Tutte Functions of Matroids

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$$F(M) = \delta_e F(M \backslash e) + \gamma_e F(M/e)$$

... and then, what?

Tutte Invariants

Tutte/Brylawski Theorems (graphs/matroids):

Given a *"Tutte–Grothendieck invariant"*
$F : \{\text{matroids}\} \to$ commutative, unital ring A, s.t.

(1) *Universal Domain*: $F(M)$ is defined for every finite matroid.
(2) *Invariance*: $F(M) = F(N)$ if $M \cong N$.
(3) *Multiplicativity*: $F(M \oplus N) = F(M) \cdot F(N)$.
(4) *Unitarity*: $F(\varnothing) = 1$.
(5) *Deletion-Contraction Law*:

$$F(M) = F(M \backslash e) + F(M/e)$$

if e neither a loop nor a coloop is.

Conclusion:
Each matroid M has a polynomial T_M (not depending on F) s.t.

$$F(M) = T_M(x, y) = \sum_{i,j \geq 0}\sum t_{ij} x^i y^j,$$

where $x = F(\text{coloop})$, $y = F(\text{loop})$.
That is: $M \mapsto T_M \in \mathbb{Z}[x, y]$.
$\mathbb{Z}[x, y]$: the **universal ring** for Tutte–Grothendieck invariants.

Given a *"Tutte invariant"*
$F : \{\text{matroids}\} \to$ module, that satisfies (1, 2, 4, 5).

Conclusion:

$$F(M) = \sum_{i,j \geq 0}\sum t_{ij} a_{ij},$$

where $a_{ij} = F(i \text{ coloops}, j \text{ loops})$.
That is, $M \mapsto \tilde{T}_M \in \mathbb{Z}\{x_{ij} : i, j \geq 0\} \cong \mathbb{Z}[x, y]^+$.
$\mathbb{Z}[x, y]^+$ is the **universal module** for Tutte invariants.

Strong Tutte Functions

Can we weaken any of the hypotheses?

(1) Restrict F to graphic matroids (Tutte). Or even further?
(2) ?
(3) Multiplicativity: Keep.
(4) ?
(5) $F(M) = \delta F(M\backslash e) + \gamma F(M/e)$: same T_M with renormalized x, y.
Weaken further?

The approach of

Thomas Zaslavsky,
Strong Tutte functions of matroids and graphs.
Trans. Amer. Math. Soc. **334** (1992), 317–347.

A *strong Tutte function*, $F : \mathcal{M} \to$ field K, s.t.
(1) *Domain*: any minor-closed class $\mathcal{M}$ that contains all 3-point matroids.
(3) *Multiplicativity*: $F(M \oplus N) = F(M) \cdot F(N)$.
(4) *Unitarity*: $F(\varnothing) = 1$.
(5) *Parametrized Deletion-Contraction*:

$$F(M) = \delta_e F(M\backslash e) + \gamma_e F(M/e)$$

if e neither a loop nor a coloop is.
($\delta_e, \gamma_e \in K$ depend on e.)

Theorem 1 (Zaslavsky)**.** *There are* 6 *types of nontrivial strong Tutte function, each having its own universal polynomial. One type ("normal") exists for all possible parameters.*

Algebras

The approach of

Bela Bollobás and Oliver Riordan,
A Tutte polynomial for coloured graphs.
Combin. Probab. Comput. **8** (1999), 45–93.

Given: $F : \{\text{graphic matroids}\} \to$ commutative ring A, s.t.

(1) *Domain*: F is defined on all graphic matroids, or on any minor-closed class $\mathcal{M}$ that contains all 3-point matroids.
(3) *Multiplicativity*: $F(M \oplus N) = F(M) \cdot F(N)$.
(4) *Unitarity*: $F(\varnothing) = 1$.
(5) *Parametrized Deletion-Contraction.* ($\delta_e, \gamma_e \in A$ depend on e.)

Conclusion: Universal scalars and algebra:

$$\tilde{A} := \mathbb{Z}[d_e, c_e : \forall\ e],$$

and the *Tutte algebra*,

$$\mathbf{W}(\mathcal{M}) := \tilde{A}[x_e, y_e : \forall\ e]/\hat{\Delta},$$

where

$$\hat{\Delta} := \langle\!\langle c_f x_e - c_e x_f - d_e y_f + d_f y_e,\ (d_e y_f - d_f y_e - d_e c_f + c_e d_f) y_g,\\ (c_e x_f - c_f x_e - d_e c_f + c_e d_f) x_g : \forall\ e, f, g \rangle\!\rangle \subseteq \tilde{A}\mathcal{M}.$$

Theorem 2 (Bollobás and Riordan)**.** *Every function that factors through $T : \mathcal{M} \to \mathbf{W}(\mathcal{M})$ is a strong Tutte function. Conversely, every strong Tutte function factors through T.*

What are the functions?

