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Tractable Partially Ordered Sets Derived from Root Systems and Biased Graphs

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Abstract. We study new posets Q obtained by removing from a geometric lattice L of a biased graph certain flats indexed by a simplicial complex $\mathcal{C}$. (One example of L is the lattice of flats of the vector matroid of a root system B_n.) We study the structure and compute the characteristic polynomial of Q. With certain choices of L and $\mathcal{C}$, including ones for which Q is a lattice interpolating between those of B_n and D_n, we observe curious relationships among the roots of the characteristic polynomials of Q, L, and $\mathcal{C}$.

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1. Introduction

We introduce a new kind of finite partially ordered set, derived from a certain type of geometric lattice via an indexing simplicial complex, whose Möbius function satisfies a simple reduction formula that in examples with high symmetry leads to exact evaluations.

Our posets originated in an attempt to understand the relationship between the exponents of the root systems B_n and D_n in $\mathbb{R}^n$. These root systems are defined by $D_n = \{\pm u_i \pm u_j : 1 \leq i < j \leq n\}$ and $B_n = D_n \cup \{\pm u_i : i \in [n]\}$, where u_i is the ith unit coordinate vector and $[n] = \{1, 2, \ldots, n\}$. Associated to each root system R of rank n there are n positive integers called its *exponents*. There are many ways to define them, some very combinatorial and some very algebraic. Best for our purposes is a definition in terms of Lat R, the lattice of flats spanned by subsets of R. The exponents of R are the roots of the characteristic polynomial

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of Lat R. (Those who are unfamiliar with this definition of exponents should see [9, 13] for a proof that it is equivalent to more familiar definitions.) One can check that the exponents of B_n and D_n are given by these lists:

$$B_n: 1, 3, 5, \ldots, 2n-3, 2n-1;$$
$$D_n: 1, 3, 5, \ldots, 2n-3, n-1.$$

We see that removal of all the basis vectors $\pm u_i$ from B_n changes only the last exponent and changes that exponent by n. The original motivation for this work was a desire to understand why this should be so. Given our definition of exponents, we must understand why only one root of the characteristic polynomial of Lat B_n changes when the lines generated by the u_i are removed.

We will consider a more general situation in which certain flats, indexed by an abstract simplicial complex, are removed from Lat B_n or more generally from matroids called "bias matroids of full biased graphs". Our results concern the characteristic polynomials of the partially ordered sets obtained in that way and, for certain matroids and indexing complexes, the behavior of their roots.

An example will give the flavor of our results. Let F be a field and ζ_M a primitive Mth root of unity in F, where $M \geq 1$. Define

$$D_n(M) = \left\{\zeta_M^k u_i - \zeta_M^l u_j : i, j \in [n],\ i \neq j, k, l \in \mathbb{Z}\right\} \subseteq F^n$$

and

$$B_n(M) = D_n(M) \cup \left\{\zeta_M^k u_i : i \in [n],\ k \in \mathbb{Z}\right\}.$$

For $R \subseteq F^n$ let Lat R be the lattice of all flats generated by subsets of R. (Then Lat $B_n(M)$ is the Dowling lattice of the cyclic group $\mathbb{Z}_M$.) If t is a subspace of F^n define $V_0(t) = \{i \in [n] : u_i \in t\}$. For $n, M \geq 0$ we call the *falling factorial with step M* the polynomial

$$(y)_{n,M} = y(y-M)\cdots(y-(n-1)M), \quad \text{with } (y)_{0,M} = 1.$$

Now, let $\mathcal{C}$ be an abstract simplicial complex on vertex set $[n]$, containing $\emptyset$ but (to keep the theorem simple) not $[n]$, and set $N_i(\mathcal{C}) = \#\{X \in \mathcal{C} : |X| = i\}$. The set of *$\mathcal{C}$-restricted flats of $B_n(M)$* is

$$Q = Q(B_n(M), \mathcal{C}) = \left\{t \in \text{Lat}\, B_n(M) : V_0(t) \notin \mathcal{C}\setminus\{\emptyset\}\right\},$$

partially ordered as in Lat $B_n(M)$. Q has zero element $\hat{0} = \{0\}$ and top element $\hat{1} = F^n$. Its characteristic polynomial is defined to be

$$p_Q(\lambda) = \sum_{t \in Q} \mu_Q(\hat{0}, t)\lambda^{n-\dim t},$$

where μ_Q is the Möbius function of Q.

THEOREM 1.1. *With $\mathcal{C}$ and Q as above, Q has rank function $r_Q(t) = \dim t$, characteristic polynomial*

$$p_Q(\lambda) = (\lambda - 1)_{n,M} + \sum_{i=1}^{n} (-1)^{i-1} N_i(\mathcal{C})(M(i-1) - 1)_{i-1,M}(\lambda - 1)_{n-i,M},$$

and Möbius invariant

$$\mu_Q(\hat{0}, \hat{1}) = (-1)^n \Big\{ (M(n-1)+1)_{n,M} - \\ - \sum_{i=1}^{n} N_i(\mathcal{C})(M(i-1) - 1)_{i-1,M}(M(n-i-1)+1)_{n-i,M} \Big\}.$$

Moreover, if $M \geq 2$ then $(-1)^n \mu_Q(\hat{0}, \hat{1}) > 0$ and the coefficients of $p_Q(\lambda)$ are all nonzero and alternate in sign.

For the proof see Section 4. Examples obtained by choosing $\mathcal{C}$ wisely include B_n, D_n, and all intermediate sets $D_n \cup \{u_1, \ldots, u_k\}$ for $k = 1, 2, \ldots, n-1$. For details on particular examples see Example 6.1.

Theorem 1.1 is a very special case of the main theorem, Theorem 4.1. Observe that, remarkably, although $Q(B_n(M), \mathcal{C})$ is not necessarily a lattice or even a semilattice, still we can give a very explicit formula for its characteristic polynomial and (in Example 4.1) its Möbius function $\mu_Q(s, t)$.

A brief outline: Section 2 provides background information, especially the definitions of the characteristic polynomial of an extrinsically graded poset $\widetilde{Q}$ and those of a biased graph Ω and its geometric lattice. Section 3 defines the poset $\widetilde{Q}(\Omega, \mathcal{C})$ and investigates its structure. (The definition is all that is needed to understand Theorem 4.1, our main theorem.) Section 4 treats the computational aspect of the poset $\widetilde{Q}(\Omega, \mathcal{C})$ when Ω is "full" and Section 5 treats non-full Ω. In Section 6 we examine numerous examples. Sections 7 and 8 discuss some problems suggested by our work.

We should mention that this article is a highly generalized version of the manuscript [7], which has been cited a few times in the literature.

2. Background Definitions

All our partially ordered sets (or "posets"), graphs, etc. are finite. For basics of posets we refer to [12], for matroids to [5], for graphs and biased graphs to [18] (but for the special case of signed graphs one may consult [15–17]).

For ready reference we list standard notations, some of which will be defined later in this section. We write $\mathbb{P} = \{1, 2, 3, \ldots\}$, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, and $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$. Also, Ω denotes a biased graph of order $n > 0$ with vertex set V and edge set E, Lat Ω the lattice of flats of its bias matroid, r the rank function of the latter, and $\mathcal{C}$ an abstract simplicial complex on vertex set V.

2A. SET POSETS AND SIMPLICIAL COMPLEXES

An (*abstract*) *simplicial complex* on vertex set V is a nonvoid class $\mathcal{C} \subseteq \mathcal{P}(V)$ such that $W \subset X \in \mathcal{C} \Rightarrow W \in \mathcal{C}$. $\mathcal{C}$ can be said to have any vertex set that contains its *support*, $\operatorname{supp} \mathcal{C} = \bigcup \mathcal{C}$. However, we need to know V in order to define the complement, $\mathcal{C}^c = \mathcal{P}(V) \backslash \mathcal{C}$.

A *poset of sets*, $\mathcal{D}$, on vertex set V is a subclass of $\mathcal{P}(V)$ to which $\emptyset$ belongs. We write $\mathcal{D}^* = \mathcal{D} \backslash \{\emptyset\}$. The *induced* subclass on $W \subseteq V$ is $\mathcal{D}{:}W = \{X \in \mathcal{C}{:}X \subseteq W\}$. The *contraction* $\mathcal{D}/\pi$, by a partition π of a subset of V, is $\{\tau \subseteq \pi : \bigcup \tau \in \mathcal{D}\}$.

2B. POSETS

Consider a poset Q with bottom element $\hat{0}$. Each element x has a *height*, $h(x) =$ length of a longest chain from $\hat{0}$ to x. The height of Q is $h(Q) = \max h(x)$. If for each x every maximal chain from $\hat{0}$ to x has length $h(x)$, we call Q *ranked*, $h(x)$ the *rank* of x, and $h(Q)$ the rank of Q. The notation Q/x means $\{y \in Q : y \geq x\}$. We write $x \lessdot y$ to mean that y covers x.

An *extrinsically graded poset* is a triple $\widetilde{Q} = (Q, \tilde{h}, \tilde{h}(Q))$ where Q is a poset with $\hat{0}$, $\tilde{h}$ is a strictly increasing function $Q \to \mathbb{N}$ such that $\tilde{h}(\hat{0}) = 0$ (called the *extrinsic grading*), and $\tilde{h}(Q)$ is an integer at least as large as the largest $\tilde{h}(x)$. By $\widetilde{Q}/x$ we mean Q/x with extrinsic grading $\tilde{h}_{\widetilde{Q}/x}(y) = \tilde{h}(y) - \tilde{h}(x)$ and with $\tilde{h}(Q/x) = \tilde{h}(Q) - \tilde{h}(x)$. We adopt the convention that a poset Q written without a tilde is "extrinsically graded" by its height function h (one might call this the *intrinsic grading*), while a tilde means there is an arbitrary extrinsic grading $\tilde{h}$.

One way to construct an extrinsically graded poset is to take a ranked poset P, such as a geometric lattice, and let Q be any subset containing $\hat{0}_P$, extrinsically graded by the rank function of P. Our posets are examples of this construction. (Another such example is the intersection lattice of an arbitrary arrangement of subspaces in a vector space, ordered by reverse inclusion and graded by codimension.)

Recall that the Möbius function of a poset Q is the function $\mu_Q : Q \times Q \to \mathbb{Z}$ defined by

$$\mu_Q(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu_Q(x, y) & \text{if } x < y, \\ 0 & \text{if } x \nleq y. \end{cases}$$

The *characteristic polynomial* of $\widetilde{Q}$ is

$$p_{\widetilde{Q}}(\lambda) = \sum_{x \in Q} \mu_Q(\hat{0}, x) \lambda^{\tilde{h}(Q) - \tilde{h}(x)}.$$

For instance, $p_Q(\lambda)$ is the characteristic polynomial associated with the height function h. If Q has $\hat{1}$, we call $\mu_Q(\hat{0}, \hat{1})$ the *Möbius invariant* of Q; it equals $p_{\widetilde{Q}}(0)$ if $\tilde{h}(Q) = \tilde{h}(\hat{1})$.

2C. GRAPHS AND BIASED GRAPHS

A *biased graph* $\Omega = (\Gamma, \mathcal{B})$ is a graph $\Gamma = (V, E)$ together with a subclass $\mathcal{B}$ of its polygons such that, if in a theta subgraph two polygons belong to $\mathcal{B}$, so does the third. In biased graphs we allow four kinds of edges: links (two distinct endpoints), loops (two coincident endpoints), half edges (one endpoint), and loose edges (no endpoints). (Neither of the latter can belong to a polygon. In matroid theory a loose edge behaves like a balanced loop and a half edge like an unbalanced one, but for technical reasons it is helpful to allow all four types of edge.)

A subgraph or edge set is *balanced* if it contains no half edge and any polygon in it belongs to $\mathcal{B}$. For $S \subseteq E$ we let $\pi(S)$ [or, respectively, $\pi_b(S)$] be the partition of V whose blocks are the vertex sets of the components [respectively, balanced components] of (V, S), let $V_0(S)$ (or just $V_0 S$) be the union of vertex sets of unbalanced components of (V, S), and set $c(S) = |\pi(S)|$, $b(S) = |\pi_b(S)|$. We write $c(\Omega) = c(E)$, $b(\Omega) = b(E)$, etc.

The *bias matroid* $G(\Omega)$ is the matroid on E whose rank function is $r(S) = n - b(S)$. For the lattice of closed sets (or flats) we write $\operatorname{Lat}\Omega$, and $\operatorname{Lat}^b\Omega$ for the subclass of balanced closed sets.

The *complete lift matroid* $L_0(\Omega)$ is the matroid on $E \cup \{e_0\}$, where e_0 is a new element, whose rank function is $n - c(S) - \varepsilon(S)$ where $\varepsilon(S) = 0$ if S is a balanced edge set and 1 otherwise.

Besides these basic ideas we need a number of technical definitions about graphs and biased graphs.

