Lectures on Signed Graphs and Geometry

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These slides are a compressed adaptation of the lecture notes.

Lecture 1

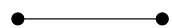
1. Graphs

Set sum, or *symmetric difference*: $A \oplus B := (A \setminus B) \cup (B \setminus A)$.

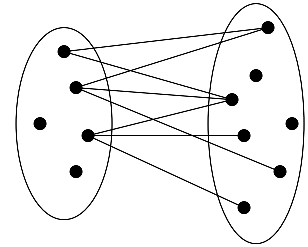
Graph:

- $\Gamma = (V, E)$, where $V := V(\Gamma)$, $E := E(\Gamma)$. *All graphs are finite.*
- $n := |V|$, the *order*.
- $V(e) :=$ multiset of vertices of edge e.
- $V(S) :=$ set of endpoints of edges in $S \subseteq E$.
- Complement of $X \subseteq V$: $X^c := V \setminus X$.
- Complement of $S \subseteq E$: $S^c := E \setminus V$.

Edges:

- Multiple edges, loops, half and loose edges.
  - *Link*: two distinct endpoints. 
  - *Loop*: two equal endpoints. 
  - *Ordinary edge*: a link or a loop.
  - *Half edge*: one endpoint. 
  - *Loose edge*: no endpoints. 
- $E_0(\Gamma) :=$ set of loose edges.
- $E_* := E_*(\Gamma) :=$ set of ordinary edges.
- *Parallel* edges have the same endpoints. 
- *Ordinary graph*: every edge is a link or a loop.
  Link graph: all edges are links.
  Simple graph: a link graph with no parallel edges.

Various:



- $E(X, Y) :=$ set of edges with one endpoint in X and the other in Y.
- *Cut* or *cutset*: any $E(X, X^c)$ that is nonempty.
- An *isolated vertex* has degree 0.
- $X \subseteq V$ is *stable* or *independent* if no edge has all endpoints in X (excluding loose edges).
- *Degree*: $d(v) = d_\Gamma(v)$. A loop counts twice.
- Γ is *regular* if $d(v) =$ constant.

Walks, trails, paths, circles:

- *Walk*: $v_0 e_1 v_1 \cdots e_l v_l$ where $V(e_i) = \{v_{i-1}, v_i\}$ and $l \geq 0$.
  Also written $e_1 e_2 \cdots e_l$ or $v_0 v_1 \cdots v_l$.
  Length: l.
- *Closed walk*: a walk with $l \geq 1$ and $v_0 = v_l$.
- *Trail*: a walk with no repeated edges.
- *Path* or *open path*: a trail with no repeated vertex.
- *Closed path*: a closed trail with no repeated vertex except $v_0 = v_l$.
  (A closed path is not a path.)
- *Circle* ('cycle', 'polygon', etc.): the graph *or* edge set of a closed path.
  Equivalently: a connected, regular graph with degree 2, or its edge set.
- $\mathcal{C} = \mathcal{C}(\Gamma)$: the class of all circles in Γ.

Examples:

- K_n: complete graph of order n.
- $K_{r,s}$: complete bipartite graph.
- Γ^c: complement of Γ, if Γ is simple.
- K_n^c: edgeless graph of order n.
- P_l: a path of length l.
- C_l: a circle of length l.

Types of subgraph: In Γ, let $X \subseteq V$ and $S \subseteq E$.

- *Component*: a maximal connected subgraph, excluding loose edges.
- $c(\Gamma) :=$ number of components of Γ.
- A *component of* S means a component of (V, S). $c(S) := c(V, S)$.
- *Spanning subgraph*: $\Gamma' \subseteq \Gamma$ such that $V' = V$.
- $\Gamma|S := (V, S)$. (A spanning subgraph.)

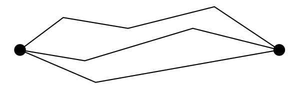
- *Induced edge set* $S{:}X := \{e \in S : \varnothing \neq V(e) \subseteq X\}$. $S{:}X := (X, S{:}X)$.
- *Induced subgraph* $\Gamma{:}X := (X, E{:}X)$. $E{:}X := (X, E{:}X)$.
- $\Gamma \setminus S := (V, E \setminus S) = \Gamma|S^c$.
- $\Gamma \setminus X$: subgraph with

  $V(\Gamma \setminus X) := X^c$,

  $E(\Gamma \setminus X) := \{e \in E \mid V(e) \subseteq V \setminus X\}$.

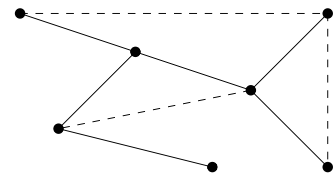
  X is *deleted* from Γ.

Graph structures and types:



- *Theta graph*: union of 3 internally disjoint paths with the same endpoints.
- *Block* of Γ: maximal subgraph without loose edges, such that every pair of edges is in a circle together. The simplest kinds of block are an isolated vertex, and $(\{v\}, \{e\})$ where e is a loop or half edge at vertex v. A loose edge is not in any block of Γ.
- *Inseparable* graph: has only one block.
- *Cutpoint*: $v \in$ more than one block.

Fundamental system of circles:



- T: a maximal forest in Γ.
- $(\forall\ e \in E_* \setminus T)$: $\exists!$ circle $C_e \subseteq T \cup \{e\}$.
- The *fundamental system of circles* for Γ is

$$\{C_e : e \in E_* \setminus T\}.$$

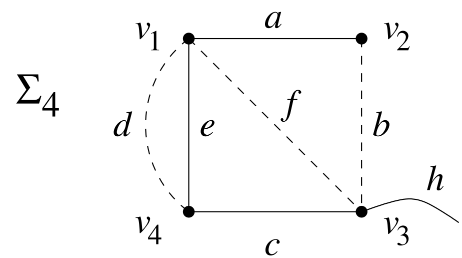
Proposition 1.1. *Choose a maximal forest T. Every circle in Γ is the set sum of fundamental circles with respect to T.*

Proof. $C = \bigoplus_{e \in C \setminus T} C_T(e)$. $\square$

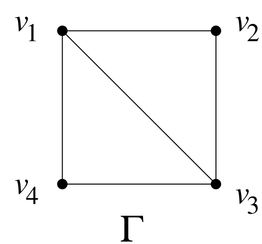
2. Signed Graphs

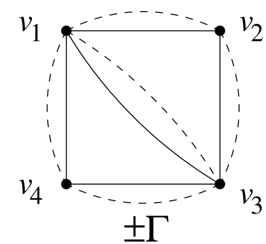
Signed graph: $\Sigma = (\Gamma, \sigma) = (V, E, \sigma)$
where $\sigma : E_* \to \{+, -\}$.

Notations: $\{+, -\}$, or $\{+1, -1\}$,
or $\mathbb{Z}_2 := \{0, 1\} \bmod 2$, or ....

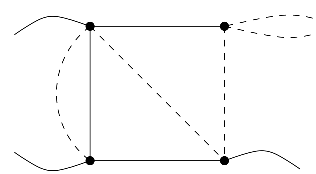


- σ: the *signature* or *sign function*.
- $|\Sigma|$: the *underlying graph*.
- $E^+ := \{e \in E : \sigma(e) = +\}$. The *positive subgraph*: $\Sigma^+ := (V, E^+)$.
  $E^- := \{e \in E : \sigma(e) = -\}$. The *negative subgraph*: $\Sigma^- := (V, E^-)$.
- $+\Gamma := (\Gamma, +)$: *all-positive* signed graph.
- $-\Gamma := (\Gamma, -)$: *all-negative* signed graph.
- $\pm\Gamma = (+\Gamma) \cup (-\Gamma)$:
  the *signed expansion* of Γ.
  $E(\pm\Gamma) = \pm E := (+E) \cup (-E)$.





