigma)$. *If from* $M(\Sigma)$ *the rows corresponding to* X^c *are deleted, the resulting matrix is* $M(\Sigma|X)$. □

Lemma 8A.2. *Suppose* Σ *has* n *nodes and* rk $G(\Sigma) = n$. *Let* $S \subseteq E(\Sigma)$ *be a basis for* $G(\Sigma)$. *Then* det $M(\Sigma|S) = \pm 2^l$, *where* $l =$ *the number of circles in* S.

For the proof it is enough to treat the case where S is connected. From Theorem 5.1(g) we know S is a spanning tree together with either a half arc or else an arc forming an unbalanced circle. If there is a node of degree 1, we can expand by minors in its row; that gives us $\pm$det $M(S:\{v\}^c)$. Thus we can prune nodes of degree 1 and their incident arcs until S is either a half arc (whose determinant is ± 1) or an unbalanced circle (whose determinant is easily seen to be ± 2). □

Lemma 8A.3.[1] *Let* Σ *be a signed graph. For any* $k \times k$ *square minor* M_1 *in* $M(\Sigma)$, det M_1 *is either* 0 *or an integer which divides* 2^k.

Proof. Say M_1 is the minor indexed by $X \times S$. By Lemma 1 we may restrict our attention to $M(\Sigma|X)$. If S is a basis of $\Sigma|X$, then the latter has rank k and we can apply Lemma 2, noting that $l \leq k$. If S is not a basis, it must have a balanced component, which (switching as necessary) can be regarded as an ordinary graph and therefore has determinant 0. That proves the theorem. □

The standard Binet-Cauchy approach now establishes:

Theorem 8A.4 (Matrix-Tree Theorem for Signed Graphs).[2] *Let* Σ *be a signed graph on* n *nodes and let* $b_l =$ *the number of independent sets of* n *arcs (bases, if* rk $\Sigma = n$) *which contain* l *circles. Then*

$$\det A'(\Sigma) = \sum_{l=0}^{n} 4^l b_l. \quad \square$$

[1] I thank Neil Robertson for a stimulating discussion of this property.
[2] I am grateful to Seth Chaiken for his clear and inspiring explanations of matrix-tree theorems.

There is also a matrix-tree theorem for the determinant of A'_{XY}, the matrix obtained by deleting from $A'(\Sigma)$ the rows corresponding to X and the columns corresponding to Y, where X and Y are fixed equicardinal node sets. This theorem is like the ordinary version (cf. [2], Lemma 7.4), differing only in that it counts independent sets $I \subseteq E(\Sigma)$ in which each balanced node component contains just one member of X and one of Y (and these members exhaust X and Y), I being weighed 4^l where l is as in Theorem 4.

A matrix criterion for balance is Proposition 5, which follows from the proof of Lemma 3 by observing that the totally unimodular case is precisely the one in which $l=0$.

Proposition 8A.5. *A signed graph Σ is a balanced graph with (possibly) added half arcs if and only if $M(\Sigma)$ is totally unimodular.* □

This result was originally stated, slightly differently, by Heller and Tompkins (the "only if" part, in [11], Theorem 1; also proved by Hoffman [12]) and by Gale (the "if" part, in [12]).

8B. Vector representations

We introduced the oriented incidence matrix $M(\Sigma)$ by way of its column vectors M_e in part so we can ask how their dependencies compare with those in the matroid $G(\Sigma)$. The answer implies that $G(\Sigma)$ is representable over any field whose characteristic is not 2.

Theorem 8B.1. *Let $\Sigma=(N,E,\sigma)$ be a signed graph and $f:E\to\mathbb{R}^N$ the mapping $f(e)=M_e$. The matroid structure G_f induced on E by the dependencies of the vectors $f(e)$ is precisely $G(\Sigma)$. This result holds if $\mathbb{R}$ is replaced by any unitary ring R in which 2 is not a zero divisor.*

Observe that G_f may be defined by any one of several statements, all equivalent by general matroid theory. Let $S\subseteq E$. We may define G_f by $\mathrm{rk}_f S=\dim f(S)$; or by:

$$S \text{ is dependent} \quad \Leftrightarrow \quad f(S) \text{ is a dependent multiset}$$

(so if $f|S$ is not one-to-one, S is dependent); etc. We can prove the theorem by appeal to Lemma 8A.3. The details are routine. □

Similarly by reference to Lemma 8A.3, noting that a minor has non-unit determinant precisely when its arcs are dependent in Σ or contain an unbalanced circle (dependent in $|\Sigma|$), we deduce the structure of G_f in a ring with characteristic 2, such as $GF(2^r)$.