Let Γ be a graph (V, E). The set of endpoints of an edge e is $V(e)$. The *degree* of a vertex is the number of incident edges. (A loop in this article counts once, not twice as in some other works.) For $W \subseteq V$ and $S \subseteq E$ we write $W^c = V \backslash W$, $S^c = E \backslash S$, and $\Gamma|S = (V, S)$. An *induced subgraph* of Γ is $\Gamma{:}W = (W, E{:}W)$ where $W \subseteq V$ and $E{:}W = \{e \in E : \emptyset \neq V(e) \subseteq W\}$. W is *stable* if $E{:}W = \emptyset$. We denote by $[\Gamma]$ the biased graph whose underlying graph is Γ and in which every polygon is balanced. If Γ has no loose or half edges, $G([\Gamma])$ is the usual polygon matroid of Γ.

We call a biased graph Ω *simply biased* if it has no loose edges, balanced loops, balanced digons, or pairs of unbalanced edges at the same vertex. We let

$$U(\Omega) = \{v \in V : v \text{ supports an unbalanced edge}\}$$

and we call Ω *full* if $U(\Omega) = V$. If $W \subseteq V$, $\Omega^{(W)}$ denotes Ω with a half edge added at each vertex in $W \backslash U(\Omega)$. We write $\Omega^\bullet$ for $\Omega^{(V)}$. By Ω^* we mean Ω with unbalanced edges removed; E^* is $E(\Omega^*)$. If $S \subseteq E$, then $\Omega{:}W$, $\Omega|S$, etc., denote subgraphs of Γ with balance of polygons the same as in Ω. The *contraction* Ω/S is the biased graph whose vertex set is $\pi_b(S)$ and whose edge set is S^c with the endpoints of an edge e modified as follows: an endpoint v is eliminated if $v \in V_0(S)$ and is replaced by the block of $\pi_b(S)$ containing it if $v \notin V_0(S)$. A polygon C in Ω/S is balanced if there is a balanced polygon $C' \subseteq C \cup S$ such that $C = C' \backslash S$. If $S \in \operatorname{Lat}\Omega$, then $\operatorname{Lat}(\Omega|S) = [\hat{0}, S]_{\operatorname{Lat}\Omega}$ and $\operatorname{Lat}(\Omega/S) = (\operatorname{Lat}\Omega)/S$.

Disjoint union is written $\Omega_1 \mathbin{\dot\cup} \Omega_2$; the balance of a polygon is the same as in Ω_1 and Ω_2, whichever it is that contains the polygon.

2D. GAIN GRAPHS AND COLORING

A gain graph can be defined in the following way (simplified from [18, Section I.5]). Take a group $\mathfrak{G}$ of order $M \geq 1$. On vertex set $[n]$ construct a graph with edges $(i, j; g)$ for all distinct $i, j \in [n]$ and $g \in \mathfrak{G}$, but identify the edge $(i, j; g)$ with $(j, i; g^{-1})$. This is the gain graph $\mathfrak{G}K_n$. We call g the *gain* of $(i, j; g)$ in the direction from i to j and we write $\varphi(i, j; g) = g$. Calling a polygon $\{(i_0, i_1; g_1), (i_1, i_2; g_2), \ldots, (i_{k-1}, i_k; g_k)\}$, where $i_0 = i_k$, *balanced* when $g_1 g_2 \cdots g_k = 1$ determines a biased graph $\langle \mathfrak{G}K_n \rangle$. Adding an unbalanced edge (say, a half edge) to each vertex gives $\mathfrak{G}K_n^\bullet$. A *gain graph* $\Phi = (V, E, \varphi)$ with gain group $\mathfrak{G}$ and gain function φ is any subgraph of $\mathfrak{G}K_n^\bullet$, φ being the restriction to E of the gain function of $\mathfrak{G}K_n^\bullet$; $\langle\Phi\rangle$ denotes the corresponding biased graph and $G(\Phi)$, Lat Φ, etc., the associated bias matroid $G(\langle\Phi\rangle)$, lattice Lat $\langle\Phi\rangle$, and so forth. If $\Delta \subseteq K_n$, then $\mathfrak{G}\Delta$ consists of all edges $(i, j; g)$ of $\mathfrak{G}K_n$ such that $ij \in E(\Delta)$, and $\mathfrak{G}\Delta^\bullet$ is the same with an unbalanced edge at each vertex. We call $\mathfrak{G}\Delta$ the $\mathfrak{G}$*-expansion of* Δ and $\mathfrak{G}\Delta^\bullet$ the *full* $\mathfrak{G}$*-expansion.* The lattice Lat $\mathfrak{G}K_n^\bullet$ is the well known Dowling lattice of $\mathfrak{G}$ of rank n [6, 12].

A *signed graph* is a gain graph whose gain group is the sign group $\{+, -\}$. Lat $(\pm K_n^\bullet)$ is the lattice of the root system B_n and Lat$(\pm K_n)$ is that of D_n.

A k*-coloring* of Φ is a mapping $\kappa\colon V \to ([k] \times \mathfrak{G}) \cup \{0\}$, where $0 \notin \mathfrak{G}$. The set of improper edges of κ is

$$I(\kappa) = \big(E{:}\kappa^{-1}(0)\big) \cup \big\{(i, j; g) \in E : \kappa(j) = \kappa(i)g \neq 0\big\},$$

where if $\kappa(i) = (m, h)$ then $\kappa(i)g = (m, hg)$, while if $\kappa(i) = 0$ then $\kappa(i)g = 0$. We call κ *proper* if $I(\kappa) = \emptyset$. When $\mathfrak{G}$ is finite there is a polynomial $\chi_\Phi(\lambda)$, called the *chromatic polynomial*, which has the property that $\chi_\Phi(kM + 1)$ is the number of proper k-colorings of Φ. There is also a *zero-free* (or *balanced*) *chromatic polynomial* $\chi_\Phi^b(\lambda)$ such that $\chi_\Phi^b(kM)$ is the number of proper k-colorings not using the color 0. These polynomials satisfy $\chi_\Phi(\lambda) = \lambda^{b(\Phi)} p_{\mathrm{Lat}\,\Phi}(\lambda)$ and $\chi_\Phi^b(\lambda) = \lambda^{c(\Phi)} p_{\mathrm{Lat}^b\,\Phi}(\lambda)$. Because of these identities we define

$$\chi_\Omega(\lambda) = \lambda^{b(\Omega)} p_{\mathrm{Lat}\,\Omega}(\lambda) \quad \text{and} \quad \chi_\Omega^b(\lambda) = \lambda^{c(\Omega)} p_{\mathrm{Lat}^b\,\Omega}(\lambda) \tag{2.1}$$

for any biased graph. (See [18, Theorems III.5.1 and III.5.3].) When Ω is balanced, $\chi_\Omega(\lambda + 1) = \chi_\Omega^b(\lambda) = \chi_\Gamma(\lambda)$, the chromatic polynomial of the underlying graph Γ. When Ω is full, $\chi_\Omega(\lambda) = p_{\mathrm{Lat}\,\Omega}(\lambda)$.

2E. FORMULAS

A biased graph obeys convolution identities

$$\begin{aligned}\chi_\Omega(\lambda+\mu) &= \sum_{W\subseteq V} \chi^b_{\Omega:W}(\lambda)\chi_{\Omega:W^c}(\mu),\\ \chi^b_\Omega(\lambda+\mu) &= \sum_{W\subseteq V} \chi^b_{\Omega:W}(\lambda)\chi^b_{\Omega:W^c}(\mu)\end{aligned} \tag{2.2}$$

(see [18, Formula (III.6.1b)]). The former with $\mu = 1$ reduces to

$$\chi_\Omega(\lambda) = \sum_{\substack{W\subseteq V\\ W^c \text{ stable}}} \chi^b_{\Omega:W}(\lambda-1), \tag{2.3a}$$

which simplifies for full Ω to

$$\chi_\Omega(\lambda) = \chi^b_\Omega(\lambda-1). \tag{2.3b}$$

For $\mathfrak{G}$-expansions there is a nice formula [18, Example III.4.6]:

$$\chi^b_{\mathfrak{G}\Delta}(\lambda) = M^n \chi_\Delta\left(\frac{\lambda}{M}\right). \tag{2.4}$$

Combining with (2.3) and setting $\Delta = K_n$, we get Dowling's formula

$$p_{\operatorname{Lat}\mathfrak{G}K_n^\bullet}(\lambda) = \chi_{\mathfrak{G}K_n^\bullet}(\lambda) = \chi^b_{\mathfrak{G}K_n}(\lambda-1) = (\lambda-1)_{n,M}. \tag{2.5}$$

Hence we can say that $\operatorname{Lat}\mathfrak{G}K_n^\bullet$ has characteristic roots

$$1,\ M+1,\ 2M+1,\ \dots,\ (n-1)M+1.$$

2F. VECTOR REPRESENTATIONS

Suppose Φ is a full gain graph on $V = [n]$ whose gain group $\mathfrak{G}$ is a subgroup of the multiplicative group of a field F. Then $\operatorname{Lat}\Phi$ is canonically isomorphic to the lattice of flats of the vector set $x(\Phi) = \{u_i : i \in [n]\} \cup \{u_i - g u_j : (i, j; g) \in E(\Phi)\}$ in F^n. (See [15, Theorem 8B.1] for a proof when $|\mathfrak{G}| = 2$, [6, proof of Theorem 10] or [18, Theorem IV.2.1] for a general proof.) Thus for example $\mathbb{Z}_M K_n^\bullet$ has a vector representation in $\mathbb{C}^n$.

3. The Poset and Its Structure

The subject of our main results is an extrinsically graded poset $\widetilde{Q}(\Omega, \mathcal{C})$ constructed from a full biased graph Ω and an indexing simplicial complex $\mathcal{C}$ on vertex set V. The definition makes sense, however, even when Ω is not full and $\mathcal{C}$ is not a simplicial complex. The poset is

$$Q(\Omega, \mathcal{C}) = \{A \in L(\Omega) : V_0 A \notin \mathcal{C}^*\}. \tag{3.1}$$

We call its members the $\mathcal{C}$*-constrained flats of* $G(\Omega)$. The grading $\tilde{h}$ is that inherited from $\mathrm{Lat}\,\Omega$:

$$\tilde{h}(A) = r(A) = n - b(A) \quad \text{and} \quad \tilde{h}(Q(\Omega, \mathcal{C})) = n. \tag{3.2}$$

Evidently, $\widetilde{Q}(\Omega, \mathcal{C})$ has $\hat{0} = \mathrm{clos}\,\emptyset$ and, if $V \notin \mathcal{C}$, $\hat{1} = E$.

As extreme instances, for any biased graph Ω we have $\widetilde{Q}(\Omega, \{\emptyset\}) = \mathrm{Lat}\,\Omega$ and $\widetilde{Q}(\Omega, \mathcal{P}(V)) = \mathrm{Lat}^b\Omega$, so that $p_{\widetilde{Q}(\Omega,\{\emptyset\})}(\lambda) = \chi_\Omega(\lambda)$ and $p_{\widetilde{Q}(\Omega,\mathcal{P}(V))}(\lambda) = \chi^b_\Omega(\lambda)$.

One naturally wonders whether $Q = Q(\Omega, \mathcal{C})$ may be a lattice or ranked, whether r (that is, $\tilde{h}$) coincides with the height function h of Q, etc. The remainder of this section concerns such questions. (The reader who wishes to see the main theorem first should turn to Section 4.). We shall not attempt to determine exactly when Q is a lattice or a ranked poset, but we do characterize the circumstances in which $h = r, h(Q) = n$, or Q is ranked with $h = r$. These questions are germane because, for instance, if $h = r$ and $h(Q) = n$ then $p_{\widetilde{Q}}(\lambda)$ equals the intrinsic characteristic polynomial $p_Q(\lambda)$, while if $h \neq r$, $p_{\widetilde{Q}}(\lambda)$ and $p_Q(\lambda)$ should be significantly different. (But just how different they might be is unknown.)

EXAMPLE 3.1. Q need not be a lattice or even a semilattice. Let $\Omega = [\pm K_4^\bullet]$ and $\mathcal{C} = \mathcal{P}([2])$. Take $B_1 = \{+12\}$ and $B_2 = \{-12\}$, where εij denotes the edge ij with sign ε, and $A_1 = E{:}\{1, 2, 3\}$, $A_2 = E{:}\{1, 2, 4\}$. Then A_1 and A_2 cover B_1 and B_2 in Q, so Q has neither meets nor joins, not even of pairs that have lower or upper bounds in Q.

LEMMA 3.2. *Suppose* $A, B \in Q = Q(\Omega, \mathcal{C})$ *and* $B < A$ *in* $\mathrm{Lat}\,\Omega$. *Then* $B \lessdot A$ *in* Q *if and only if either:*

(i) $V_0A = \emptyset$, *and* $B \lessdot A$ *in* $\mathrm{Lat}\,\Omega$; *or*
(ii) $V_0B \notin \mathcal{C}$, *and* $B \lessdot A$ *in* $\mathrm{Lat}\,\Omega$; *or*
(iii) $V_0B = \emptyset \subset V_0A \notin \mathcal{C}$, *and* A *and* B *satisfy these properties:* $B{:}(V_0A)^c = A{:}(V_0A)^c$, $B{:}V_0A$ *is a maximal element of* $\mathrm{Lat}^b(\Omega{:}V_0A)$, *and for every* $X \in \pi(\Omega{:}V_0A)$ *we have* $(V_0A)\backslash X \in \mathcal{C}$.

