Six Signed Petersen Graphs

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The Petersen graph with signed edges makes a fascinating example of many aspects of signed graph theory. I will show that there are exactly six essentially different ways to sign P, find their automorphisms, show some ways in which their signs matter, color them, and mention two applications of signed graphs in which signed Petersen graphs have not yet made an appearance.

The Petersen graph is $P = (V, E)$ with vertex set $V = \{v_{ij} : 1 \leq i < j \leq 5\}$ and edge set $E = \{v_{ij}v_{kl} : \{i, j\} \cap \{k, l\} = \varnothing\}$. A *signed graph* is a pair $\Sigma := (\Gamma, \sigma)$ where Γ is a graph and $\sigma : E(\Gamma) \to \{+1, -1\}$ is an *(edge) signature* that labels each edge positive or negative. Hence, a *signed Petersen graph* is (P, σ); two examples are $+P := (P, +1)$, where every edge is positive, and $-P := (P, -1)$, where every edge is negative.

The sign of a circle (cycle, circuit, polygon) C is $\sigma(C) :=$ the product of the signs of the edges in C. The most essential fact about a signed graph is the set of circles that have negative sign. If this set is empty we call the signed graph *balanced*. Such a signed graph is equivalent to an unsigned graph in most ways. Harary [4] introduced signed graphs and balance and proved:

Proposition 1 *Σ is balanced if and only if its vertices can be divided into two sets so the positive edges are within a set and the negative edges are between the sets.*

Cartwright and Harary used signed graphs to model positive and negative relations in groups of people in social psychology [2].

Two signed graphs, $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$, are *switching equivalent* if $\Gamma_1 = \Gamma_2$ and there is a *switching function* $\tau : V_1 \to \{+1, -1\}$ such that $\sigma_2(vw) = \tau(v)\sigma_1(vw)\tau(w)$ for every edge vw; then we write $\sigma_2 = \sigma_1^\tau$. Σ_1 and Σ_2 are *isomorphic* if there is a graph isomorphism that preserves edge signs. They are *switching isomorphic* if Σ_2 is isomorphic to a switching of Σ_1; that is, there are a graph isomorphism θ :

$\Gamma_1 \to \Gamma_2$ and a switching function $\tau : V_1 \to \{+1, -1\}$ such that $\sigma_2(\theta(vw)) = \sigma_1^{\tau}(vw)$ for every edge. Switching equivalence and switching isomorphism are equivalence relations on signed graphs. Sozański (1976) and Zaslavsky (1981) proved:

Proposition 2 *Switching preserves circle signs. If two signatures of* Γ *have the same circle signs, then one is a switching of the other.*

For instance, Σ is balanced if and only if it is switching equivalent to the all-positive signature. Because of this lemma, switching equivalent signed graphs are in most ways the same.

Theorem 3 *There are exactly six signed Petersen graphs up to switching isomorphism.*

Two of them are $+P$ and $-P$. Two more are P_1, which has only one negative edge, and its negative $-P_1$, with only one positive edge. A fifth is $P_{2,2}$, which has two negative edges at distance 2. The last, $P_{3,2}$, has three negative edges, all at distance 2 from each other.

The proof of Theorem 3 makes use of properties of the class of balanced circles. One property is the degree of imbalance as measured by the *frustration index*, introduced by Abelson and Rosenberg [1] (the name is drawn from physics); it is defined as $l(\Sigma) :=$ the smallest number of edges whose deletion leaves a balanced signed graph. For instance, $l(\Sigma) = 0$ if and only if Σ is balanced.

The frustration index is probably the most popular measure of how unbalanced a signed graph is. In social psychology $l(\Sigma)$ is the minimum number of interpersonal relations to change to achieve balance. In the non-ferromagnetic Ising model of spin glass theory it determines the ground state energy of the spin glass.

Proposition 4 *Switching does not change* $l(\Sigma)$*.*

Thus, we can distinguish switching isomorphism classes by their having different frustration indices. This helps to prove the six signed P's are not switching isomorphic.

(P, σ)	$+P$	P_1	$P_{2,2}$	$-P_1$	$P_{3,2}$	$-P$
$l(P, \sigma)$	0	1	2	2	3	3

We need another method to distinguish $P_{2,2}$ from P_1 and $P_{3,2}$ from $-P_1$. One way is by their switching automorphisms; another is by the zero-free chromatic number.

An automorphism of a signed graph is an isomorphism with itself. A *switching automorphism* is a switching isomorphism with itself. Here are the automorphism and switching automorphism groups of our six signed Petersen graphs. The former is not invariant under switching, but the latter is. $\mathfrak{S}_k$ and $\mathfrak{A}_k$ are the symmetric and alternating groups on k letters; $\mathfrak{D}_k$ is the dihedral group of symmetries of the k-gon.

(P,σ)	$+P,\ -P$	$P_1,\ -P_1$	$P_{2,2}$	$P_{3,2}$
$\mathrm{Aut}(P,\sigma)$	$\mathfrak{S}_5$	$\mathfrak{D}_4$	$\mathfrak{S}_2 \times \mathfrak{S}_2$	$\mathfrak{S}_3$
$\mathrm{SwAut}(P,\sigma)$	$\mathfrak{S}_5$	$\mathfrak{D}_4$	$(\mathfrak{S}_2 \times \mathfrak{S}_3) \rtimes \mathfrak{S}_2$	$\mathfrak{A}_5$

A signed graph has a *chromatic polynomial* $\chi_\Sigma(y)$, which generalizes the chromatic polynomial of a graph. $\chi_\Sigma(2k+1)$ is defined as the number of proper colorings using colors $0, \pm 1, \ldots, \pm k$. Correspondingly there is a chromatic number $\chi(\Sigma)$. There is also a second, *zero-free chromatic polynomial* $\chi^*_\Sigma(2k)$, which counts proper colorings without the color 0, and a zero-free chromatic number $\chi^*(\Sigma)$. These polynomials and numbers are invariant under switching and isomorphism. Here are the two chromatic numbers for the six switching isomorphism types of signed Petersen graph:

(P,σ)	$+P$	P_1	$P_{2,2}$	$-P_1$	$P_{3,2}$	$-P$
$\chi(P,\sigma)$	1	1	1	1	1	1
$\chi^*(P,\sigma)$	2	2	2	1	2	1

A signed graph Σ is *clusterable* if its vertices can be partitioned into sets, called clusters, so that each edge within a cluster is positive and each edge between two clusters is negative. Davis [3] proposed this as a possibly more realistic alternative to balance as an ideal state of a social group, and he proved:

Proposition 5 *A signed graph is clusterable if and only if no circle has exactly one negative edge.*

Clusterability is the second property I discuss that is not invariant under switching. There are two ways to measure it. When Σ is clusterable, the smallest possible number of clusters is the *cluster number* $\mathrm{clu}(\Sigma)$. This is 1 when Σ is all positive and ≤ 2 if and only if Σ is balanced; in the all-negative case, $\mathrm{clu}(-\Gamma) = \chi(\Gamma)$, the chromatic number. Even if a signed graph is not clusterable, it becomes clusterable when enough edges are deleted; the smallest such number is the *clusterability index* $Q(\Sigma)$. The values of these numbers for signed Petersen graphs are given in the next table.

(P,σ)	$+P$	P_1	$P_{2,2}$	$-P_1$	$P_{3,2}$	$-P$
$Q(P,\sigma)$	0	1	2	0	3	0
$\mathrm{clu}(P,\sigma)$	1	–	–	3	–	3

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