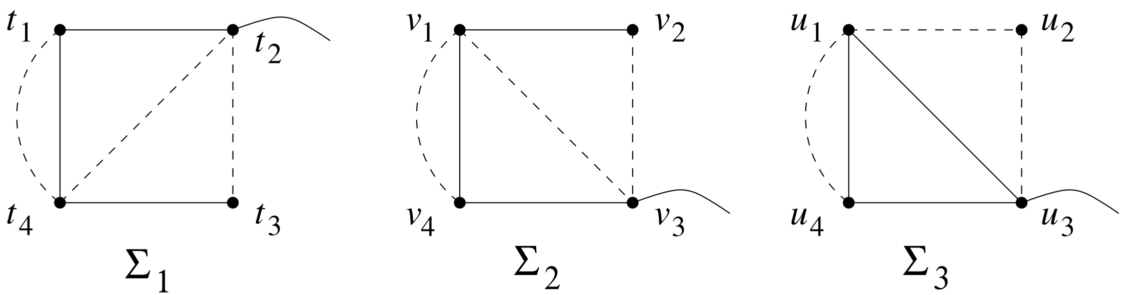
- $\Sigma^\bullet = \Sigma$ with a half edge or negative loop at every vertex.
  $\Sigma^\bullet$ is called a *full* signed graph.
  $\Sigma^\circ := \Sigma$ with a negative loop at every vertex.



Isomorphism.

Σ_1 and Σ_2 are *isomorphic*, $\Sigma_1 \cong \Sigma_2$, if $\exists\ \theta : |\Sigma_1| \cong |\Sigma_2|$ that preserves signs.

Example: $\Sigma_1 \cong \Sigma_2 \ncong \Sigma_3$.



2.1. **Balance.**

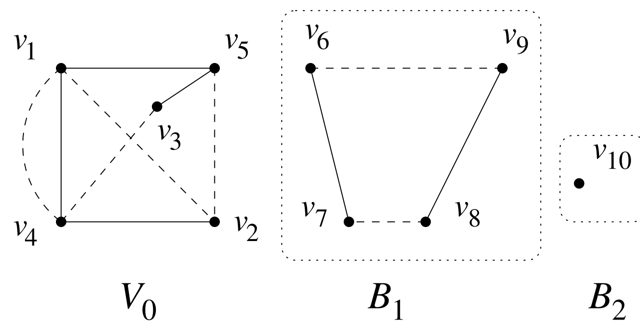
- $\sigma(W) := \prod_{i=1}^{l} \sigma(e_i)$ = product of signs of edges in walk W, with repetition.
- $\sigma(S)$:= product of the signs of edges in set S, without repetition.
- The *class of positive circles*:

$$\mathcal{B} = \mathcal{B}(\Sigma) := \{C \in \mathcal{C}(|\Sigma|) : \sigma(C) = +\}.$$

- Σ, or a subgraph, or an edge set, is *balanced* if:
  no half edges, and every circle is positive.
- A circle is balanced $\iff$ it is positive.
  A walk cannot be balanced because it is not a graph or edge set.

- $\pi_{\mathrm{b}}(\Sigma) := \{V(\Sigma') : \Sigma' \text{ is a balanced component of } \Sigma\}$. $\qquad \pi_{\mathrm{b}}(S) := \pi_{\mathrm{b}}(\Sigma|S)$.
- $b(\Sigma) := |\pi_{\mathrm{b}}(\Sigma)| = \#$ of balanced components of Σ. $\qquad b(S) := b(\Sigma|S)$.
- $V_0(\Sigma) := V \setminus \bigcup_{W \in \pi_{\mathrm{b}}(\Sigma)} W$
  $= \{\text{vertices of unbalanced components of } \Sigma\}$. $\qquad V_0(S) := V_0(\Sigma|S)$.

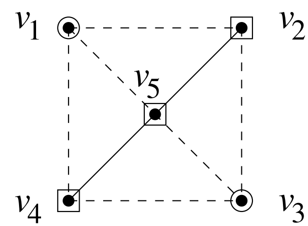
Example:
$\pi_{\mathrm{b}}(\Sigma) = \{B_1, B_2\}$ and
$V_0(\Sigma) = V \setminus (B_1 \cup B_2)$.



A *bipartition* of a set X is $\{X_1, X_2\}$ such that $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$. X_1 or X_2 could be empty.

Theorem 2.1 (Harary's Balance Theorem, 1953).
Σ is balanced $\iff$ it has no half edges and there is a bipartition $V = V_1 \,\dot\cup\, V_2$ such that $E^- = E(V_1, V_2)$.

I like to call $\{V_1, V_2\}$ a *Harary bipartition* of Σ.



$V_1 = \{v_1, v_3\}$,
$V_2 = \{v_2, v_4, v_5\}$

Corollary 2.2. *$-\Gamma$ is balanced $\iff$ Γ is bipartite.*

Thus, balance is a generalization of bipartiteness.

Proposition 2.3. *Σ is balanced $\iff$ every block is balanced.*

Deciding balance:
Deciding whether Σ is balanced is easy. (Soon!)

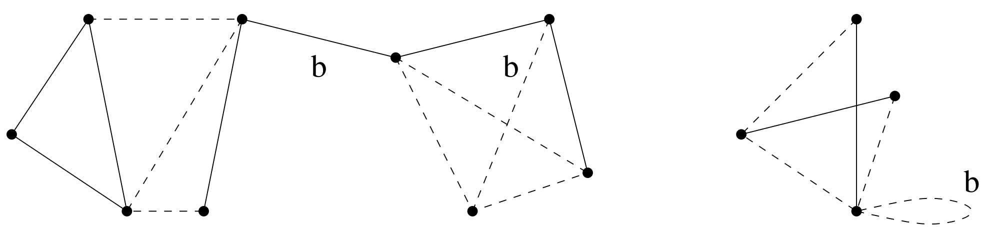
Types of vertex and edge:

- *Balancing vertex*: v such that $\Sigma \setminus v$ is balanced but Σ is unbalanced.
- *Partial balancing edge*: e such that $b(\Sigma \setminus e) > b(\Sigma)$.
- *Total balancing edge*: e such that $\Sigma \setminus e$ is balanced but Σ is not balanced.

Proposition 2.4. *e is a partial balancing edge of $\Sigma \iff$ it is*

(a) *an isthmus between two components of $\Sigma \setminus e$, of which at least one is balanced, or*
(b) *a negative loop or half edge in a component Σ' such that $\Sigma' \setminus e$ is balanced, or*
(c) *a link with endpoints v, w, which is not an isthmus, in a component Σ' such that $\Sigma' \setminus e$ is balanced and every vw-path in $\Sigma' \setminus e$ has sign $-\sigma(e)$.*

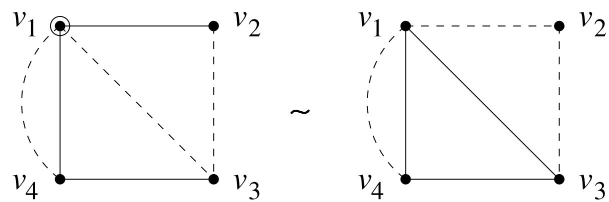
In the diagram, ‘b’ denotes a partial balancing edge.



Determining whether Σ has a partial balancing edge is easy.

Lecture 2

2.2. Switching.



- *Switching function*: $\zeta : V \to \{+, -\}$.
- *Switched signature*:
  $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where $e = vw$.
- *Switched signed graph*: $\Sigma^\zeta := (|\Sigma|, \sigma^\zeta)$.
  Note: $\Sigma^\zeta = \Sigma^{-\zeta}$.
- *Switching* $X \subseteq V$ means: negate every edge in $E(X, X^c)$.
- The switched graph is $\Sigma^X = \Sigma^{X^c}$.
  $\Sigma^X = \Sigma^\zeta$ where $\zeta(v) := -$ iff $v \in X$.