Proof. Assume $B \lessdot A$ in Q. If A is balanced or B is not, the entire interval $[B, A]_{\mathrm{Lat}\,\Omega}$ lies in Q, so $A \gtrdot B$ in Q.

Henceforth let $V_0B = \emptyset$ and $V_0A \notin \mathcal{C}$. Set $A_0 = A{:}V_0A$ and $B_0 = B{:}V_0A$. Then $B_0 \subset A_0$. Since $A' = A_0 \cup (B\backslash B_0)$ is a flat of $G(\Omega)$ and has $V_0A' = V_0A \notin \mathcal{C}$, A' is in Q and $A \geq A' > B$. Thus $A' = A$; that is, A and B are the same except that $B_0 \subset A_0$. Furthermore, B_0 and A_0 are in Q and $[B_0, A_0] \cong [B, A]$, both in Q and in $\mathrm{Lat}\,\Omega$.

If B_0 were not a maximal balanced flat in $G(\Omega{:}V_0A) = G(\Omega)|A_0$, then it could be enlarged to a balanced flat B_0' and we would have $A_0 > B_0' > B_0$ in Q. So B_0 is maximal.

If there were $X \in \pi(\Omega{:}V_0A)$ such that $Y = (V_0A)\backslash X \notin \mathcal{C}$, then $A_0 > (B{:}X) \cup (E{:}Y) > B_0$ in Q, contrary to assumption.

We have proved the necessity of the conditions of the lemma.

Now we assume the conditions hold and we prove $A \gtrdot B$ in Q. Again the nontrivial case is (iii), where B but not A is balanced, and again $[B, A] \cong [B_0, A_0]$. Suppose $A_0 > F > B_0$ in Q. Then F cannot be balanced, or it would equal B_0. Since $\pi(B_0) = \pi(\Omega{:}V_0A) = \pi(A_0)$, $\pi(F)$ must equal $\pi(\Omega{:}V_0A)$, whence $V_0(F)$ is a union of blocks of $\pi(\Omega{:}V_0A)$. But then V_0F, if not in $\mathfrak{C}^*$, equals $\emptyset$ or V_0A, whence $F = B$ or A, a contradiction. Thus $A \gtrdot B$. □

LEMMA 3.3. *Suppose $B \lessdot A$ in $\widetilde{Q} = \widetilde{Q}(\Omega, \mathfrak{C})$. The step length $r(A) - r(B)$ of the interval $[B, A]$ is one except that when B is balanced and A is not, it equals $c(\Omega{:}V_0A)$.*

Proof. With notation as in the preceding proof, note that $[B, A]$ and $[B_0, A_0]$ have the same step length in $\widetilde{Q}$. Moreover, $b(A_0)$ and $b(B_0)$, calculated in $\Omega{:}V_0A$, respectively equal 0 and $c(\Omega{:}V_0A)$. Therefore $r(A_0) - r(B_0) = c(\Omega{:}V_0A)$, as claimed. □

LEMMA 3.4. *Let $A \in Q = Q(\Omega, \mathfrak{C})$. There exists $B \in Q$ such that $A \gtrdot B$ in* Lat Ω *if and only if either*

(i) *$A \supset E{:}V_0A$, or*

(ii) *$A = E{:}W$ where $b(\Omega{:}W) = 0$, and there exist $X \in \pi(\Omega{:}W)$ and $Y \subseteq X$ such that $c(\Omega{:}Y) = 1$, $b(\Omega{:}(X\backslash Y)) = 0$, and $W\backslash Y \notin \mathfrak{C}^*$.*

Proof. First we show the conditions are sufficient. If $A \supset E{:}V_0A$, then A has a nontrivial balanced component $A{:}X$, where $|X| > 1$. Let D be a cutset of $A{:}X$ and $B = A\backslash D$. Then $V_0B = V_0A$ so $B \in Q$, while $b(B) = b(A) + 1$ so $r(B) = r(A) - 1$, whence $A \gtrdot B$ in Lat Ω.

If $A = E{:}W$ where $b(\Omega{:}W) = 0$ (so $W = V_0A$), let B_Y be a maximal balanced flat in $\Omega{:}Y$ and let $B = E{:}(W\backslash Y) \cup B_Y$. Thus $r(B) = |W| - b(B{:}W) = |W| - b(B_Y) = |W| - 1$. Thus $B \lessdot A$ in Lat Ω. The fact that $W\backslash Y \notin \mathfrak{C}^*$ implies $B \in Q$.

To prove necessity we assume B exists. Let $W = V_0A$. If we are not in case (i), then $V_0B \subset W$ and $A = E{:}W$. Set $Y = W\backslash V_0B$. Thus $r(B) = |W| - b(B{:}Y) - b(B{:}V_0B) = |W| - b(B{:}Y)$. The covering relation $A \gtrdot B$ in Lat Ω implies that $b(B{:}Y) = 1$, whence we conclude that $c(\Omega{:}Y) = 1$, so that Y is contained in a block X of $\pi(\Omega{:}W)$. The fact that $V_0B = W\backslash Y$ implies that $b(\Omega{:}(X\backslash Y)) = 0$ and $W\backslash Y \notin \mathfrak{C}^*$. □

The case of full Ω is much simpler:

LEMMA 3.5. *Let Ω be full and $A \in Q = Q(\Omega, \mathfrak{C})$. There exists $B \in Q$ such that $A \gtrdot B$ in* Lat Ω *if and only if*

(i) *$A \supset E{:}V_0A$, or*

(ii) *$A = E{:}V_0A$ and either $c(\Omega{:}V_0A) = 1$ or V_0A is not minimal in $\mathfrak{C}^c$.*

Proof. In Lemma 3.4(ii), for any singleton $Y \subseteq W$ we have $c(\Omega{:}Y) = 1$ and $b(\Omega{:}(X \backslash Y)) = 0$ because Ω is full.

If $V_0 A$ is nonminimal in $\mathcal{C}^c$ we can find Y to make $W \backslash Y \notin \mathcal{C}^*$. If $c(\Omega{:}W) = 1$ then $Y = X = W$ makes $W \backslash Y \notin \mathcal{C}^*$.

Conversely, suppose Y exists as in Lemma 3.4(ii). If $Y = W$ then $c(\Omega{:}W) = 1$ so B exists. If $Y \subset W$, Y is nonvoid because $c(\Omega{:}Y) = 1$ so we can choose $y \in Y$. Then $W \backslash \{y\} \notin \mathcal{C}$ because $W \backslash Y \notin \mathcal{C}$. □

Now we can answer questions about height in Q when Ω is full.

THEOREM 3.6. *Suppose Ω full. For $Q(\Omega, \mathcal{C})$ to have height n it is necessary and sufficient that $\pi(\Omega) \nsubseteq \mathcal{C}$.*

Proof. If there is an $X \in \pi(\Omega) \backslash \mathcal{C}$, one easily constructs a chain of length n from $\hat{0}$ to E in $Q(\Omega, \mathcal{C})$.

Suppose contrariwise that such a chain exists, say $\hat{0} = A_0 \lessdot A_1 \lessdot \cdots \lessdot A_n = E$. There is some index k for which A_k is unbalanced but A_{k-1} is balanced. Thus $V_0 A_k \notin \mathcal{C}$. By Lemma 3.3, $\Omega{:}V_0 A$ is connected. It therefore lies in a component $\Omega{:}X$ of Ω, and X cannot belong to $\mathcal{C}$. □

COROLLARY 3.7. *If Ω is full and connected, then $Q(\Omega, \mathcal{C})$ has height n if and only if $V \notin \mathcal{C}$.*

The criterion of Corollary 3.7 obviously fails to be sufficient if Ω is disconnected.

We call $\mathcal{C}$ *disconnection-closed* with respect to Ω if, whenever X and Y are disjoint members of $\mathcal{C}$ and no edge of Ω joins a vertex in X to one in Y, then $X \cup Y \in \mathcal{C}$.

THEOREM 3.8. *Supposing Ω full, the following properties are equivalent.*

(i) *$Q(\Omega, \mathcal{C})$ has height function h equal to $r = r_{\mathrm{Lat}\,\Omega}$.*
(ii) *For every minimal member W of $\mathcal{C}^c$, $\Omega{:}W$ is connected.*
(iii) *$\mathcal{C}$ is disconnection-closed with respect to Ω.*
(iv) *For every $W \in \mathcal{C}^c$, $\pi(\Omega{:}W) \nsubseteq \mathcal{C}$.*

Proof. We show (iv) is equivalent to each other property.

That (i) $\Leftrightarrow$ (iv) is an easy consequence of Theorem 3.6.

Obviously (iv) $\Rightarrow$ (ii). To prove the converse assume (ii) and choose a minimal $W \notin \mathcal{C}$ such that $\pi(\Omega{:}W) \subseteq \mathcal{C}$, if any such W exists. W cannot be minimal in $\mathcal{C}^c$ because of (ii). Therefore we can strip off a vertex $w \in W$ so that $W \backslash w \notin \mathcal{C}$, but clearly $\pi(\Omega{:}(W \backslash w)) \subseteq \mathcal{C}$, contradicting the minimality of the counterexample.

That (iii) $\Leftrightarrow$ (iv) is easy to prove by induction on $c(\Omega{:}(X \cup Y))$. □

THEOREM 3.9. *Let Ω be any biased graph. For $Q(\Omega, \mathcal{C})$ to be a ranked poset with rank function r it is necessary and sufficient that, whenever $W \in \mathcal{C}^c$ is such* *that* $b(\Omega{:}W) = 0$, *then either* $\Omega{:}W$ *is connected or there is an* $X \in \pi(\Omega{:}W)$ *for which* $W \setminus X \notin \mathcal{C}$.

Proof. Every maximal element of Q has the same extrinsic height because there is only one such element unless $\mathcal{C} = \mathcal{P}(V)$, and then $Q = \mathrm{Lat}^b\,\Omega$, in which every maximal element F has $r(F) = n - c(\Omega)$ (from [18, Theorem II.2.1(j)]).

Now, the key to the proof is that Q is ranked with $h = r$ if and only if every covering interval has step length 1. Lemmas 3.2 and 3.3 imply that step length 1 is equivalent to the criterion of our theorem. □

COROLLARY 3.10. *Suppose* Ω *complete. Then* $Q(\Omega, \mathcal{C})$ *is ranked with rank function* r.

In the rest of this section we develop three lemmas that describe order filters and intervals in $\widetilde{Q}(\Omega, \mathcal{C})$, for use in calculating the Möbius function in Section 4. The lemmas remain valid with any poset of sets in place of the simplicial complex $\mathcal{C}$.

LEMMA 3.11. *Let* $A \in Q(\Omega, \mathcal{C})$. *The interval* $[\hat{0}, A]$ *in* $\widetilde{Q}(\Omega, \mathcal{C})$ *equals* $\widetilde{Q}(\Omega|A, \mathcal{C})$ *and is canonically isomorphic to*

$$\widetilde{Q}(\Omega{:}V_0A, \mathcal{C}{:}V_0A) \times \bigtimes_{Z \in \pi_b(A)} \mathrm{Lat}((\Omega|A){:}Z).$$

The proof is immediate. The canonical isomorphism is that by which $(B_0, B_1, \ldots, B_{b(A)})$, an element of the Cartesian product set, corresponds to $B_0 \cup B_1 \cup \cdots \cup B_{b(A)} \in \mathrm{Lat}\,\Omega$. Note that $(\Omega|A){:}Z$ is the biased graph of the balanced component $A{:}Z$ of A, and its underlying graph is $(Z, A{:}Z)$.

LEMMA 3.12. *Let* $A \in Q(\Omega, \mathcal{C})$. *The order filter* $\widetilde{Q}(\Omega, \mathcal{C})/A$ *is canonically isomorphic to* $\widetilde{Q}(\Omega/A, \mathcal{C}/\pi_b(A))$.

Proof. The isomorphism here is that induced by the canonical partial function $V \to \pi_b(A)$, which carries $v \in V$ to the block $X \in \pi_b(A)$ such that $v \in X$, if $v \in \bigcup \pi_b(A)$, and is undefined otherwise.