Theorem 8B.2. *Let $\Sigma=(N,E,\sigma)$ be a signed graph, R a unitary ring having characteristic 2, and $f:E\to R^N$ the mapping $f(e)=M_e$. The matroid structure G induced by f is precisely the graphic matroid $G(|\Sigma|)$.* □

The matrix $M(\Sigma)$ acts as a linear operator $R^E\to R^N$. Its kernel induces a chain-group matroid on E. It is a standard theorem (cf. [19], Section 9.4) that when R is a field that chain-group matroid is exactly the matroid of columns of the matrix—that is, the matroid G_f, whose nature we established in Theorems 1 and 2. $M(\Sigma)$ also acts as a linear operator $R^N\to R^E$; its image induces a chain-group matroid, which by another standard theorem is the dual matroid $G_f^\perp$.

Corollary 8B.3. *Let $\Sigma=(N,E,\sigma)$ be a signed graph and F a field. The kernel of $M(\Sigma)$ acting on F^E induces a chain-group matroid which is $G(\Sigma)$ if* char $F\neq 2$, *$G(|\Sigma|)$ if* char $F=2$. *The image of $M(\Sigma)$ acting on F^N induces a matroid which is $G(\Sigma)^\perp$ if* char $F\neq 2$, *$G(|\Sigma|)^\perp$ if* char $F=2$. □

From Theorem 1 we have also:

Corollary 8B.4. *The five-point line L_5 and the Fano plane and their dual matroids are minimal excluded minors for the matroids of signed graphs.*

Proof. To prove these are excluded minors it suffices (by Theorem 5.2) to show they are not matroids of signed graphs. But none of them is ternary. Now apply Theorem 1.

To prove they are minimal we observe first that the one-point deletions and contractions of the Fano plane are graphic and cographic and that the four-point line L_4 is $G(\pm K_2^\bullet)$. Either deleting a point from the five-point line L_5 or contracting $L_5^\perp$ by a point gives L_4. And contracting L_5 by a point gives a matroid that is graphic and cographic. □

The failure of G_f to represent $G(\Sigma)$ in characteristic 2 is sometimes, but not always, the fault of the representing map f. For instance $G(\pm K_3)$, which is isomorphic to $G(K_4)$ (see Section 5), is therefore binary, but $G_f(\pm K_3)$ equals $G(K_3)$ with elements doubled in parallel. And $G(-K_5)$ is represented in $GF(2)^5$ by mapping e_{ij} to the vector with all 1's except in positions i and j, but $G_f(-K_5)=G(K_5)$. On the other hand the matroid of $\pm K_3^{(2)}$ (that is, $\pm K_3$ with two full nodes) cannot be represented over any field of characteristic 2. (One can see this by solving the representation equations, which require $1\neq -1$.) Thus the same holds true of all $G(\Sigma)$ for Σ that contain $\pm K_3^{(2)}$ as a minor, such $\pm K_n^\bullet$ for $n\geq 3$.

This suggests the problem of characterizing exactly, e.g. by excluded minors, the signed graphs with different kinds of representability over fields of characteristic 2. Five categories of signed graphs Σ are readily distinguishable:

I. Those for which $G_f(\Sigma)=G(\Sigma)$ over every field. They are the balanced signed graphs with (possibly) added half arcs (Proposition 8A.5). Their matroids are graphic (by Corollary 5.4 and Proposition 7A.1).

II. Those whose matroids are graphic but are not represented by f in characteristic 2. An example is $\pm K_3$.

III. Those whose matroids are binary but not graphic. An example is $-K_5$.

IV. Those whose matroids are representable over GF(4) but are not binary. An example is $\pm K_2^{\bullet}$.

V. Those whose matroids cannot be represented over any field of characteristic 2. An example is $\pm K_3^{(2)}$.

Conjecture 8B.1. *Every signed graph is in one of the categories above.* That is, any signed-graphic matroid which is representable over a field of characteristic 2 is representable over GF(4). This conjecture is certainly true at least for signed graphs in which every component is balanced or has an unbalanced arc. (A proof will appear elsewhere.)