Proposition 2.5.
(a) *Switching preserves the signs of closed walks. So,* $\mathcal{B}(\Sigma^\zeta) = \mathcal{B}(\Sigma)$.
(b) *If* $|\Sigma_1| = |\Sigma_2|$ *and* $\mathcal{B}(\Sigma_1) = \mathcal{B}(\Sigma_2)$, *then* $\exists\ \zeta$ *such that* $\Sigma_2 = \Sigma_1^\zeta$.

Proof of (a) *by formula.*
Let $W = v_0e_0v_1e_1v_2 \cdots v_{n-1}e_{n-1}v_0$ be a closed walk. Then

$$\begin{aligned}\sigma^\zeta(W) &= \big[\zeta(v_0)\sigma(e_0)\zeta(v_1)\big]\big[\zeta(v_1)\sigma(e_1)\zeta(v_2)\big]\ldots\big[\zeta(v_{n-1})\sigma(e_{n-1})\zeta(v_0)\big]\\ &= \sigma(e_0)\sigma(e_1)\cdots\sigma(e_{n-1}) = \sigma(W).\end{aligned}$$ □

Proof of (b) *by defining a switching function.*
Pick a spanning tree T and a vertex v_0. Define

$$\zeta(v) := \sigma_1(T_{v_0v})\sigma_2(T_{v_0v})$$

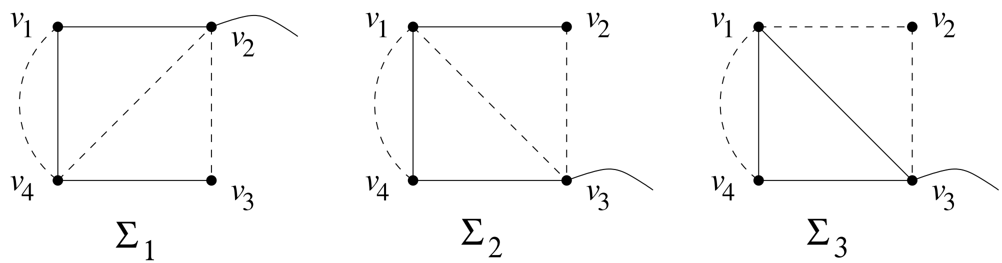
where T_{v_0v} is the path in T from v_0 to v. It is easy to calculate that $\Sigma_1^\zeta = \Sigma_2$. □

Equivalence relations:

- Σ_1 and Σ_2 are *switching equivalent*, $\Sigma_1 \sim \Sigma_2$, if $|\Sigma_1| = |\Sigma_2|$ and $\exists\ \zeta$ such that $\Sigma_1^\zeta \cong \Sigma_2$.
- The equivalence class $[\Sigma]$ is the *switching class* of Σ.

- Σ_1 and Σ_2 are *switching isomorphic*, $\Sigma_1 \simeq \Sigma_2$, if Σ_1 is isomorphic to a switching of Σ_2.
- The equivalence class of Σ is its *switching isomorphism class*.

Example: $\Sigma_2 \sim \Sigma_3$ but $\Sigma_1 \not\sim \Sigma_2, \Sigma_3$. $\Sigma_1 \simeq \Sigma_2 \simeq \Sigma_3$.



Proposition 2.6.
$\sim$ is an equivalence relation on signatures of a fixed underlying graph.
$\simeq$ is an equivalence relation on signed graphs.

Proof. Obvious! □

Corollary 2.7. *Σ is balanced $\iff$ it has no half edges and it is $\sim +|\Sigma|$.*

Two consequences of Corollary 2.7.

Short Proof of Harary's Balance Theorem.
Σ has the form stated in the theorem $\iff$ it is $(+|\Sigma|)^{V_1}$
$\iff$ it is a switching of $+|\Sigma|$ $\iff$ (by Proposition 2.5) it is balanced. □

Algorithm to detect balance.
Assume Σ is connected.
Apply the proof of Proposition 2.5(ii) to determine whether Σ can be switched to all positive. That is:

(1) Choose a spanning tree T and a vertex v_0.
(2) Calculate the function $\zeta(v) = \sigma(T_{v_0 v})$ of that proof.
(3) Switch by ζ.
(4) Look for negative non-tree edges.

$$\Sigma \text{ is balanced} \iff \text{all non-tree edges are positive.}$$

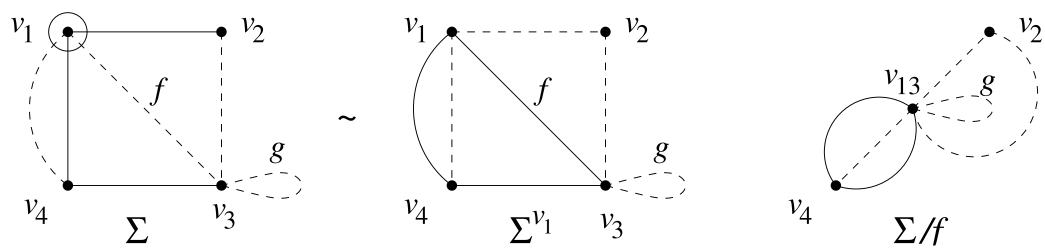
2.3. **Deletion, contraction, and minors.**

$R, S \subseteq E$.

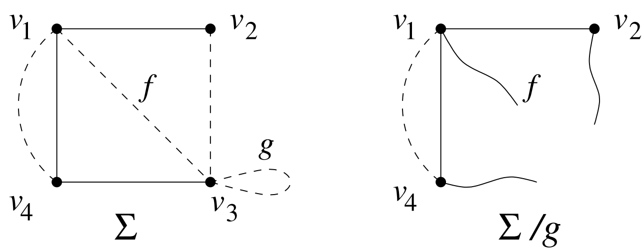
- The *deletion* of S is $\Sigma \setminus S := (V, S^c, \sigma|_{S^c})$.
- The *contraction* of S is Σ/S, to be defined in the next slides.

2.3.1. *Contracting an edge e.*

- A positive link:
  Delete e, identify its endpoints;
  do not change any edge signs.
  (= contraction in an unsigned graph.)
- A negative link:
  Switch Σ by a switching function ζ,
  chosen so e is positive in Σ^ζ.
  Then contract e (as a positive link).



- A positive loop or a loose edge: Delete e.
- A negative loop or half edge at v:
  Delete v and e.
  Other edges at v lose their endpoint v.



Lemma 2.8. *In Σ any two contractions of a link e are switching equivalent. The contraction of a link in a switching class is a well defined switching class.*

2.3.2. *Contracting an edge set S.*

$$E(\Sigma/S) := E \setminus S,$$
$$V(\Sigma/S) := \pi_{\mathrm{b}}(\Sigma|S) = \pi_{\mathrm{b}}(S),$$
$$V_{\Sigma/S}(f) = \{W \in \pi_{\mathrm{b}}(S) : w \in V_\Sigma(f) \text{ and } w \in W \in \pi_{\mathrm{b}}(S)\}.$$

Switch Σ to Σ^ζ so every balanced component of S is all positive. Then

$$\sigma_{\Sigma/S}(e) := \sigma^\zeta(e).$$

Lemma 2.9.

(a) *All contractions Σ/S (by different choices of ζ) are switching equivalent. Any switching of one contraction is another contraction. Any contraction of a switching of Σ is a contraction of Σ.*

(b) *If $|\Sigma_1| = |\Sigma_2|$, $S \subseteq E$ is balanced in Σ_1 and Σ_2, and $\Sigma_1/S \sim \Sigma_2/S$, then $\Sigma_1 \sim \Sigma_2$.*

(c) *For $e \in E$, $[\Sigma/e] = [\Sigma/\{e\}]$.*

2.3.3. *Minors.*

A *minor* is any contraction of any subgraph.

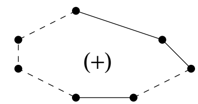
Theorem 2.10 (Zaslavsky, 1982)**.** *The result of any sequence of deletions and contractions of edge and vertex sets of* Σ *is a minor of* Σ.