We know from [18, Theorem II.2.5] that $(\mathrm{Lat}\,\Omega)/A \cong \mathrm{Lat}(\Omega/A)$; it follows that the extrinsic grading of $\widetilde{Q}(\Omega, \mathcal{C})/A$ equals that of $\widetilde{Q}(\Omega/A, \mathcal{C}/\pi_b(A))$. We need to prove that $\mathcal{C}/\pi_b(A)$ is the correct indexing complex. A flat $F \geq A$ in $\mathrm{Lat}\,\Omega$ is excluded from $\widetilde{Q}(\Omega, \mathcal{C})/A$ when $V_0(F) \in \mathcal{C}$. The corresponding flat $F' \in \mathrm{Lat}(\Omega/A)$ satisfies $V_0(F') = \{X \in \pi_b(A) : X \subseteq V_0(F)\}$, by [18, Lemma I.4.4]. The desired conclusion follows at once. □

Combining these two results gives the main lemma:

LEMMA 3.13. *Let* $A_1 \leq A_2$ *in* $Q(\Omega, \mathcal{C})$ *and let* $W = V_0A_2$. *For* $Z \in \pi_b(A_2)$, *let* $\Gamma_Z = (Z, A_2{:}Z)/(A_1{:}Z)$, *the underlying graph of* $(\Omega|A_2){:}Z$ *contracted by* $A_1{:}Z$. *The interval* $[A_1, A_2]$ *in* $\widetilde{Q}(\Omega, \mathcal{C})$ *is canonically isomorphic to*

$$\widetilde{Q}((\Omega{:}W)/(A_1{:}W), (\mathcal{C}{:}W)/\pi_b(A_1{:}W)) \times \bigtimes_{Z \in \pi_b(A_2)} \mathrm{Lat}(\Gamma_Z).$$

4. Characteristic Polynomial and Möbius Function

Here at last is the centerpiece of our work. For the theorem, note that $\chi_\emptyset(\lambda) = \chi^b_\emptyset(\lambda) = 1$.

THEOREM 4.1. *Let Ω be a full biased graph without loose edges or balanced loops and let $\mathcal{C}$ be an abstract simplicial complex on vertex set V. The characteristic polynomial of $\widetilde{Q} = \widetilde{Q}(\Omega, \mathcal{C})$ is given by*

$$p_{\widetilde{Q}}(\lambda) = \sum_{W \in \mathcal{C}} \chi^b_{\Omega:W}(1)\chi^b_{\Omega:W^c}(\lambda - 1). \tag{4.1}$$

If $V \notin \mathcal{C}$, the Möbius invariant of $\widetilde{Q}$ is

$$\mu_{\widetilde{Q}}(\hat{0}, \hat{1}) = \sum_{W \in \mathcal{C}} \chi^b_{\Omega:W}(1)\mu_{L(\Omega:W^c)}(\hat{0}, \hat{1}). \tag{4.2}$$

We have written $\chi^b_{\Omega:W}(1)$ and $\chi^b_{\Omega:W^c}(\lambda - 1)$ but because Ω is full those quantities equal $\chi_{\Omega:W}(2)$ and $\chi_{\Omega:W^c}(\lambda)$, by the identity (2.3b). Sometimes one form is preferable, sometimes the other. For nonempty W, moreover,

$$\chi^b_{\Omega:W}(1) = (-1)^{|W|-c(\Omega:W)}\beta\big(L_0(\Omega^*{:}W)\big) \tag{4.3}$$

by [18, Theorem III.5.2], where β denotes Crapo's invariant, which is nonnegative [4].

We do not know any enumerative interpretation of $p_{\widetilde{Q}}(\lambda)$. It would be quite interesting if there is one.

Proof. The form of the characteristic polynomial makes Möbius inversion the natural choice for proving Theorem 4.1. The model is the proof by counting colorings (see [10, p. 362]) that the chromatic polynomial of a graph equals the characteristic polynomial of its polygon matroid (up to a factor of a power of λ). We face two difficulties. One is that the function to be inverted is relatively complicated. The other is that we can color only when Ω has gains in a finite group. Thus we give two proofs: one depends on counting colorings to set up the Möbius inversion; the second is an algebraic adaptation of the first that permits us to dispense with coloring and be completely general.

We write Q for $Q(\Omega, \mathcal{C})$.

Coloring Proof. We assume $\Omega = \langle\Phi\rangle$, the biased graph of a gain graph Φ whose gain group $\mathfrak{G}$ is finite. Write $M = |\mathfrak{G}|$. A *$\mathcal{C}$-restricted coloring* of Φ is a coloring κ whose zero set $\kappa^{-1}(0)$ is not in $\mathcal{C}^*$.

Step 1. We set up a Möbius inversion. Let $k \in \mathbb{N}$ and $\lambda = kM+1$. For $A \in \operatorname{Lat}\Omega$ let $f(A)$ be the number of $\mathcal{C}$-restricted k-colorings κ with $I(\kappa) \geq A$ and let $g(A)$ be the number with $I(\kappa) = A$. Obviously,

$$f(A) = \sum_{A' \geq A} g(A'), \tag{4.4}$$

the sum being taken indifferently in Q or Lat Ω since $g(A') = 0$ if $A' \notin Q$. It is also clear that $g(\emptyset)$ is the number of proper k-colorings of Φ; that is, $g(\emptyset) = \chi_\Omega(\lambda)$.

Before inverting (4.4) we need an explicit formula for f. When we color so that $I(\kappa) \geq A$ there are three points to remember: we must color 0 on $V_0(A)$, the coloring of each block $X \in \pi_b(A)$ is determined by the color on one vertex, and each such block is all zero or all non-zero. If $V_0(A) \notin \mathcal{C}$, then $f(A) = \lambda^{b(A)}$, the number of k-colorings with $I(\kappa) \geq A$. If $V_0(A) = \emptyset$, $f(A)$ equals $\lambda^{b(A)}$ less the number of k-colorings (with $I(\kappa) \geq A$) whose zero set is in $\mathcal{C}^*$. To evaluate the latter number, let $\zeta \subseteq \pi_b(A)$ be the set of zero-colored blocks. Then there are $kM = \lambda - 1$ ways to color each of the remaining blocks, so

$$f(A) = \begin{cases} \lambda^{b(A)} & \text{if } V_0(A) \notin \mathcal{C}, \\ \lambda^{b(A)} - \displaystyle\sum_{\substack{\zeta \subseteq \pi_b(A) \\ \bigcup \zeta \in \mathcal{C}^*}} (\lambda - 1)^{b(A) - |\zeta|} & \text{if } V_0(A) = \emptyset. \end{cases} \tag{4.5}$$

Step 2. Now we invert (4.4), regarded as a sum over Q, employing (4.5) to evaluate f. The inversion yields

$$g(A') = \sum_{A \geq A'} \mu_Q(A', A) f(A),$$

from which we conclude that

$$g(\emptyset) = \sum_{A \in Q} \mu_Q(\emptyset, A)\lambda^{b(A)} - \sum_{\substack{A \in Q \\ V_0(A) = \emptyset}} \sum_{\substack{\zeta \subseteq \pi_b(A) \\ \bigcup \zeta \in \mathcal{C}^*}} \mu_Q(\emptyset, A)(\lambda - 1)^{b(A) - |\zeta|}. \tag{4.6}$$

The first sum equals $p_{\tilde{Q}}(\lambda)$. To simplify the second we write $Z = \bigcup \zeta$ and sum over $Z \in \mathcal{C}^*$: then for each Z we sum over $A \in \mathrm{Lat}^b\,\Omega$ such that Z is a union of blocks of $\pi_b(A)$. Thus ζ is determined: $\zeta = \{X \in \pi_b(A) : X \subseteq Z\}$. We further split A into $A_1 = A{:}Z$ and $A_2 = A{:}Z^c$. Since $(V, A) = (Z, A_1) \mathbin{\dot\cup} (Z^c, A_2)$, the Möbius function factors and the whole sum becomes

$$\begin{aligned} &\sum_{Z \in \mathcal{C}^*} \sum_{A_1 \in \mathrm{Lat}^b(\Omega:Z)} \sum_{A_2 \in \mathrm{Lat}^b(\Omega:Z^c)} \mu(\emptyset, A_1)\mu(\emptyset, A_2)(\lambda - 1)^{b(A_2)} \\ &\quad = \sum_{Z \in \mathcal{C}^*} \chi^b_{\Omega:Z}(1)\chi^b_{\Omega:Z^c}(\lambda - 1). \end{aligned}$$

Substituting this expression for the double sum in (4.6) and recalling that $g(\emptyset) = \chi_\Omega(\lambda)$, we easily deduce (4.1).

Step 3. To prove (4.2) we set $\lambda = 0$ in (4.1). □

Algebraic Proof. To generalize the first proof to any biased graph Ω we distill the algebraic essence of f and g. For $A \in \mathrm{Lat}\,\Omega$ let us redefine

$$f(A) = \begin{cases} \lambda^{b(A)} & \text{if } V_0(A) \notin \mathcal{C}, \\ \lambda^{b(A)} - \displaystyle\sum_{Z \in \mathcal{C}^*} \sum_{\substack{\zeta \subseteq \pi_b(A) \\ V_0 A \cup \bigcup \zeta = Z}} (\lambda - 1)^{b(A) - |\zeta|} & \text{if } V_0(A) \in \mathcal{C}, \end{cases}$$

and

$$g(A) = \begin{cases} \chi_{\Omega/A}(\lambda) = \chi^b_{\Omega/A}(\lambda - 1) & \text{if } V_0(A) \notin \mathcal{C}^*, \\ 0 & \text{if } V_0(A) \in \mathcal{C}^*. \end{cases}$$

Once we prove that (4.4) holds we can proceed as in Steps 2 and 3 of the first proof.

First, suppose $V_0(A) \notin \mathcal{C}$. Then in (4.4) we are summing over A' in $\operatorname{Lat}\Omega$. Bearing in mind that $\operatorname{Lat}(\Omega/A') \cong [A', E]_{\operatorname{Lat}\Omega}$, we have

$$\begin{aligned}\sum_{A' \ge A} g(A') &= \sum_{A' \ge A} \chi_{\Omega/A'}(\lambda) = \sum_{A \le A' \le A''}\sum \mu(A', A'')\lambda^{b(A'')} \\ &= \lambda^{b(A)} = f(A).\end{aligned}$$

Now assume $V_0(A) \in \mathcal{C}$. Again summing over A' in $\operatorname{Lat}\Omega$, we have

$$\begin{aligned}\sum_{A' \ge A} g(A') &= \sum_{A' \ge A} \chi_{\Omega/A'}(\lambda) - \sum_{\substack{A' \ge A \\ V_0 A' \in \mathcal{C}^*}} \chi_{\Omega/A'}(\lambda) \\ &= \lambda^{b(A)} - \sum_{Z \in \mathcal{C}^*} \sum_{\substack{A' \ge A \\ V_0 A' = Z}} \chi^b_{\Omega/A'}(\lambda - 1).\end{aligned}$$

Therefore,

$$f(A) - \sum_{A' \ge A} g(A') = \sum_{Z \in \mathcal{C}^*} \left\{ \sum_{\substack{A' \ge A \\ V_0 A' = Z}} \chi^b_{\Omega/A'}(\lambda - 1) - \sum_{\substack{\zeta \subseteq \pi_b(A) \\ V_0 A \cup \bigcup \zeta = Z}} (\lambda - 1)^{b(A) - |\zeta|} \right\}.$$

We wish to show that the quantity in braces equals zero.

If Z is not the union of $V_0(A)$ and $\bigcup \zeta$ for some $\zeta \subseteq \pi_b(A)$, then no $A' \ge A$ can have $V_0(A') = Z$, so both sums are empty.

If Z is such a union, then ζ is unique so the second sum equals $(\lambda - 1)^{b(A:Z^c)}$. As for the first sum, let $A'_0 = E{:}Z$, $A'_1 = A'{:}Z^c$, and $A_1 = A{:}Z^c$. Then $A' = A'_0 \cup A'_1$, and $A_1 \le A'_1$ in $\operatorname{Lat}^b(\Omega{:}Z^c)$. By the definition of contraction, $\operatorname{Lat}^b(\Omega/A') \cong \operatorname{Lat}^b((\Omega{:}Z^c)/A'_1)$; thus $\chi^b_{\Omega/A'}(\lambda - 1) = \chi_{(\Omega:Z^c)/A'_1}(\lambda - 1)$. It follows that

$$\begin{aligned}\sum_{\substack{A' \ge A \text{ in } \operatorname{Lat}\Omega \\ V_0 A' = Z}} \chi^b_{\Omega/A'}(\lambda - 1) &= \sum_{A' \le A'_1 \in \operatorname{Lat}^b(\Omega:Z^c)} \chi^b_{(\Omega:Z^c)/A'_1}(\lambda - 1) \\ &= \sum_{\substack{A_1 \le A'_1 \le A''_1 \\ \in \operatorname{Lat}^b(\Omega:Z^c)}}\sum \mu(A'_1, A''_1)(\lambda - 1)^{b(A''_1)} \\ &= (\lambda - 1)^{b(A_1)} = (\lambda - 1)^{b(A:Z^c)}.\end{aligned}$$

The two sums in the braces are consequently equal.

Hence (4.4) is valid and the proof can continue through Steps 2 and 3. □

Quite a different proof can be given, depending on guessing the correct formula, induction, the convolution identities (2.2), and intricate calculations. (This was essentially Hanlon's original proof in [7] of a form of Theorem 1.1 for B_n.) We omit it because we feel that the inversion proofs are more natural and illuminating.