9. Voltage graphs

A *voltage graph* $\Phi=(\Gamma,\varphi)$ consists of a graph Γ and a *voltage*, a mapping $\varphi:E(\Gamma)\to\mathfrak{G}$, where $\mathfrak{G}$ is a group implicitly associated with Φ. We assume that $\varphi(e^{-1})=\varphi(e)^{-1}$, where e^{-1} denotes the arc e taken in the opposite direction; that φ is undefined on half arcs; and that $\varphi\,|\,\{\text{free loops}\}\equiv 1$. It is often useful to assume that $\mathfrak{G}$ comes with a specific permutation representation; to stress this aspect of $\mathfrak{G}$ the map φ can be called a *permutation voltage*. (The terminology is drawn from Gross [7] and Gross and Tucker [8].)

Most of Sections 1 to 5 generalizes to voltage graphs with appropriate obvious changes. For instance switching Φ by a function $v:N(\Gamma)\to\mathfrak{G}$ means replacing φ by φ^v, whose definition is $\varphi^v(e)=v(v)^{-1}\varphi(e)v(w)$ when $e:v\to w$ is an arc from v to w. But the results relating switching equivalence to equal balance (they are 3.2 and 5.4) are valid only for signed graphs. The matroid results generalize (except for Corollary 5.3), but a general voltage graph has a fourth type of circuit: a theta graph in which the three paths from one trivalent node to the other all have different values.

The idea of covering extends totally to voltage graphs, as Gross and Tucker showed in [8]. They proved that every k-sheeted covering graph $\tilde{\Gamma}$ of Γ is derived from a permutation voltage on k letters and every such voltage on Γ comes from a k-sheeted covering graph. Their results can be strenghtened slightly to the following version of Theorem 6.3, which shows that coverings are classified by switching classes (or vice versa).

Theorem 9.1. *Let Γ be an unsigned graph. There is a one-to-one correspondence between isomorphism types of k-sheeted covering graphs of Γ and switching classes of voltage assignments with values in the symmetric group on k letters.* □

The results of Section 6 on balance and the matroid extend more or less obviously. Of the examples in Section 7, full graphs and signed expansions generalize directly. In particular Dowling's lattice $Q_n(\mathfrak{G})$ (cf. [6]) equals Lat $\mathfrak{G}K_n^{\bullet}$, where $\mathfrak{G}K_n^{\bullet}$ is the full $\mathfrak{G}$-expansion of K_n, that is K_n with each arc replaced by one arc with each voltage $g\in\mathfrak{G}$ and with an unbalanced arc at each node. The proof is just like that of Proposition 7C.3.

The question of linear representation is complicated by the need to map $\mathfrak{G}$ into the coefficient field. For work on this problem in the case $\Phi=\mathfrak{G}K_n^{\bullet}$, see Dowling [6], Section 5.

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Citations are provided to reviews in Mathematical Reviews (MR) and Zentralblatt für Mathematik (ZM).

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~~Example 8.1, which illustrates Theorem 8.1, contains some typographical errors. In the expressions for β_6, β_3 and β_5 on page 37, min and max should be interchanged. Besides, the entries a_{46} and a_{45} of Table 2 (also on page 37) should read $-\frac{1}{3}$ and $\frac{1}{3}$ instead of $-\frac{2}{3}$ and $-\frac{1}{3}$ respectively. As a result of this and the change in (8.8) and (8.9), the values of $m_0^*(1)$ and $m_0^*(2)$ are $\frac{3}{5}$ and $-\frac{3}{5}$, instead of $\frac{2}{3}$ and $-\frac{2}{3}$, respectively, and the cut becomes~~

$$x_1 + x_2 + x_5 + \tfrac{1}{2}x_6 \geq 1.$$

ERRATUM

To: T. Zaslavsky, Signed Graphs, Discrete Appl. Math. 4 (1982) 47–74.

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The characterizations of bases of $G(\Sigma)$ and $G(-\Gamma)$ in the above mentioned paper are incorrectly stated. They should read as follows.

Theorem 5.1. (g) *An arc set S is a basis $\Leftrightarrow$ for each $B \in \pi_b(\Sigma)$, $S:B$ is a spanning tree; while for each $B \in \pi(\Sigma) \setminus \pi_b(\Sigma)$, $S:B$ is a spanning forest plus, in each of its trees T, either a half arc or an arc forming an unbalanced circle with T.*

Corollary 7D.3. (g) *An arc set S is a basis $\Leftrightarrow$ in each bipartite component of Γ it is a spanning tree, and in each other component it is a spanning forest plus, in each of its trees T, either a half arc or an arc which forms an odd circle with T.*

NB: Parts (f) are similarly incorrect.

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