Proof. Technical but not deep. $\square$

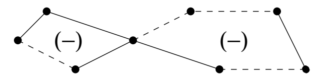
2.4. **Frame circuits.**

A *frame circuit* of Σ is

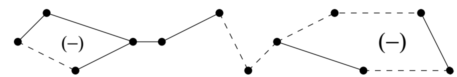
- a positive circle or a loose edge, or



- a pair of negative circles that intersect in precisely one vertex and no edges (a *tight handcuff circuit*), or



- a pair of disjoint negative circles together with a minimal path that connects them (a *loose handcuff circuit*).



A half edge = a negative loop in everything that concerns frame circuits.

A frame circuit in $+\Gamma$ is a circle.

Proposition 2.11. *Σ contains a loose handcuff circuit $\iff$ there is a component of Σ that contains two disjoint negative circles.*

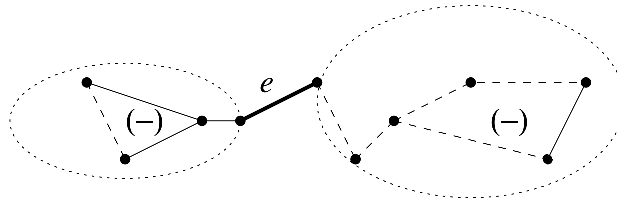
Proof. Elementary (my dear Watson). $\square$

But the next is less elementary.

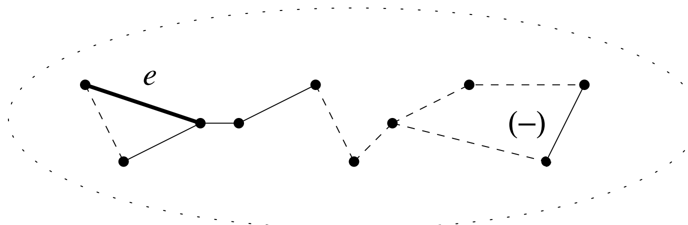
Proposition 2.12. *Let $e \in$ an unbalanced component of Σ. Then $e \in$ a frame circuit $\iff$ e is* not *a partial balancing edge.*

Proof. Nec. Suppose $e \in$ frame circuit C.

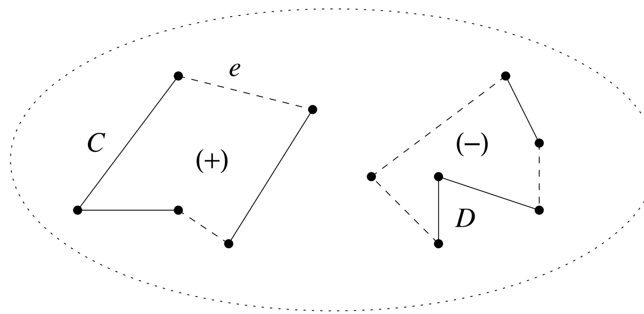
- If e is an isthmus of C: If $\Sigma \setminus e$ is connected, it contains the negative circles of C. If $\Sigma \setminus e$ is disconnected, each of its two components contains one negative circle of C. Therefore, e is not a partial balancing edge.



- If $e \in$ a circle in C, then $\Sigma \setminus e$ is connected. C is unbalanced $\implies$ $C \setminus e$ is unbalanced $\implies$ $\Sigma \setminus e$ is unbalanced $\implies$ e is not a partial balancing edge.



- But suppose C is a positive circle. As there is a negative circle D in Σ', for e to be a partial balancing edge it must belong to D; this leads to a contradiction.



Suff. If e is not a partial balancing edge; we produce a frame circuit C containing e. $\square$

2.5. **Closure and closed sets.**

Ordinary graphs: Closure of an edge set is an important operation, and is easy.
Signed graphs: Closure exists, but more complicated.

2.5.1. *Closure in signed graphs.*

For $S \subseteq E$:

Balance-closure:

$$\operatorname{bcl}(S) := S \cup \{e \in S^c : \exists \text{ a positive circle } C \subseteq S \cup e \text{ such that } e \in C\} \cup E_0(\Sigma).$$

Closure: $S_1, \ldots, S_k$ are the balanced components of S:

$$\operatorname{clos}(S) := \big(E{:}V_0(S)\big) \cup \Big(\bigcup_{i=1}^{k} \operatorname{bcl}(S_i)\Big) \cup E_0(\Sigma).$$

S is *closed* if $\operatorname{clos} S = S$. We write

$$\operatorname{Lat}\Sigma := \{S \subseteq E : S \text{ is closed}\},$$

$\operatorname{Lat}\Sigma$ is a lattice, partially ordered by set inclusion.

A half edge = a negative loop in everything that concerns closure.

Properties.

Lemma 2.14.
$\mathrm{bcl}(S)$ *is balanced* $\iff$ S *is balanced.*
If S is balanced, $\mathrm{bcl}(\mathrm{bcl}\,S) = \mathrm{bcl}(S) = \mathrm{clos}(S)$.

Lemma 2.15. $\pi_{\mathrm{b}}(\mathrm{clos}\,S) = \pi_{\mathrm{b}}(\mathrm{bcl}\,S) = \pi_{\mathrm{b}}(S)$ *and* $V_0(\mathrm{clos}\,S) = V_0(\mathrm{bcl}\,S) = V_0(S)$.

Power set $\mathcal{P}(E)$: the class of all subsets of E.

An *abstract closure operator* is $J : \mathcal{P}(E) \to \mathcal{P}(E)$ such that
(C1) $J(S) \supseteq S$ for every $S \subseteq E$ (increase).
(C2) $R \subseteq S \implies J(R) \subseteq J(S)$ (isotonicity).
(C3) $J(J(S)) = J(S)$ (idempotence).

Theorem 2.16. clos *is an abstract closure operator on* $E(\Sigma)$.

clos_Σ has the *exchange property* of matroid theory, which means:

Theorem 2.17. *For* $S \subseteq E$,

$$\mathrm{clos}\,S = S \cup \{e \notin S : \exists \textit{ a frame circuit } C \textit{ such that } e \in C \subseteq S \cup e\}.$$

Proof. Necessity. Assume $e \in \mathrm{clos}\,S$. We must find C. It takes some effort.

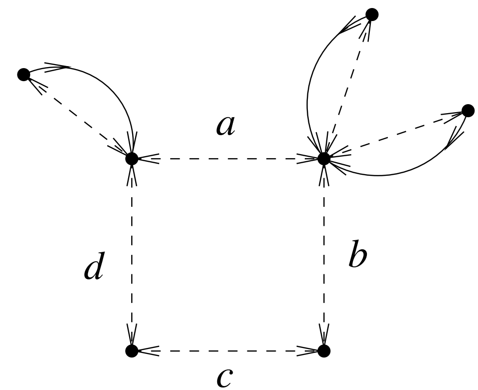
Sufficiency. Assuming a circuit C exists, we want to prove that $e \in \mathrm{clos}\,S$. Not difficult.

Both parts depend on Proposition 2.12. □

2.6. Oriented signed graphs = bidirected graphs.

- *Bidirected graph*: each end of each edge has an independent direction.
  - B ('Beta') $= (\Gamma, \tau)$ where $\tau : \{\text{edge ends}\} \to \{+, -\}$.
  - The directions on e *agree* when $\tau(v, e) = -\tau(w, e)$.
  - $|\mathrm{B}|$ = underlying graph.

- *Orientation* of Σ: a direction for each *end* of each edge.
  - Positive e: the directions on e agree.
  - Negative e: the directions on e disagree:
    - Both point towards the middle of e (an *introverted* edge) or
    - both away from the middle (an *extraverted* edge).