Proof of Theorem 1.1. By Section 2F, $Q(B_n(M), \mathcal{C})$ is canonically isomorphic to $\widetilde{Q}(\mathbb{Z}_M K_n^\bullet, \mathcal{C})$, in which $h = \tilde{h} = r$ (by Theorem 3.8; also easy to check directly). The characteristic polynomial of the latter is evaluable by Theorem 4.1 and Equation (2.5). The sign properties follow from the continuation of Example 4.1, after Corollary 4.7. □

We present next two variations on Theorem 4.1: the first one elementary, the second an intriguing formula that expresses $p_{\widetilde{Q}}(\lambda)$ in terms of principal simplicial complexes though at the cost of having to compute a new Möbius function. The first variation is useful when $\mathcal{C}$ is very large.

COROLLARY 4.2. *If Ω is full, then*

$$p_{\widetilde{Q}(\Omega,\mathcal{C})}(\lambda) = \chi^b_\Omega(\lambda) - \sum_{W \notin \mathcal{C}} \chi^b_{\Omega:W}(1)\chi^b_{\Omega:W^c}(\lambda - 1).$$

Proof. Rewriting (4.1) slightly,

$$p_{\widetilde{Q}}(\lambda) = \sum_{W \subseteq V} \chi^b_{\Omega:W}(1)\chi^b_{\Omega:W^c}(\lambda - 1) - \sum_{W \notin \mathcal{C}} \chi^b_{\Omega:W}(1)\chi^b_{\Omega:W^c}(\lambda - 1).$$

The former sum equals $\chi^b_\Omega(\lambda)$ by (2.2). □

Given $\mathcal{C}$, let $\mathcal{M}(\mathcal{C})$ consist of all intersections of one or more maximal elements of $\mathcal{C}$ and let $\widehat{M}(\mathcal{C}) = \mathcal{M}(\mathcal{C}) \cup \{\hat{1}\}$ where $\hat{1}$ is not in $\mathcal{M}(\mathcal{C})$.

COROLLARY 4.3. *If Ω is full, then*

$$p_{\widetilde{Q}(\Omega,\mathcal{C})}(\lambda) = -\sum_{W \in \mathcal{M}(\mathcal{C})} \mu_{\widehat{\mathcal{M}}(\mathcal{C})}(W, \hat{1})\, p_{\widetilde{Q}(\Omega,\mathcal{P}(W))}(\lambda).$$

Proof. The right-hand side expands to

$$-\sum_{X \subseteq W \in \mathcal{M}(\mathcal{C})} \sum \mu(W, \hat{1})\chi^b_{\Omega:X}(1)\chi^b_{\Omega:X^c}(\lambda - 1)$$

$$= \sum_{X \in \mathcal{C}} \chi^b_{\Omega:X}(1)\chi^b_{\Omega:X^c}(\lambda - 1)\left\{-\sum_{X \subseteq W \in \mathcal{M}(\mathcal{C})} \mu(W, \hat{1})\right\}.$$

Let $\overline{X}$ be the intersection of all maximal elements of $\mathcal{C}$ that contain X. Then $\overline{X} \in \mathcal{M}(\mathcal{C})$, so the quantity in braces reduces to $\mu(\hat{1}, \hat{1}) = 1$. Then by Theorem 4.1 the outer sum equals $p_{\widetilde{Q}}(\lambda)$. □

Knowing the formula (4.2) for the Möbius invariant of $Q(\Omega, \mathcal{C})$, we can write an expression for the Möbius function of any interval $[A_1, A_2]$ using the notation of Lemmas 3.11–3.13.

COROLLARY 4.4. *Let Ω be a full biased graph and let $A_1 \le A_2$ in $Q(\Omega, \mathcal{C})$. Setting $W = V_0 A_2$, $Q' = Q((\Omega{:}W)/(A_1{:}W), \mathcal{C}/\pi_b(A_1{:}W))$, and $\Gamma_Z = (Z, A_2{:}Z)/(A_1{:}Z)$ for $Z \in \pi_b(A_2)$, we have*

$$\mu_Q(A_1, A_2) = \mu_{Q'}(\hat{0}, \hat{1}) \prod_{Z \in \pi_b(A)} \mu_{\mathrm{Lat}\,\Gamma_Z}(\hat{0}, \hat{1}).$$

Proof. We appeal to Lemma 3.13 for the structure of $[A_1, A_2]$. □

EXAMPLE 4.1. We show how Corollary 4.4 applies to $\Omega = \langle \mathfrak{G} K_n^\bullet \rangle$. That gives the Möbius function of $B_n(M)$ if we take group $\mathbb{Z}_M$. As usual, we set $M = |\mathfrak{G}|$.

Let $Q = Q(\mathfrak{G} K_n^\bullet, \mathcal{C})$ and let $A_1 \le A_2$ in Q. Let $Z_1, Z_2, \ldots, Z_k$ be the balanced components of (V, A_2), let $Z_0 = V_0(A_2) \backslash V_0(A_1)$, and let n_i be the number of components of (V, A_1) contained in Z_i. Thus, n_0 is the number of balanced components of (V, A_1) that are contained in $V_0(A_2)$, and $n_0 + n_1 + \cdots + n_k - k = r(A_2) - r(A_1)$. Then

$$\begin{aligned} \mu_Q(A_1, A_2) \;=\; & (-1)^{n_0+n_1+\cdots+n_k-k}(n_1-1)!\cdots(n_k-1)! \times \\ & \times \Bigg\{ (M(n_0-1)+1)_{n_0,M} - \\ & - \sum_{i=1}^{n} N_i(\mathcal{C}/\pi_b(A_1{:}Z_0))(M(i-1)-1)_{i-1,M} \times \\ & \times (M(n_0-i-1)+1)_{n_0-i,M} \Bigg\}. \end{aligned} \tag{4.7}$$

Proof. In Corollary 4.4, $(\Omega{:}W)/(A_1{:}W)$ simplifies, by elimination of multiple unbalanced edges and multiple links having the same gain, to $\langle \mathfrak{G} K_{n_0}^\bullet \rangle$, and Γ_{Z_i} simplifies to K_{n_i} for $i \ge 1$. Then Theorem 1.1 gives $\mu_{Q'}(\hat{0}, \hat{1})$. (That $\mu_{\mathrm{Lat}\,K_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$ is a well known theorem of Schutzenberger and of Frucht and Rota; see [10, p. 359].) □

See the continuation of this example following Corollary 4.7.

One would like to determine the sign of $\mu_Q(\hat{0}, \hat{1})$. This seems difficult, but in many cases we can show that $(-1)^n \mu_Q(\hat{0}, \hat{1})$ is positive or at least non-negative. We being with some examples.

EXAMPLE 4.2. If $\mathcal{C} = \{\emptyset\}$ or $\mathcal{C} = \mathcal{P}(V)\backslash\{V\}$, then $(-1)^n \mu_Q(\hat{0}, \hat{1}) \ge 0$; it is positive except when $\mathcal{C} = \mathcal{P}(V)\backslash\{V\}$ and Ω^* has a balanced block.

Proof. In the former case $Q = \operatorname{Lat}\Omega$. We know the sign of μ by Rota's theorem [10, p. 357, Theorem 4].

In the latter case, $\mu_Q(\hat{0},\hat{1}) = -\chi^b_\Omega(1) = (-1)^n\beta(L_0(\Omega^*))$, where β is known to be nonnegative always and positive if and only if $L_0(\Omega^*)$ is inseparable [4]. Inseparability holds if and only if Ω has no balanced block [18, Theorem II.3.8]. □

EXAMPLE 4.3. Suppose T is a tree of order n and $\Omega = \langle T^\bullet\rangle$. The sign of $\mu_Q(\hat{0},\hat{1})$ is very sensitive to the choice of simplicial complex. Since Ω^* is balanced, $\chi^b_{\Omega:Y}(\lambda) = \chi_{T:Y}(\lambda) = \lambda^{c(T:Y)}(\lambda-1)^{|E(T:Y)|}$. We conclude that $\chi^b_{\Omega:W}(1) = 1$ if W is stable in T and 0 otherwise. Thus from Theorem 4.1 we deduce that

$$\mu_Q(\hat{0},\hat{1}) = \sum_{\substack{W\in\mathcal{C}\\ \text{stable in } T}} (-1)^{|W^c|}2^{|E(T:W^c)|}.$$

Since W is stable, $|E(T:W^c)| = n-1-d_T(W)$, where d_T means degree in T and $d_T(W) = \sum_{v\in W} d_T(v)$. Therefore,

$$(-1)^n\mu_Q(\hat{0},\hat{1}) = 2^{n-1}\sum_{\substack{W\in\mathcal{C}\\ \text{stable in } T}} (-1)^{|W|'}2^{-d_T(W)}. \tag{4.8}$$

We can now produce an indexing complex for which $(-1)^n\mu_Q(\hat{0},\hat{1})$ is negative. Take $\mathcal{C} = \mathcal{C}_V = \{\emptyset, \{v\} : v \in V\}$ and assume $n \geq 3$. Because at least two vertices have degree one, (4.8) becomes negative.

For instance, if T is a star then $(-1)^n\mu_Q(\hat{0},\hat{1}) = -(n-3)2^{n-2} - 1$.

We can predict non-negativity if Ω meets some rather stringent conditions.

THEOREM 4.5. *Assume that Ω is full, $V \notin \mathcal{C}$, and, for every $W \in \mathcal{C}^c$, $\Omega{:}W$ is connected. Then $(-1)^n\mu_{Q(\Omega,\mathcal{C})}(\hat{0},\hat{1}) \geq 0$.*

Furthermore, $(-1)^n\mu_{Q(\Omega,\mathcal{C})}(\hat{0},\hat{1}) > 0$ if $\operatorname{supp}\mathcal{C} \neq V$ or if $\Omega^*{:}W$ has no balanced blocks for some $W \in \mathcal{C}^c$ (in particular, if Ω^* has no balanced blocks).

Proof. First we establish a lemma.

LEMMA 4.6. *Let Ω be full and $V \notin \mathcal{C}$. Then*

$$\begin{aligned}(-1)^n&\mu_{Q(\Omega,\mathcal{C})}(\hat{0},\hat{1})\\ &= \sum_{W\in\mathcal{C}^c}(-1)^{c(\Omega:W)-1}\beta(L_0(\Omega^*{:}W))|\mu_{\operatorname{Lat}(\Omega:W^c)}(\hat{0},\hat{1})|.\end{aligned} \tag{4.9}$$

Proof. From Corollary 4.2 and Equation (4.3) we obtain

$$\mu_Q(\hat{0},\hat{1}) = 0 - \sum_{W\notin\mathcal{C}}(-1)^{|W|-c(\Omega:W)}\beta(L_0(\Omega{:}W))\mu_{\operatorname{Lat}(\Omega:W^c)}(\hat{0},\hat{1}).$$

Since $\operatorname{Lat}(\Omega{:}W^c)$ is a geometric lattice of rank $|W^c|$, we know the Möbius invariant has sign $(-1)^{|W^c|}$. The lemma follows. □

Now it is clear that, assuming $c(\Omega{:}W) = 1$ for all $W \notin \mathcal{C}$, the right-hand side of (4.9) is nonnegative. Under the same assumption, it is positive if any one $\Omega^*{:}W$ has no balanced block or, since $\beta(\emptyset) = 1$, if there is a vertex v such that $\{v\} \notin \mathcal{C}$.

COROLLARY 4.7. *If Ω is full and complete and Ω^* is unbalanced, then* $(-1)^n \mu_{Q(\Omega,\mathcal{C})}(\hat{0}, \hat{1}) > 0$ *for every indexing complex* $\mathcal{C}$.

It seems that Theorem 4.5 is far from covering all cases in which one would expect $(-1)^n \mu_Q(\hat{0}, \hat{1})$ to be nonnegative. It does not apply to Example 4.2, for instance.

EXAMPLE 4.1, continued. If $\Omega = \langle \mathfrak{G} K_n^\bullet \rangle$, where $|\mathfrak{G}| \geq 2$, then sgn $\mu_{Q(\Omega,\mathcal{C})}(A_1, A_2) = (-1)^{n_0+n_1+\cdots+n_k-k} = (-1)^{r(A_2)-r(A_1)}$.