- $\sigma_{\mathrm{B}}(e) := -\tau(v, e)\tau(w, e)$.
- Σ_{B} = signed graph $(|\mathrm{B}|, \sigma_{\mathrm{B}})$.
- *Switching*: $\mathrm{B}^{\zeta} := (|\mathrm{B}|, \tau^{\zeta})$ where $\tau^{\zeta}(v, e) := \tau(v, e)\zeta(v)$.

Lemma 2.18. $\Sigma_{\mathrm{B}^{\zeta}} = (\Sigma_{\mathrm{B}})^{\zeta}$.

- *Source* vertex: All arrows point in: $\tau(v, e) = +,\ \forall\ (v, e)$.
- *Sink* vertex: All arrows point away: $\tau(v, e) = -,\ \forall\ (v, e)$.
- *Acyclic orientation*: Every frame circuit C in (Σ, τ) has a source or a sink.

3. Geometry and Matrices

$V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$. $\mathbf{F}$ is any field.

3.1. **Vectors for edges.** $e \mapsto$ vector $\mathbf{x}(e) \in \mathbf{F}^n$:

$$
\begin{array}{c}
\\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\
\end{array}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \\ \mp\sigma(e) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\quad
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \\ \mp 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\quad
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\qquad
\begin{array}{c}
\\ \\ \\ i \\ \\ \\ \\
\end{array}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \mp \sigma(e) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\qquad
\begin{array}{c}
\\ \\ \\ i \\ \\ \\ \\
\end{array}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\qquad
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

link $e{:}v_iv_j$, $\quad + $ link, $\quad -$ link, $\quad$ loop $e{:}v_iv_i$, $\quad$ half edge $e{:}v_i$, $\quad$ loose edge.

Define $\mathbf{x}(S) := \{\mathbf{x}(e) : e \in S\} \subseteq \mathbf{F}^n$.

Theorem 3.1. *Let $S \subseteq E(\Sigma)$.*
(a) *If* $\operatorname{char} \mathbf{F} \neq 2$, *$\mathbf{x}(S)$ is linearly dependent $\iff$ S contains a frame circuit.*
(b) *If* $\operatorname{char} \mathbf{F} = 2$, *$\mathbf{x}(S)$ is linearly dependent $\iff$ S contains a circle or loose edge.*

Corollary 3.2. *If* $\operatorname{char} \mathbf{F} \neq 2$, *the minimal linearly dependent subsets of $\mathbf{x}(E)$ are the sets $\mathbf{x}(C)$ where C is a frame circuit.*

Call $S \subseteq E(\Sigma)$ *independent* if $\mathbf{x}(S)$ is linearly independent over $\mathbf{F}$ when $\operatorname{char} \mathbf{F} \neq 2$.

Corollary 3.3. *$S \subseteq E(\Sigma)$ is independent $\iff$ it does not contain a frame circuit.*

Define $\langle X \rangle :=$ vector subspace generated by $X \subseteq \mathbf{F}^n$.
Then the set of subspaces generated by subsets of E,

$$\mathcal{L}_{\mathbf{F}}(\Sigma) := \{\langle X \rangle : X \subseteq \mathbf{x}(E)\},$$

is a lattice, partially ordered by set inclusion.

Corollary 3.4. *Assume* $\operatorname{char} \mathbf{F} \neq 2$.
Then $\mathbf{x}(E) \cap \langle \mathbf{x}(S) \rangle = \mathbf{x}(\operatorname{clos} S)$.
Thus, $\mathcal{L}_{\mathbb{R}}(\Sigma) \cong \operatorname{Lat} \Sigma$.

Rank function:

$$\operatorname{rk} S := n - b(S) \qquad \text{for } S \subseteq E.$$
$$\operatorname{rk} \Sigma := \operatorname{rk} E = n - b(\Sigma).$$

Theorem 3.5. *If* $\operatorname{char} \mathbf{F} \neq 2$, $\dim\langle \mathbf{x}(S) \rangle = \operatorname{rk} S$.

Proof. Use Corollary 3.3 to compare

- the minimum number of edges required to generate S by closure in Σ,
- the minimum number of vectors $\mathbf{x}(e)$ required to generate $\langle \mathbf{x}(S) \rangle$. □

Orientation.

Choosing $\mathbf{x}(e)$ or $-\mathbf{x}(e)$ $\longleftrightarrow$ choosing an orientation of Σ.

Orient Σ as $\mathrm{B} = (|\Sigma|, \tau)$, and define

$$\eta(v,e) := \sum_{\text{incidences } (v,e)} \tau(v,e). \tag{3.1}$$

Then $\mathbf{x}(e)_v = \eta(v,e)$.

Conversely, if we choose $\mathbf{x}(e)$ first and then define τ to orient Σ, τ will satisfy (3.1).

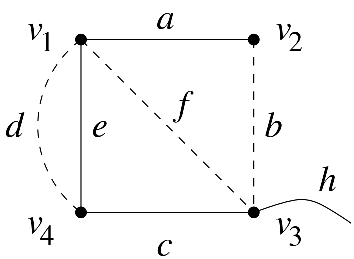
Lecture 3

3.2. **The incidence matrix** $\mathrm{H}(\Sigma)$. (‘Eta’.)

$$\mathrm{H}(\Sigma) = \begin{bmatrix} \mathbf{x}(e_1) & \mathbf{x}(e_2) & \cdots & \mathbf{x}(e_m) \end{bmatrix},$$

where $m := |E|$.

Σ_4



$$\mathrm{H}(\Sigma_4) = \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & h \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Theorem 3.6. *If* $\operatorname{char} \mathbf{F} \neq 2$, *then*

$$\operatorname{rank}(\mathrm{H}(\Sigma)) = \operatorname{rk} \Sigma := n - b(\Sigma) \quad \textit{and} \quad \operatorname{rank}(\mathrm{H}(\Sigma|S)) = \operatorname{rk} S.$$

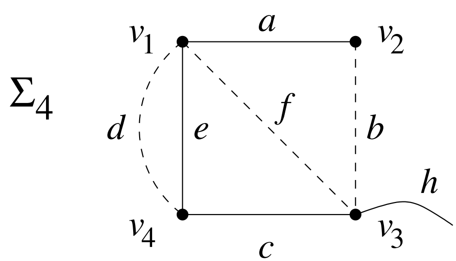
Proof.
Column rank = dim(span of the columns corresponding to S) = dim(span of $\mathbf{x}(S)$).
Apply Theorem 3.5. □

3.3. **Frame matroid** $G(\Sigma)$.

An abstract way of describing vector-like closure properties including closure in signed graphs.

3.4. **Adjacency and Laplacian (Kirchhoff) matrices.**

- *Adjacency matrix* $A(\Sigma) = (a_{ij})_{n\times n}$, where
  $a_{ii} := 0$, and $a_{ij} :=$ (# positive edges v_iv_j) $-$ (# negative edges v_iv_j) if $i \neq j$.
  - A does not change if a a negative digon is deleted from Σ.
  - Σ is *reduced* if it has no negative digon.
  - $\bar{\Sigma}$: the reduced signed graph with $A(\bar{\Sigma}) = A(\Sigma)$.
- *Degree matrix* $D(|\Sigma|)$: the diagonal matrix with $d_{ii} = d_{|\Sigma|}(v_i)$.
- *Laplacian matrix* $L(\Sigma) := D(|\Sigma|) - A(\Sigma)$.



$$A(\Sigma_4) = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad L(\Sigma_4) = \begin{pmatrix} 4 & -1 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

Graphic examples:

- $A(\Gamma) = A(+\Gamma)$.
- *Laplacian* matrix of Γ: $L(+\Gamma)$.
- *Signless Laplacian* matrix of Γ: $L(-\Gamma)$.

Proposition 3.7. $L(\Sigma) = \mathrm{H}(\Sigma)\mathrm{H}(\Sigma)^{\mathrm{T}}$.

Theorem 3.8.
The eigenvalues of $A(\Sigma)$ are real.
The eigenvalues of $L(\Sigma)$ are real and non-negative.

Proof.
$A(\Sigma)$ is symmetric.
$\mathrm{H}(\Sigma)\mathrm{H}(\Sigma)^{\mathrm{T}}$ is positive semidefinite. □

A use for the Laplacian (off topic).

Theorem 3.9 (Matrix-Tree Theorem for Signed Graphs)**.** *Let*
$b_i :=$ *number of sets of n independent edges in Σ that contain exactly i circles.*
Then $\det L(\Sigma) = \sum_{i=0}^{n} 4^i b_i$.

3.5. Arrangements of hyperplanes.

- *Arrangement of hyperplanes* $\mathcal{H} = \{h_1, h_2, \dots, h_m\}$: finite set of hyperplanes in $\mathbb{R}^n$.
- *Region* of $\mathcal{H}$: a connected component of $\mathbb{R}^n \setminus \big(\bigcup_{k=1}^m h_k\big)$.
- $r(\mathcal{H}) :=$ number of regions.
- *Intersection lattice* $\mathcal{L}(\mathcal{H})$: set of all intersections of subsets of $\mathcal{H}$, partially ordered by $s \leq t \iff t \subseteq s$.
- *Characteristic polynomial*:

$$p_{\mathcal{H}}(\lambda) := \sum_{\mathcal{S} \subseteq \mathcal{H}} (-1)^{|\mathcal{S}|} \lambda^{\dim \mathcal{S}}, \qquad \text{where } \dim \mathcal{S} := \dim \Big(\bigcap_{h_k \in \mathcal{S}} h_k \Big). \tag{3.2}$$

Theorem 3.10. $r(\mathcal{H}) = (-1)^n p_{\mathcal{H}}(-1)$.

(In T.Z.'s Ph.D. thesis.)

Signed-graphic hyperplane arrangement:

Σ with $E = \{e_1, e_2, \ldots, e_m\}$ forms $\mathcal{H}[\Sigma] := \{h_1, h_2, \ldots, h_m\}$ where

$$h_k = \mathbf{x}(e_k)^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}(e_k) \cdot \mathbf{x} = 0\};$$

$$h_k \text{ has the equation } \begin{cases} x_j = \sigma(e_k)x_i, & \text{if link or loop } e_k{:}v_iv_j, \\ x_i = 0, & \text{if half edge } e_k{:}v_i, \\ \ 0 = 0, & \text{if loose edge } e_k{:}\varnothing. \end{cases}$$

($0 = 0$ gives $\mathbb{R}^n$, the 'degenerate hyperplane'.)

Lemma 3.11.

Let $\mathcal{S} = \{h_{i_1}, \ldots, h_{i_l}\} \subseteq \mathcal{H}[\Sigma] \longleftrightarrow S = \{e_{i_1}, \ldots, e_{i_l}\}$. *Then* $\dim \bigcap \mathcal{S} = b(S)$.

Proof. Apply vector space duality to Theorem 3.5. □

Theorem 3.12. $\mathcal{L}(\mathcal{H}[\Sigma]) \cong \mathcal{L}_{\mathbb{R}}(\Sigma) \cong \operatorname{Lat} \Sigma$.

Proof.

$\mathcal{L}(\mathcal{H}[\Sigma]) \cong \mathcal{L}_{\mathbb{R}}(\Sigma)$ is standard vector-space duality.

$\mathcal{L}_{\mathbb{R}}(\Sigma) \cong \operatorname{Lat} \Sigma$ is from Corollary 3.4. □

Acyclic orientations reappear:

The regions of $\mathcal{H}[\Sigma] \longleftrightarrow$ *the acyclic orientations of* Σ.

Define

$$R(\tau) := \big\{ \mathbf{x} \in \mathbb{R}^n : \tau(v_i, e)x_i + \tau(v_j, e)x_j > 0 \text{ for every edge } e, \text{ where } V(e) = \{v_i, v_j\} \big\}.$$

Theorem 3.13.

(a) *$R(\tau)$ is nonempty $\iff$ τ is acyclic.*

(b) *Every region is an $R(\tau)$ for some acyclic τ.*

4. Coloring

- Color set: $\Lambda_k := \{\pm 1, \pm 2, \ldots, \pm k\} \cup \{0\}$
- Zero-free color set: $\Lambda_k^* := \{\pm 1, \pm 2, \ldots, \pm k\}$.
- A *k-coloration* (or *k-coloring*) of Σ: a function $\gamma : V \to \Lambda_k$.
- γ is *zero free* if it does not use the color 0.
- γ is *proper* if

$$\begin{cases} \gamma(v_j) \neq \sigma(e)\gamma(v_i), & \text{for a link or loop } e = v_i v_j, \\ \gamma(v_i) \neq 0, & \text{for a half edge } e \text{ at } v_i, \end{cases}$$

and there are no loose edges.

4.1. Chromatic polynomials.

For an integer $k \geq 0$, define

$$\chi_\Sigma(2k+1) := \ \# \text{ proper } k\text{-colorations},$$

and

$$\chi_\Sigma^*(2k) := \ \# \text{ proper zero-free } k\text{-colorations}.$$

The following theorem (except (2)) is the same as with ordinary graphs.

Theorem 4.1. *Properties of the chromatic polynomials:*

(1) *Unitarity:*

$$\chi_{\emptyset}(2k+1) = \chi^*_{\emptyset}(2k) = 1 \quad \textit{for all } k \geq 0.$$

(2) *Switching Invariance: If* $\Sigma \sim \Sigma'$*, then*

$$\chi_{\Sigma}(2k+1) = \chi_{\Sigma'}(2k+1) \quad \textit{and} \quad \chi^*_{\Sigma}(2k) = \chi^*_{\Sigma'}(2k).$$

(3) *Multiplicativity: If* Σ *is the disjoint union of* Σ_1 *and* Σ_2*, then*

$$\chi_{\Sigma}(2k+1) = \chi_{\Sigma_1}(2k+1)\chi_{\Sigma_2}(2k+1)$$

and

$$\chi^*_{\Sigma}(2k) = \chi^*_{\Sigma_1}(2k)\chi^*_{\Sigma_2}(2k).$$

(4) *Deletion-Contraction: If* e *is a link, a positive loop, or a loose edge, then*

$$\chi_{\Sigma}(2k+1) = \chi_{\Sigma\setminus e}(2k+1) - \chi_{\Sigma/e}(2k+1)$$

and

$$\chi^*_{\Sigma}(2k) = \chi^*_{\Sigma\setminus e}(2k) - \chi^*_{\Sigma/e}(2k).$$

Theorem 4.2.
$\chi_\Sigma(\lambda)$ is a polynomial function of $\lambda = 2k+1 > 0$:

$$\chi_\Sigma(\lambda) = \sum_{S \subseteq E} (-1)^{|S|} \lambda^{b(S)}. \tag{4.1}$$

$\chi^*_\Sigma(\lambda)$ is a polynomial function of $\lambda = 2k \geq 0$:

$$\chi^*_\Sigma(\lambda) = \sum_{S \subseteq E:\, balanced} (-1)^{|S|} \lambda^{b(S)}. \tag{4.2}$$

Proof. Apply Theorem 4.1 and induction on $|E|$ and n. □

Therefore, we can evaluate $\chi_\Sigma(-1)$.

A geometrical application of the chromatic polynomial.

Lemma 4.3. $\chi_\Sigma(\lambda) = p_{\mathcal{H}[\Sigma]}(\lambda)$.

Proof. Compare (4.1) and (3.2). □

Theorem 4.4. *The number of acyclic orientations of Σ and the number of regions of $\mathcal{H}[\Sigma]$ are both equal to $(-1)^n \chi_\Sigma(-1)$.*

The lecture notes present some ways to simplify the computation of chromatic polynomials.

4.2. **Chromatic numbers.**

The lectures are short; see the lecture notes.

Almost any question about chromatic numbers is open.