Proof. Referring back to Lemma 3.13, we see that the task is to show that sgn $\mu_{Q'}(\hat{0}, \hat{1}) = (-1)^{n_0}$, where Q' is as in Corollary 4.4. But $\Omega{:}W = \langle \mathfrak{G} K_{|W|}^\bullet \rangle$; contracting by $A_1{:}W$ and neglecting multiple edges as in the proof of (4.7), we see that $Q' = Q(\Omega', \mathcal{C}')$ for $\Omega' = \langle \mathfrak{G} K_{n_0}^\bullet \rangle$ and suitable $\mathcal{C}'$. Now Corollary 4.7 applies (unless $n = 1$, which is trivial). □

It follows that, if w_i denotes the coefficient of λ^{n-i} in $p_{\tilde{Q}}(\lambda)$, then

$$|w_i| = (-1)^i w_i = \sum_{\substack{A \in Q(\Omega,\mathcal{C}) \\ r(A)=i}} |\mu_Q(\hat{0}, A)|.$$

Thus the coefficients of $p_{\tilde{Q}}(\lambda)$ alternate in sign, and they are all nonzero if $\mathcal{C} \subset P(V)$. In particular,

$$(-1)^n \mu_Q(\hat{0}, \hat{1}) > 0 \quad \text{if } \mathcal{C} \subset \mathcal{P}(V).$$

(If $\mathcal{C} = \mathcal{P}(V)$, then $Q = \text{Lat}^b(\mathfrak{G} K_n^\bullet)$, which has rank $n - 1$. Thus $w_n = 0$ but the other w_i are all nonzero.) These observations complete the proof of Theorem 1.1, which concerns the case in which $\mathfrak{G} = \mathbb{Z}_M$. □

We have one more topic to develop in this section: a deletion-contraction identity.

THEOREM 4.8. *Suppose Ω is full and e is a link in Ω. Then*

$$p_{\tilde{Q}(\Omega,\mathcal{C})}(\lambda) = p_{\tilde{Q}(\Omega\setminus e,\mathcal{C})}(\lambda) - p_{\tilde{Q}(\Omega/e,\mathcal{C}/\pi_b(e))}(\lambda).$$

Proof. In the formula of Theorem 4.1 there are three kinds of sets $W \in \mathcal{C}$: those for which $W \supseteq V(e)$, those for which $W^c \supseteq V(e)$, and the rest. In the former cases let $\overline{W}$ denote the element of $\mathcal{C}/\pi_b(e)$ that corresponds to W. We apply the balanced deletion-contraction identity of [18, Corollary III.3.3], which says that $\chi^b_\Omega(\lambda) = \chi^b_{\Omega\setminus e}(\lambda) - \chi^b_{\Omega/e}(\lambda)$.

If $V(e) \subseteq W$, then the W term of (4.1) expands to

$$\chi^b_{(\Omega\backslash e):W}(1)\chi^b_{\Omega:W^c}(\lambda - 1) - \chi^b_{(\Omega/e):\overline{W}}(1)\chi^b_{\Omega:W^c}(\lambda - 1).$$

But $\Omega{:}W^c = (\Omega\backslash e){:}W^c$ and also $= (\Omega/e){:}\overline{W}^c$, so we get

$$\chi^b_{(\Omega\backslash e):W}(1)\chi^b_{(\Omega\backslash e):W^c}(\lambda - 1) - \chi^b_{(\Omega/e):\overline{W}}(1)\chi^b_{(\Omega/e):\overline{W}^c}(\lambda - 1).$$

Similarly, the term corresponding to W for which $V(e) \subseteq W^c$ transforms to

$$\chi^b_{(\Omega\backslash e):W}(1)\chi^b_{(\Omega\backslash e):W^c}(\lambda - 1) - \chi^b_{(\Omega/e):\overline{W}}(1)\chi^b_{(\Omega/e):\overline{W}^c}(\lambda - 1).$$

The term of a set W of the third type can be rewritten as

$$\chi^b_{(\Omega\backslash e):W}(1)\chi^b_{(\Omega\backslash e):W^c}(\lambda - 1).$$

Now applying (4.1) again, one has the right-hand side of the formula in Theorem 4.8. □

We hope that Theorem 4.8 with $\lambda = 0$ might permit a generalization of Theorem 4.5, but we have not seen how to do that.

5. Polynomials from Non-Full Biased Graphs

Most of the results of Section 4 were proved only when Ω was full, but by a little trick we can apply them to some non-full biased graphs, and there are variants of Theorem 4.1 that hold for others.

The little trick is that, if $Q(\Omega, \mathcal{C}) = Q(\Omega^\bullet, \mathcal{C}')$ for some $\mathcal{C}'$, then $Q(\Omega, \mathcal{C})$ can be treated as though Ω were full. (We want $\Omega^\bullet$ rather than some other full biased graph, in part because the balanced polynomials, appearing in the right-side of (4.1) and in other identities, are unaffected by the presence or absence of unbalanced edges but tend to change when one alters the biased graph in other ways.) The first results characterize when this is possible, either for all $\mathcal{C}$ (with $\mathcal{C}'$ constructed in a certain systematic way) or for a particular $\mathcal{C}$. Let

$$\mathcal{U}(\Omega) = \{X \subseteq V : b(\Omega{:}X) = 0\}.$$

PROPOSITION 5.1. *Given Ω and $\mathcal{C}$, let $\mathcal{D}$ be a subset of $\mathcal{P}(V)$ with $\emptyset \in \mathcal{D}$. Then $Q(\Omega, \mathcal{C}) = Q(\Omega^\bullet, \mathcal{D})$ if and only if $\mathcal{D} = \mathcal{C} \cup \mathcal{U}(\Omega)^{*c}$.*

Proof. It is clear that $\operatorname{Lat}\Omega = Q(\Omega^\bullet, \mathcal{U}(\Omega)^{*c})$. It follows that $Q(\Omega, \mathcal{C}) = Q(\Omega^\bullet, \mathcal{U}(\Omega)^{*c} \cup \mathcal{C})$. Evidently $\mathcal{D}$ must contain $\mathcal{U}(\Omega)^{*c}$ and $\mathcal{C}$ if $Q(\Omega^\bullet, \mathcal{D})$ is not to have extra flats; meanwhile, it cannot contain any more than $\mathcal{C} \cup \mathcal{U}(\Omega)^{*c}$ or it will have too few flats. □

Propostion 5.1 shows what the form of $\mathcal{D}$ ought to be. The hitch is that $\mathcal{U}(\Omega)^{*c}$ may not be a simplicial complex. Then $\mathcal{C}' = \mathcal{C} \cup \mathcal{U}(\Omega)^{*c}$ may not be a simplicial complex so (4.1) may fail. Thus we want to know when $\mathcal{C}'$ is a simplicial complex; but that may depend on $\mathcal{C}$, so we also want to know when $\mathcal{C}'$ is a simplicial complex for every $\mathcal{C}$ – that is, when $\mathcal{U}(\Omega)^{*c}$ itself is a simplicial complex.

PROPOSITION 5.2. *Let Ω be a simply biased graph. $\mathcal{U}(\Omega)^{*c}$ is a simplicial complex if and only if*

(a) *every $u \in U(\Omega)$ and $w \in U(\Omega)^c$ are adjacent, and*
(b) *for every unbalanced digon or induced unbalanced polygon C in $\Omega{:}U(\Omega)^c$ and every vertex $w \in U(\Omega)^c \backslash V(C)$, there is an edge from w to a vertex of C.*

Proof. $\mathcal{U}(\Omega)^{*c}$ is an order ideal $\Leftrightarrow$ each induced subgraph $\Omega{:}W$ that has a balanced component is balanced $\Leftrightarrow$

(c) for each balanced, connected $\Omega{:}W$ and unbalanced, connected $\Omega{:}X$ with $X \cap W = \emptyset$, there is an edge between X and W.

This last condition implies (a) and (b). Conversely, suppose (a) and (b) and let W and X be as in (c). Pick $w \in W$ and either an unbalanced edge at a vertex $u \in X$, if possible, or a vertex-minimal unbalanced polygon C in $\Omega{:}X$. C must be an induced polygon or a digon. Therefore in either case there is an edge between $w \in W$ and a vertex in X. □

COROLLARY 5.3. *$\mathcal{U}(\Omega)^{*c}$ is a simplicial complex if Ω is complete.*

Let $\widetilde{N}(y) = \{z \in V \backslash y : z \text{ is not adjacent to } y\}$.

PROPOSITION 5.4. *Given Ω and $\mathcal{C}$, $\mathcal{C} \cup \mathcal{U}(\Omega)^{*c}$ is a simplicial complex if and only if $\mathcal{C}$ contains $V_0(\Omega{:}\widetilde{N}(y))$ for every $y \in U(\Omega)^c$.*

Proof. Let $\mathcal{A}$ be the class of sets $X \in \mathcal{U}(\Omega)^*$ that are subsets of sets in $\mathcal{U}(\Omega)^{*c}$. Call X *critical* if it is maximal in $\mathcal{A}$. Evidently, for $\mathcal{C}' = \mathcal{C} \cup \mathcal{U}(\Omega)^{*c}$ to be a simplicial complex it is necessary and sufficient that $\mathcal{C}$ contain every critical set.

It is clear that, if $w \notin U(\Omega)$, then $V_0(\Omega{:}\widetilde{N}(w)) \in \mathcal{A}$.

Suppose X is critical and W is a minimal superset of X in $\mathcal{U}(\Omega)^{*c}$. Thus $W = X \cup \{y\}$, $b(\Omega{:}X) = 0$, and $b(\Omega{:}W) > 0$. Therefore $\Omega{:}W = (\Omega{:}X) \,\dot\cup\, (\Omega{:}\{y\})$ and $y \notin U(\Omega)$. It follows that $X \subseteq V_0(\Omega{:}\widetilde{N}(y))$. But then X, being critical, equals $V_0(\Omega{:}\widetilde{N}(y))$. We conclude that the critical sets are the maximal sets of the form $V_0(\Omega{:}\widetilde{N}(y))$ where $y \in U(\Omega)^c$. □

EXAMPLE 5.1. Let us apply this theory to $\Omega = \langle \mathfrak{G}\Delta^{(U)} \rangle$ where Δ is a simple graph of order n, $\mathfrak{G}$ is a nontrivial group, and $U \subseteq V = V(\Delta)$.

Here $U(\Omega) = U$ and $\mathcal{U}(\Omega) = \{X \subseteq V : \Omega{:}X \text{ has no isolated vertices in } U^c\}$. Also, $\widetilde{N}(y)$ is the same in Δ as in Ω, and $V_0(\Omega{:}\widetilde{N}(y)) = \widetilde{N}(y) \backslash \{z \in \widetilde{N}(y) \backslash U : z$ is isolated in $\Delta{:}\widetilde{N}(y)\}$. This makes the application of Proposition 5.4 obvious.

Let $\Delta_1 \vee \Delta_2$ denote the *join* of graphs: the disjoint union together with all edges between Δ_1 and Δ_2. It is an exercise to deduce from Proposition 5.2 that $\mathcal{U}(\Omega)^{*c}$ is a simplicial complex if and only if $\Delta = (\Delta{:}U) \vee K^c_{n_1} \vee K^c_{n_2} \vee \cdots \vee K^c_{n_l}$, where $\Delta{:}U$ is any simple graph on vertex set U, $l \geq 0$, and $n_1 + n_2 + \cdots + n_l = |U^c|$.

When computing the right-hand sides of (4.1) and (4.2), etc., keep in mind Formula (2.4).

A weaker version of Theorem 4.1 can be stated for all biased graphs that come from gain graphs Φ with finite gain group $\mathfrak{G}$. Let $M = |\mathfrak{G}|$ as usual. For $\lambda = kM + 1$ and $W \subseteq V$, let $\xi_{\Phi,W}(\lambda)$ be the number of proper k-colorings γ of $\Phi{:}W^c$ such that $\gamma^{-1}(0)$ is nonadjacent to W.

LEMMA 5.5. *Given Φ with gain group $\mathfrak{G}$ and $W \subseteq V$. Let Φ_1 be a $\mathfrak{G}$-gain graph on vertex set $V_1 = W^c \cup \{v_0\}$, where $v_0 \notin V$, such that $\Phi_1{:}W^c = \Phi{:}W^c$, v_0 supports no unbalanced edge, and a vertex $v \in W^c$ is adjacent to v_0 in Φ_1 if and only if it is adjacent to a vertex of W in Φ.* (*The gains on edges vv_0 are arbitrary.*) *Let Φ_2 be Φ_1 with an unbalanced edge attached to v_0. Then*

$$\xi_{\Phi,W}(\lambda) = \chi_{\Phi_1}(\lambda) - \chi_{\Phi_2}(\lambda)$$

for all λ of the form $Mk + 1$ with $k \in \mathbb{N}$.

The proof is straightforward. □

Consequently, $\xi_{\Phi,W}(\lambda)$ is a polynomial in λ and we can evaluate it, for example, at $\lambda = 0$.

THEOREM 5.6. *If Φ is a gain graph whose gain group is finite of order M, then*

$$p_{\tilde{Q}(\Phi,\mathcal{C})}(\lambda) = \chi_\Omega(\lambda) + \sum_{W \in \mathcal{C}^* \cap \mathcal{U}(\Phi)} \chi^b_{\Phi:W}(1)\xi_{\Phi,W}(\lambda), \tag{5.1}$$

and if $V \notin \mathcal{C}$ and $E = \hat{1}$ has height n in $Q(\Phi, \mathcal{C})$, then

$$\mu_{\tilde{Q}(\Phi,\mathcal{C})}(\hat{0}, \hat{1}) = \mu_{\operatorname{Lat}\Phi}(\hat{0}, \hat{1}) + \sum_{W \in \mathcal{C}^* \cap \mathcal{U}(\Phi)} \chi^b_{\Phi:W}(1)\xi_{\Phi,W}(0). \tag{5.2}$$

Proof. We adapt the coloring proof of Theorem 4.1. Let $\lambda = kM + 1$ where $k \in \mathbb{N}$. The crucial difference is in (4.5) for balanced A. Instead of that formula we get an expression in terms of ξ.