5. Catalog of Examples

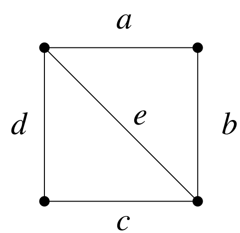
The lecture notes present several general examples, for which there is no time in the lectures.

6. Line Graphs

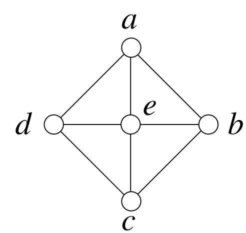
The *line graph* of a graph is $\Lambda(\Gamma)$: $\begin{cases} V(\Lambda(\Gamma)) = E(\Gamma), \\ e \sim f \text{ if they have a common endpoint.} \end{cases}$

(Link graphs only!)

Γ



$\Lambda(\Gamma)$

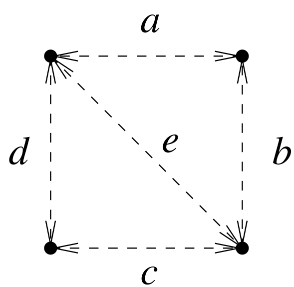


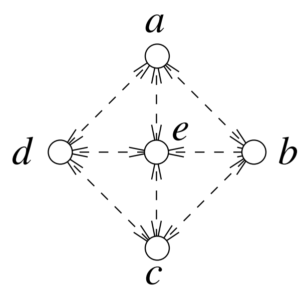
6.1. **Bidirected line graphs and switching classes.**

Line graph of a bidirected graph B:
$\Lambda(\mathrm{B}) := (\Lambda(|\mathrm{B}|), \tau_\Lambda)$ where

$$\tau_\Lambda(e, ef) := \tau(v, e) \qquad \text{if } e \sim f \text{ at } v.$$

Line graph of Σ: Orient Σ as $\mathrm{B} = (|\Sigma|, \tau)$. Form $\Lambda(\mathrm{B})$.





Different τ give different $\Lambda(\mathrm{B})$, which may have different signed graphs $\Sigma_{\Lambda(\mathrm{B})}$.

Lemma 6.1. *Any orientations of any two switchings of* Σ *have line graphs that are switching equivalent.*

Proof: See the lecture notes.

Reorienting e as edge in B $\longleftrightarrow$ switching e as vertex in $\Lambda(\mathrm{B})$.
Therefore, $\Lambda(\Sigma)$ must be a *switching class* of signatures of $\Lambda(|\Sigma|)$.

Theorem 6.2.
$\Lambda($*switching class of signed graphs*$)$ = *switching class of signed graphs.*

Proof. $\Sigma_1 \sim \Sigma_2 \implies \Lambda(\Sigma_1, \tau_1) \sim \Lambda(\Sigma_2, \tau_2)$ by Lemma 6.1.
The converse follows from Proposition 2.5(ii). ∎

Notation:
$\Lambda[\Sigma] :=$ switching class of line graphs of the signed graphs in the switching class $[\Sigma]$.

All-negative signature:

In connection with line graphs, an ordinary graph Γ is $-\Gamma$.

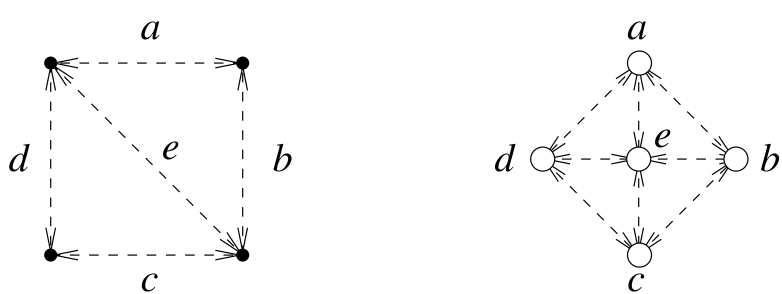
Reason:

$$\Lambda(-\Gamma) = -\Lambda(\Gamma)$$

because:

Proposition 6.3. *If* Γ *is a link graph, then* $\Lambda[-\Gamma] = [-\Lambda(\Gamma)]$.

Proof. Orient $-\Gamma$ so every edge is extraverted; that is, $\tau(v, e) \equiv +$. Then in $\Lambda(-\Gamma, \tau)$, every edge is extraverted; thus, the signed graph underlying $\Lambda(-\Gamma, \tau)$ has all negative edges. $\square$



6.2. Adjacency matrix and eigenvalues.

Theorem 6.4. *For a bidirected link graph Σ, $A(\Lambda(\Sigma)) = 2I - \mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma)$.*

Proof by matrix multiplication.

$$\big[\mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma)\big]_{(j,j)} = \sum_{v_i} \eta(v_i, e_j)^2 = 1 + 1 = 2,$$

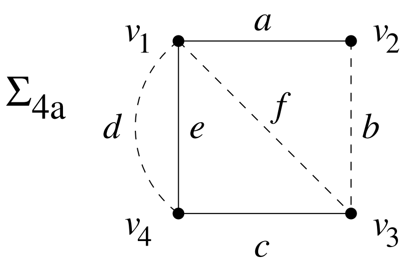
$$\begin{aligned}\big[\mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma)\big]_{(j,k)} &= \sum_{v_i} \eta(v_i, e_j)\eta(v_i, e_k) \\ &= \begin{cases} 0 & \text{if } e_j \not\sim e_k, \\ \tau(v_i, e_j)\tau(v_i, e_k) = -\sigma(e_j e_k) & \text{if they are adjacent at } v_m. \end{cases}\end{aligned}$$ □

Therefore, $\mathbf{x}(e_j) \cdot \mathbf{x}(e_k)$ equals 2 if $j = k$ and $-\sigma(e_j e_k)$ if $j \neq k$. Thus, $2I - A(\Lambda(\Sigma))$ is a Gram matrix of vectors with length $\sqrt{2}$.

Corollary 6.5. *All the eigenvalues of a line graph of a signed graph are ≤ 2.*

Proof. $\mathrm{H}^{\mathrm{T}}\mathrm{H}$ has non-negative real eigenvalues. Apply Proposition 6.4. □

Example (using a particular choice of orientation of Σ):



$$A(\Lambda(\Sigma_{4\mathrm{a}})) = \begin{pmatrix} 0 & 1 & 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 1 & 0 \end{pmatrix}$$

6.3. Reduced line graphs and induced non-subgraphs.

Σ has a negative digon $\{e, f\} \implies \Lambda(\Sigma)$ has a negative digon between e and f. Thus $e \not\sim f$ in $\bar{\Lambda}(\Sigma)$, and $A(\Lambda(\Sigma))_{e,f} = 0$.

Conclusion: For eigenvalues, one should look at reduced line graphs.

1970 (Beineke, Gupta): A simple graph is a line graph $\iff$ it has no induced subgraph that is one of 9 graphs (of order ≤ 6).

1990: Chawathe and Vijayakumar found the 49 excluded induced switching classes (all of order ≤ 6) for reduced line graphs of signed graphs.

7. Angle Representations

All graphs are simple.
$\hat{\mathbf{y}}$:= unit vector in the same direction: $\hat{\mathbf{y}} := \|\mathbf{y}\|^{-1}\mathbf{y}$.

Angle representation of Σ:
$\boldsymbol{\rho} : V \to \mathbb{R}^d$ such that

$$\hat{\boldsymbol{\rho}}(v) \cdot \hat{\boldsymbol{\rho}}(w) = \frac{a_{vw}}{\nu} = \begin{cases} 0, & \text{if } vw \text{ is not an edge and } v \neq w, \\ +1/\nu, & \text{if } vw \text{ is a positive edge, and} \\ -1/\nu, & \text{if } vw \text{ is a negative edge,} \end{cases}$$

where $\nu > 0$.
One can multiply $\boldsymbol{\rho}(v)$ by any positive real number.
E.g., make all $\|\boldsymbol{\rho}(v)\| = 1$, or 2.