Let A be a balanced flat in $\operatorname{Lat}\Phi$. Then

$$f(A) = \lambda^{b(A)} - \sum_{\substack{A' \geq A \\ V_0 A' \in \mathcal{C}^*}} (\#\text{ of } k\text{-colorings } \kappa \text{ of } \Phi \text{ with } I\kappa = A').$$

Thus after inverting (4.4), rather than (4.6) we get

$$\begin{aligned} g(\emptyset) &= p_{\tilde{Q}}(\lambda) - \sum_{\substack{A \leq A' \\ V_0 A = \emptyset,\, V_0 A' \in \mathcal{C}^*}} \sum \mu(\emptyset, A)(\#\text{ of } \kappa \text{ with } I(\kappa) = A') \\ &= p_{\tilde{Q}}(\lambda) - \sum_{W \in \mathcal{C}^*} \sum_{\substack{A' \in \operatorname{Lat}\Phi \\ V_0 A' = W}} \chi^b_{\Phi|A'}(1)(\#\text{ of } \kappa \text{ with } I(\kappa) = A'). \end{aligned}$$

We may write A' as $(E{:}W) \cup A'_1$ where $A'_1 \in \mathrm{Lat}^b(\Phi{:}W^c)$. If $A'_1 \neq \emptyset$, then $\chi^b_{\Phi|A'}(1) = \chi^b_{\Phi:W}(1)\chi^b_{(\Phi:W^c)|A'_1}(1)$ and, because A'_1 is balanced, $\chi^b_{(\Phi:W^c)|A'_1}(1) = \chi_{(\Gamma:W^c)|A'_1}(1) = 0$, where Γ is the underlying graph of Φ. Consequently

$$g(\emptyset) = p_{\tilde{Q}}(\lambda) - \sum_{\substack{W \in \mathcal{C}^* \\ b(\Phi:W)=0}} \chi^b_{\Phi:W}(1)(\#\text{ of } \kappa \text{ with } I(\kappa) = E{:}W). \tag{5.3}$$

Now, a coloring κ that is counted here must be 0 on W, because $b(\Phi{:}W) = 0$, and can be any proper k-coloring on W^c except that it may not color a vertex 0 that is adjacent to a vertex in W. Hence the quantity in parentheses in (5.3) equals $\xi_{\Phi,W}(\lambda)$. Since $g(\emptyset) = \chi_\Phi(\lambda)$, we have (5.1) for $\lambda = kM + 1$.

Being a polynomial identity, (5.1) therefore holds good for all numbers λ. Setting $\lambda = 0$ and observing that, if $h(E) = n$ then $r(E) = n$ so $\chi_\Phi(0) = \mu_Q(\hat{0}, \hat{1})$, we get (5.2). □

The question is: what is $\xi_{\Phi,X}(\lambda)$? If we can evaluate it we have a formula for $p_{\tilde{Q}}(\lambda)$. Lemma 5.5 can provide one means of evaluation. Lemmas 5.7 and 5.8 provide others.

LEMMA 5.7. *Given Φ with gain group of finite order $M, \lambda = kM + 1$ where $k \in \mathbb{N}$, and $W \subseteq V$, we have:*

(a) $\chi^b_{\Phi:W^c}(\lambda - 1) \le \xi_{\Phi,W}(\lambda) \le \chi_{\Phi:W^c}(\lambda)$,
(b) $\xi_{\Phi,W}(\lambda) =$ *the upper bound* $\Leftrightarrow U(\Omega)^c \backslash W$ *is nonadjacent to* W,
(c) $\xi_{\Phi,W}(\lambda) =$ *the lower bound* $\Leftrightarrow$ *every* $v \in U(\Omega)^c \backslash W$ *is adjacent to every vertex in* W.

Proof. Straightforward. □

LEMMA 5.8. *Let Φ have finite gain group. A necessary and sufficient condition that $\xi_{\Phi,W}(\lambda) = \chi^b_{\Phi:W^c}(\lambda - 1)$ for all $W \in \mathcal{C}^*$ is that each $v \in \operatorname{supp} \mathcal{C}$ be adjacent to every vertex in $U(\Omega)^c \backslash \{v\}$.*

Proof. Easy, using Lemma 5.7(c). □

COROLLARY 5.9. *If Φ has finite gain group and satisfies the property in Lemma* 5.8, *for instance if it is complete, then*

$$p_{\tilde{Q}(\Phi,\mathcal{C})}(\lambda) = \chi_\Phi(\lambda) + \sum_{W \in \mathcal{C}^* \cap \mathcal{U}(\Phi)} \chi^b_{\Phi:W}(1)\chi^b_{\Phi:W^c}(\lambda - 1).$$

Let us compare this with the formula obtained from Proposition 5.2 or 5.4. If we employ (2.3a) to expand $\chi_\Phi(\lambda)$ in terms of balanced chromatic polynomials, Corollary 5.9 takes the form

$$p_{\tilde{Q}(\Phi,\mathcal{C})}(\lambda) = \left(\sum_{W \in \mathcal{C} \cup \mathcal{U}(\Phi)} + \sum_{\substack{W \neq \emptyset \\ \text{stable}}} \right) \chi^b_{\Phi:W}(1)\chi^b_{\Phi:W^c}(\lambda - 1).$$

That is rather different from what we get by setting $\widetilde{Q}(\Omega, \mathcal{C}) = \widetilde{Q}(\Phi^\bullet, \mathcal{C} \cup \mathcal{U}(\Phi)^{*c})$ in (4.1).

6. Examples

We shall investigate the posets associated with several types of indexing complexes. In the first example, as a foretaste of the general ones to follow, for each type of complex we examine $Q(\mathfrak{G}K_n^\bullet, \mathcal{C})$. Most of the examples show instances of a curious root-shifting phenomenon that we discuss briefly in the next section.

Throughout this section Ω will be a full biased graph.

EXAMPLE 6.1. We take $\Omega = \langle \mathfrak{G}K_n^\bullet \rangle$ so that, as usual denoting by M the order of the group,

$$p_{\mathrm{Lat}\,\Omega}(\lambda) = \chi_\Omega(\lambda) = (\lambda - 1)_{n,M}, \tag{6.1a}$$

whose roots are

$$1, M+1, \ldots, (n-2)M+1, (n-1)M+1. \tag{6.1b}$$

We write $Q = Q(\Omega, \mathcal{C})$, which equals $\widetilde{Q}(\Omega, \mathcal{C})$ since by Corollary 3.10 the extrinsic grading r is the actual rank function. Note that the statement and proof of Theorem 1.1 remain valid even when $\mathfrak{G}$ is not a cyclic group.

EXAMPLE 6.1A. For our first instance we take $\mathcal{C}$ to be an m-point 0-dimensional complex $\mathcal{C}_m$, say $\mathcal{C}_m = \{\emptyset, \{i\} : n - m < i \le n\}$, and we assume $M \ge 2$. Here Q is the geometric lattice $\mathrm{Lat}\,\mathfrak{G}K_n^{(n-m)}$, where $\mathfrak{G}K_n^{(n-m)}$ is $\mathfrak{G}K_n^\bullet$ with the unbalanced edges removed from vertices $n - m + 1, n - m + 2, \ldots, n$. Geometrically, if $\mathfrak{G} = \mathbb{Z}_M$, Q is the lattice of flats spanned by the subsystem of $B_n(M)$ defined by $D_n^{(n-m)}(M) = \{u_1, \ldots, u_{n-m}\} \cup D_n(M)$, which as m varies generates a chain of geometric lattices interpolating between $D_n(M)$ and $B_n(M)$. The characteristic polynomials are easy to calculate: from Theorem 1.1 we get

$$\begin{aligned} p_Q(\lambda) &= (\lambda - 1)_{n,M} + m(\lambda - 1)_{n-1,M} \\ &= (\lambda - 1)_{n-1,M}(\lambda - (n-1)M - 1 + m). \end{aligned}$$

So the characteristic roots of $D_n^{(n-m)}(M)$, or of $\mathfrak{G}(K_n^{(n-m)})$ for any group of order M, are

$$1, M+1, \ldots, (n-2)M+1, (n-1)M+1-m. \tag{6.1c}$$

Compare to (6.1b): the largest root is reduced by m while the others remain as they were.

The characteristic polynomial of $\mathcal{C}_m$ is

$$p_{\mathcal{C}_m}(y) = y - m,$$

whose one root is precisely the amount by which the largest root of Lat Ω decreases in passing to Q.

The characteristic polynomials of these "interpolating" lattices have been treated elsewhere by different methods, three of which are described in [1, Section 5; 8; 14, Theorem 7].

EXAMPLE 6.1B. Next, let $\mathcal{C} = \mathcal{P}(X)$ where $|X| = k < n$. The characteristic roots of Q are

$$1, M + 1, \ldots, (n - k - 1)M + 1, (n - k)M, \ldots, (n - 1)M.$$

(This list can be deduced from Theorem 1.1; it is also a special case of Example 6.3.) The indexing complex has characteristic polynomial

$$p_{\mathcal{P}(X)}(y) = (y - 1)^k,$$

whose k roots all equal 1, precisely the amount by which the k largest roots of Lat Ω are decreased as we pass to Q.

Geometrically, when $\mathfrak{G}$ is cyclic, Q consists of those flats generated by $B_n(M)$ that contain either no coordinate vector u_i at all or at least one u_i for which $i \notin X$.

EXAMPLE 6.1C. Now we let $\mathcal{C} = \mathcal{P}(X_1) \cup \cdots \cup \mathcal{P}(X_m)$ where $m \geq 2$ and the X_i are nonempty, pairwise disjoint subsets of $[n]$. Let $k_i = |X_i|$. Then $N_i(\mathcal{C})$ in Theorem 1.1 equals $\sum_{i=1}^{m} \binom{k_i}{i}$. If $k = \max k_i$, then $(\lambda - 1)_{n-k,M}$ is a factor of $p_Q(\lambda)$ so $1, M + 1, \ldots, (n - k - 1)M + 1$ are characteristic roots, but there seem to be no general further factorization and no transformation of the roots in terms of the roots of $p_{\mathcal{C}}(y)$ unless all $k_i = 1$, which is Example 6.1A.

This suggests that cases like those in Examples 6.1A and B are rather special. We can, though, generalize them.

EXAMPLE 6.1D. Given two simplicial complexes $\mathcal{D}$ and $\mathcal{D}'$, define $\mathcal{D} * \mathcal{D}' = \{A \cup B : A \in \mathcal{D} \text{ and } B \in \mathcal{D}'\}$. If the supports of $\mathcal{D}$ and $\mathcal{D}'$ are disjoint, then $p_{\mathcal{D}*\mathcal{D}'}(y) = p_{\mathcal{D}}(y) p_{\mathcal{D}'}(y)$. Suppose $\mathcal{C} = \mathcal{D} * \mathcal{P}(X)$ where $X = [n - k]^c$ and $\mathcal{D}$ is a simplicial complex on vertex set $[n - k]$. Then $p_{\mathcal{C}}(y) = (y - 1)^k p_{\mathcal{D}}(y)$. From Proposition 6.1 below we get

$$\begin{aligned} p_Q(\lambda) \;=\; & (\lambda - (n - k)M)(\lambda - (n - k + 1)M) \cdots \times \\ & \times (\lambda - (n - 1)M) p_{Q(\mathfrak{G}K^\bullet_{n-k}, \mathcal{D})}(\lambda). \end{aligned}$$

So, whatever $\mathcal{D}$ does to the characteristic roots of $\mathfrak{G}K^\bullet_{n-k}$, the effect of $\mathcal{C}$ is to do the same to the smallest $n - k$ roots of $\mathfrak{G}K^\bullet_n$ while shrinking each of the largest k by 1.

For instance, when $\mathcal{D} = \emptyset$, $\mathcal{C}$ merely reduces by 1 the k largest roots. When $\mathcal{D} = \mathcal{C}_{[m]} = \{\emptyset, \{1\}, \ldots, \{m\}\}$, then in addition the $(k+1)$st largest root is lowered by m.

EXAMPLE 6.2. In treating more general examples we need a certain property of a vertex which is related to modularity of coatoms in $G(\Omega)$. In a full biased graph Ω a vertex v is *bias simplicial* if, for any two links e_{vw} and e_{vx} from v to distinct vertices w and x, there is an edge e_{wx} forming a balanced triangle $\{e_{vw}, e_{vx}, e_{wx}\}$. For instance, in $\mathfrak{G}K_n^\bullet$ every vertex is bias simplicial. In $\mathfrak{G}\Delta^\bullet$ a vertex is bias simplicial if and only if it is simplicial in Δ (i.e., its neighbors form a clique). From [19, Theorem 2.1] we know that $E(\Omega\backslash v)$ is a modular flat if and only if v is bias simplicial. Then by Stanley's factorization theorem [11, Theorem 2], provided that Ω is simply biased, we have

$$\chi_\Omega(\lambda) = (\lambda - d_v)\chi_{\Omega\backslash v}(\lambda), \tag{6.2a}$$

where d_v = degree of v in Ω; also

$$\chi^b_\Omega(\lambda) = (\lambda + 1 - d_v)\chi^b_{\Omega\backslash v}(\lambda) \tag{6.2b}$$

by (2.3b) applied to (6.2a).