Switching v in Σ:

$$\text{replaces } \boldsymbol{\rho}(v) \text{ by } -\boldsymbol{\rho}(v).$$

A *Gramian* angle representation of Σ means:

$$\boldsymbol{\rho}(v) \cdot \boldsymbol{\rho}(w) = a_{vw}.$$

Therefore, $\|\boldsymbol{\rho}(v)\| \cdot \|\boldsymbol{\rho}(w)\| = \nu$ for adjacent vertices.

(*Anti-Gramian*: Gramian angle representation of $-\Sigma$.
Vijayakumar et al. use anti-Gramian representations.)

Proposition 7.1. *Let $\boldsymbol{\rho}$ be a Gramian angle representation of connected Σ.*
(a) *If Σ is not bipartite: all $\|\boldsymbol{\rho}(v)\| = \sqrt{\nu}$.*
(b) *If Σ is bipartite:* $\|\boldsymbol{\rho}(v)\| = \begin{cases} \alpha & \text{if } v \in V_1, \\ \nu/\alpha & \text{if } v \in V_2. \end{cases}$

Then $\boldsymbol{\rho}'(v) = \hat{\boldsymbol{\rho}}(v)\sqrt{\nu}$ *is an angle representation with all* $\|\boldsymbol{\rho}'(v)\| = \sqrt{\nu}$.

Normalized Gramian angle representation: all vectors have the same length. Then

$$\big(\boldsymbol{\rho}(v) \cdot \boldsymbol{\rho}(w)\big)_{v,w \in V} = A(\Sigma) + \nu I.$$

Proposition 7.1 $\implies$ we can normalize any Gramian angle representation.

Theorem 7.2. *Σ has a Gramian (anti-Gramian) angle representation with constant ν $\iff$ the eigenvalues of Σ are $\geq -\nu$ (respectively, $\leq \nu$).*

Example 7.3.

The vector representation of Σ,

$$\mathbf{x} : E(\Sigma) \to \mathbb{R}^n,$$

is an anti-Gramian angle representation of $\bar{\Lambda}(\Sigma)$, the *reduced* line graph:

- $\boldsymbol{\rho} := \mathbf{x}$ since $V(\bar{\Lambda}(\Sigma)) = E(\Sigma)$.
- $\nu = 2$ and the angle $\theta = \pi/3$.
- Every $\|\mathbf{x}(e)\| = \sqrt{2}$.
- Inner products: $+1$ if $\sigma_\Lambda(ef) = -$, -1 if $\sigma_\Lambda(ef) = +$.
  (The signs reverse because the representation is anti-Gramian.)

Cameron, Goethals, Seidel, and Shult (1976a) used Gramian angle representations of unsigned graphs to find all graphs with eigenvalues ≥ -2. G.R. Vijayakumar et al. extended that to signed graphs (anti-Gramian).

The root system E_8 is

$$E_8 := D_8 \cup \left\{ \tfrac{1}{2}(\varepsilon_1, \ldots, \varepsilon_8) \in \mathbb{R}^8 : \varepsilon_i \in \{\pm 1\},\ \varepsilon_1 \cdots \varepsilon_8 = +1 \right\}.$$

Theorem 7.4. *Take an anti-Gramian angle representation of Σ with $\nu = 2$.*
(a) *It is $\mathbf{x}$ for $\bar{\Lambda}(\Sigma')$, or*
(a) *The representation is in E_8 and $|V(\Sigma)| \leq 184$.*

Proof. Vijayakumar (1987a) observed:
Cameron et al. $\implies$ an anti-Gramian angle representation having $\nu = 2$ is in D_n or E_8.

If in D_n: $\exists\ \Sigma'$ such that $\boldsymbol{\rho}$ is $\mathbf{x} : E(\Sigma') \to \mathbb{R}^n$. Then $\Sigma = \bar{\Lambda}(\Sigma')$.

If in E_8: $\|V(\Sigma)\| \leq$ number of pairs of opposite vectors in E_8, which $= 184$. $\square$

Corollary 7.5. *Σ (a signed simple graph) has all eigenvalues $\leq 2 \iff$ it is a reduced line graph of a signed graph or it has order ≤ 184.*

Eigenvalues $\implies$ whether Σ is a (reduced) line graph, with a finite number of exceptions!

Example 7.6.

Γ with $V = \{v_1, \dots, v_n\}$.

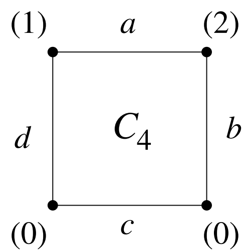
Cocktail party graph $CP_m := K_{2m} \setminus$ perfect matching.

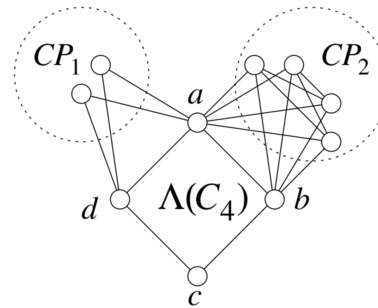
Generalized line graph

$$\Lambda(\Gamma; m_1, \dots, m_n) := \Lambda(\Gamma) \mathbin{\dot\cup} CP_{m_1} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} CP_{m_n}$$

with edges from every vertex in CP_{m_i} to every $v_iv_j \in V(\Lambda(\Gamma))$.

Example: C_4 and $\Lambda(C_4; 1, 2, 0, 0)$.





Hoffman (1977a): A generalized line graph has eigenvalues ≥ -2, just like a line graph.

Cameron et al.: no other graphs have eigenvalues ≥ -2 *except* a handful with anti-Gramian angle representations in E_8.

Corollary 7.5 $\implies$ *this fact*, because

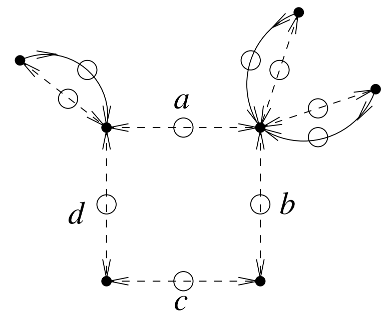
$$\bar{\Lambda}(\Gamma(m_1,\dots,m_n)) \text{ (signed l.g.) } = -\Lambda(\Gamma;m_1,\dots,m_n) \text{ (all-negative g.l.g.)},$$

where $\Gamma(m_1,\dots,m_n) := -\Gamma$ with m_i negative digons attached to v_i.

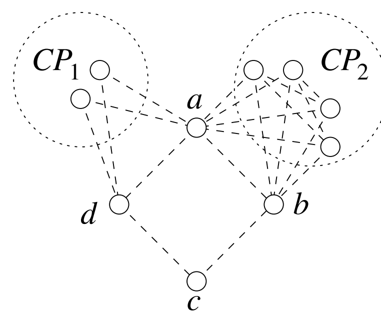
That is, $\Lambda(\Gamma;m_1,\dots,m_n)$ is a reduced line graph of a signed graph.

Example: $\Lambda(C_4;1,2,0,0) = -\bar{\Lambda}(C_4(1,2,0,0))$.

Extraverted C_4 in $C_4(1,2,0,0)$



$-\Lambda(C_4;1,2,0,0) = \bar{\Lambda}(C_4(1,2,0,0))$



The End

Keys to the Literature

Thomas Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas. *Electronic Journal of Combinatorics*, Dynamic Surveys in Combinatorics (1998), No. DS8 (electronic), vi + 151 pp. MR 2000m:05001a. Zbl 898.05001.
Current version (about 290 pp.) available at `http://www.math.binghamton.edu/zaslav/Bsg/`.

——, Glossary of signed and gain graphs and allied areas. *Electronic Journal of Combinatorics* (1998), Dynamic Surveys in Combinatorics, #DS9 (electronic), 41 pp. MR 2000m:05001b. Zbl 898.05002.