We call Ω a *bias-simplicial extension of* $\Omega{:}Y$ by $(v_{l+1}, v_{l+2}, \ldots, v_n)$ if Y^c can be ordered as $(v_{l+1}, v_{l+2}, \ldots, v_n)$ so that each v_i is bias simplicial in $\Omega_i = \Omega{:}(Y \cup \{v_{l+1}, \ldots, v_i\})$. Define

$$d_i = \text{degree of } v_i \text{ in } \Omega_i. \tag{6.2c}$$

Then (6.2a) and (6.2b) imply

$$\chi_\Omega(\lambda) = (\lambda - d_{l+1})(\lambda - d_{l+2})\cdots(\lambda - d_n)\chi_{\Omega:Y}(\lambda), \tag{6.2d}$$

$$\chi^b_\Omega(\lambda) = (\lambda + 1 - d_{l+1})(\lambda + 1 - d_{l+2})\cdots(\lambda + 1 - d_n)\chi^b_{\Omega:Y}(\lambda), \tag{6.2e}$$

as long as Ω is simply biased.

Recall from Example 6.1D the definition of $\mathcal{D} * \mathcal{D}'$. As before, we are interested in examples where $\mathcal{D}' = \mathcal{P}(X)$ for $X \subseteq V$.

PROPOSITION 6.1. *Let Ω be a full, simply biased graph and $X = \{v_{n-k+1}, v_{n-k+2}, \ldots, v_n\}$. Suppose Ω is a bias-simplicial extension by $(v_{n-k+1}, v_{n-k+2}, \ldots, v_n)$ of $\Omega{:}X^c$. Let $\mathcal{D}$ be a simplicial complex on vertex set X^c and $\mathcal{C} = \mathcal{D} * \mathcal{P}(X)$. Then*

$$\begin{aligned} p_{\tilde{Q}(\Omega,\mathcal{C})}(\lambda) \;=\; & (\lambda + 1 - d_{n-k+1})(\lambda + 1 - d_{n-k+2})\cdots \times \\ & \times(\lambda + 1 - d_n)\,p_{\tilde{Q}(\Omega:X^c,\mathcal{D})}(\lambda), \end{aligned}$$

where d_i is defined by (6.2c).

Proof. Clearly, we may apply the case $k = 1$ iteratively to deduce the whole proposition. So we assume $k = 1$. For $W \subseteq V\backslash v_n$ we write $W' = W \cup \{v_n\}$ and $d_n(W')$ for the degree of v_n in $\Omega{:}W'$.

Since v_n is bias simplicial in Ω it has that same property in any induced subgraph $\Omega{:}W'$. Thus

$$\begin{aligned} p_{\widetilde{Q}(\Omega,\mathcal{C})}(\lambda) &= \sum_{W\in\mathcal{D}} \left\{\chi^b_{\Omega:W}(1)\chi_{\Omega:W^c}(\lambda) + \chi^b_{\Omega:W'}(1)\chi_{\Omega:W'^c}(\lambda)\right\} \\ &= \sum_{W\in\mathcal{D}} \big\{[\lambda - d_n(W^c)]\chi^b_{\Omega:W}(1)\chi_{\Omega:W'^c}(\lambda) + \\ &\qquad + [2 - d_n(W')]\chi^b_{\Omega:W}(1)\chi_{\Omega:W'^c}(\lambda)\big\} \\ &= (\lambda + 1 - d_n) \sum_{W\in\mathcal{D}} \chi^b_{\Omega:W}(1)\chi_{\Omega:W'^c}(\lambda) \\ &= (\lambda + 1 - d_n)\, p_{\widetilde{Q}(\Omega\setminus v_n,\mathcal{D})}(\lambda). \end{aligned}$$

□

We see that, whatever may be the effect on the characteristic roots of changing $\mathrm{Lat}(\Omega{:}X^c)$ to $\widetilde{Q}(\Omega{:}X^c, \mathcal{D})$, our construction of $\mathcal{C}$ superimposes a further effect of subtracting 1 from the additional roots of $\mathrm{Lat}\,\Omega$.

EXAMPLE 6.3. We get $\mathcal{C} = \mathcal{P}(X)$ by taking $\mathcal{D} = \{\emptyset\}$ in Example 6.2. So when Ω is a bias-simplicial extension of $\Omega{:}X^c$ (and assuming $X \neq V$) we have a simple root-shifting phenomenon: certain k roots of $\mathrm{Lat}\,\Omega$, namely $d_{n-k+1}, \dots, d_n$, are reduced by 1, while the others are unchanged.

When $\Omega = \langle \mathfrak{G} K_n^\bullet \rangle$ the shifted roots are the largest ones because the degrees are given by $d_i = Mi + 1$ for all i.

EXAMPLE 6.4. Take $\mathcal{C}$ to be the 0-dimensional complex $\mathcal{C}_Z = \{\emptyset, \{v\} : v \in Z\}$ where $Z \subseteq V$ and $|Z| = m \geq 1$. Theorem 4.1 reduces to

$$p_{\widetilde{Q}(\Omega,\mathcal{C})}(\lambda) = \chi_\Omega(\lambda) + \sum_{v\in Z} \chi_{\Omega\setminus v}(\lambda). \tag{6.4a}$$

If it happens that $\Omega^*{:}\{v, w\}$ is unbalanced for every pair $\{v, w\} \subseteq Z$, then $\widetilde{Q}(\Omega, \mathcal{C}_Z)$ is the geometric lattice $\mathrm{Lat}\,\Omega^{*(Z^c)}$, where $\Omega^{*(Z^c)}$ is Ω with the unbalanced edges removed from the vertices in Z. This applies for instance to Example 6.1A and more generally to $\Omega = \langle \mathfrak{G}\Delta^\bullet \rangle$ if Z is a clique in Δ and $|\mathfrak{G}| \geq 2$.

Suppose every vertex in Z is bias simplicial and has degree l. Then

$$\chi_\Omega(\lambda) = (\lambda - l)\chi_{\Omega\setminus v}(\lambda)$$

for $v \in Z$ by Proposition 6.1, so (6.4a) simplifies to

$$p_{\widetilde{Q}(\Omega,\mathcal{C}_Z)}(\lambda) = \frac{\lambda - l + m}{\lambda - l}\, p_{\mathrm{Lat}\,\Omega}(\lambda), \tag{6.4b}$$

which we interpret as saying that a certain characteristic root of $\mathrm{Lat}\,\Omega$ has been reduced by m.

EXAMPLE 6.5. We combine Examples 6.3 and 6.4. Let Ω be a bias-simplicial extension of $\Omega{:}X^c$ as in Proposition 6.1 and let $\mathcal{C} = \mathcal{P}(X) * \mathcal{C}_Z$ with $\mathcal{C}_Z$ as in Example 6.4, where $\emptyset \neq Z \subseteq X^c$ and each vertex $v \in Z$ is bias simplicial in $\Omega{:}X^c$ and has degree l in Ω. Let $m = |Z|$ and $k = |X|$. Comparing the characteristic roots of $\operatorname{Lat}\Omega$ and $\widetilde{Q}(\Omega, \mathcal{C})$, we observe that each of the former's roots $d_{n-k+1}, \ldots, d_n$ is lowered by 1, while one of the roots l is reduced by m.

EXAMPLE 6.6. Now let $\mathcal{C} = \mathcal{P}(X_1) \cup \mathcal{P}(X_2) \cup \cdots \cup \mathcal{P}(X_m)$ where $m \geq 2$ and the sets $X_1, X_2, \ldots, X_m$ are nonvoid and mutually disjoint. Then

$$p_{\widetilde{Q}(\Omega,\mathcal{C})}(\lambda) = \chi_\Omega(\lambda) + \sum_{i=1}^{m} \sum_{\emptyset \subset W \subseteq X_i} \chi^b_{\Omega:W}(1)\chi_{\Omega:W^c}(\lambda), \tag{6.6a}$$

which has no obvious factorization except in one special case.

Suppose all the flats $A_i = E{:}X_i^c$ are modular in $\operatorname{Lat}\Omega$ and let $Y = (X_1 \cup \cdots \cup X_m)^c$. Then $A = A_1 \cap \cdots \cap A_m = E{:}Y$ is modular because in a geometric lattice the meet of modular flats is modular (a result due to Stanley; see [3, Proposition 3.6] and [11, Lemma 2]). By Stanley's factorization theorem [11], the characteristic polynomial of $\operatorname{Lat}(\Omega|A)$, which is $\chi_{\Omega:Y}(\lambda)$, is a common factor of all terms in (6.6a), hence a factor of $p_{\widetilde{Q}}(\lambda)$.

In Example 6.1C we noticed the factor $(\lambda - 1)_{n-k,M}$. That is an instance of the modular factorization just discussed, because $E{:}Y$ is a modular flat of $\operatorname{Lat}\mathfrak{G}K_n^\bullet$ for any $Y \subseteq [n]$. (See [6, Theorem 4].)

7. Root Shifting and Other Problems

In most of the examples in Section 6 we observed that under certain circumstances characteristic roots are shifted as we pass from $\operatorname{Lat}\Omega$ to $\widetilde{Q}(\Omega, \mathcal{C})$. The number of shifted roots and the amount of shift in these examples are precisely the number and size of the characteristic roots of $\mathcal{C}$. Why this should be so, and whether it represents a general phenomenon or an accident, we do not know.

We summarize the data. In each case the shifting is known to take place only for certain nice Ω's.

EXAMPLES 6.1A and 6.4. The characteristic polynomial of $\mathcal{C}$ is $y - m$. We observe that one root of $p_{\operatorname{Lat}\Omega}(\lambda)$ is decreased by m, the root of $p_{\mathcal{C}}(y)$.

EXAMPLES 6.1B and 6.3. Here $p_{\mathcal{C}}(y) = (y-1)^k$, which has k roots equal to 1, and k roots of $p_{\operatorname{Lat}\Omega}(\lambda)$ are decreased by 1.

EXAMPLES 6.1D and 6.2. We have $p_{\mathcal{C}}(y) = (y-1)^k p_{\mathcal{D}}(y)$ where $k = |X|$, so the characteristic roots of $\mathcal{C}$ are those of $\mathcal{D}$ and k additional 1's. The effect on the roots of $\operatorname{Lat}\Omega$ corresponds: whatever $\mathcal{D}$ does to $p_{\operatorname{Lat}\Omega:X^c}(\lambda)$, each additional root of $p_{\operatorname{Lat}\Omega}(\lambda)$ suffers a decrease of 1.

EXAMPLE 6.5. Here $p_{\mathcal{C}}(y) = (y-1)^k(y-m)$, whose roots correspond to the effect of $\mathcal{C}$-restriction on the characteristic roots of Lat Ω.

EXAMPLES 6.1C and 6.6. Here, if $m > 1$ and some $k_i > 1$, the characteristic polynomial of $\mathcal{C}$ has no obvious factorization and the effect of $\mathcal{C}$ on the characteristic roots of Lat Ω, while obscure, is probably not simply related to the roots of $p_{\mathcal{C}}(y)$.

Based on these examples it is hard to tell whether or not there is an interesting answer to the problem of root shifting, which we state formally as

PROBLEM 1. Is there any theory of a relationship among the roots of $\chi_\Omega(\lambda)$, $p_{\mathcal{C}}(y)$, and $p_{\widetilde{Q}(\Omega,\mathcal{C})}(\lambda)$?

Our work suggests other theoretical questions in addition to the root-shifting problem. There are the questions of generalizing Theorems 3.6 and 3.8 to all biased graphs, determining more completely when $\mu_{Q(\Omega,\mathcal{C})}(\hat{0}, \hat{1}) \geq 0$ (generalizing Theorem 4.5), clarifying the relationship (if any) between $p_{\widetilde{Q}}(\lambda)$ and $p_Q(\lambda)$, and gaining more understanding of $\xi_{\Phi,W}(\lambda)$ in Theorem 5.6. Two broader problems are these:

PROBLEM 2. Can one extend the construction of $\widetilde{Q}(\Omega, \mathcal{C})$ from bias matroids of biased graphs to an arbitrary matroid M with prescribed basis V, in such a way as to generalize any of our results?

PROBLEM 3. Can algebraic constructions that lead to the exponents of a root system be carried out more generally, either for $Q(\pm K_n^\bullet, \mathcal{C})$ or $Q(\mathfrak{G}K_n^\bullet, \mathcal{C})$, or for $\widetilde{Q}(\Omega, \mathcal{C})$ in general?

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