What is $\hat{\Delta}$? What is the structure of $\mathbf{W}(\mathcal{M})$?

We extend it not quite slightly.
A *multiplicative Tutte function*, $F : \mathcal{M} \to$ commutative ring A s.t.

(1) *Domain*: any minor-closed class $\mathcal{M}$.
(3) *Multiplicativity*: $F(M \oplus N) = F(M) \cdot F(N)$.
(4) Unitarity: Give it up.
(5) *Parametrized Deletion-Contraction.*

Conclusion:
Universal scalars and algebra:

$$\tilde{A} := \mathbb{Z}[d_e, c_e : \forall\ e],$$

and the *Tutte algebra*,

$$\mathbf{W}(\mathcal{M}) := \tilde{A}[x_e, y_e : \forall\ e]/\hat{\Delta},$$

where

$$\hat{\Delta} := \left\langle\!\!\left\langle \left\{ \begin{array}{c} \begin{vmatrix} c_e & c_f \\ x_e & x_f \end{vmatrix} + \begin{vmatrix} d_e & d_f \\ y_e & y_f \end{vmatrix} \text{ for } (ef)_1 \in \mathcal{M}, \\ \left(\begin{vmatrix} d_e & d_f \\ y_e & y_f \end{vmatrix} - \begin{vmatrix} d_e & d_f \\ c_e & c_f \end{vmatrix} \right) y_g \text{ for } (efg)_1 \in \mathcal{M}, \\ \left(\begin{vmatrix} c_e & c_f \\ x_e & x_f \end{vmatrix} - \begin{vmatrix} d_e & d_f \\ c_e & c_f \end{vmatrix} \right) x_g \text{ for } (efg)_2 \in \mathcal{M}, \end{array} \right\} \right\rangle\!\!\right\rangle \subseteq \tilde{A}\mathcal{M}.$$

Theorem 3 (Bollobás and Riordan, extended by us)**.** *Every function that factors through $T : \mathcal{M} \to \mathbf{W}(\mathcal{M})$ is a multiplicative Tutte function. Conversely, every strong Tutte function factors through T.*

What are the functions? What is $\hat{\Delta}$? The structure of $\mathbf{W}(\mathcal{M})$?

Tutte Functions

Can we weaken the hypotheses more drastically?

(1) *Domain*: any minor-closed class $\mathcal{M}$.
(3) Multiplicativity: Give it up.
(5) *Parametrized Deletion-Contraction.*

A *Tutte function*:
$F : \mathcal{M} \to$ any module over any commutative, unital ring A, s.t.

- *Domain*: any minor-closed class $\mathcal{M}$.
- *Parametrized Deletion-Contraction*:

$$F(M) = \delta_e F(M \backslash e) + \gamma_e F(M/e)$$

if e neither a loop nor a coloop is.
($\delta_e, \gamma_e \in A$ depend on e.)

Universal ring and module:

$$\tilde{A} := \mathbb{Z}[d_e, c_e : e],$$

and the *Tutte module*,

$$\mathbf{w}(\mathcal{M}) := \tilde{A}\mathcal{M}/\Gamma,$$

where

$$\Gamma := \left\langle M - d_e(M \backslash e) - c_e(M/e) : M, e \right\rangle \subseteq \tilde{A}\mathcal{M}.$$

Theorem 4. *Every function that factors through $t : \mathcal{M} \to \mathbf{w}(\mathcal{M})$ is a strong Tutte function. Conversely, every strong Tutte function factors through t.*

Classify all Tutte functions!
What is the structure of $\mathbf{w}(\mathcal{M})$? What is Γ?

SIMPLIFICATION

A discrete matroid is all loops and coloops.

$$\mathcal{D} := \text{ set of discrete matroids in } \mathcal{M},$$

$$\Delta := \Gamma \cap \tilde{A}\mathcal{D}.$$

Theorem 5. $\mathbf{w}(\mathcal{M}) := \tilde{A}\mathcal{M}/\Gamma = \tilde{A}\mathcal{D}/\Delta$.

$\therefore$ *What is Δ? Use it to get the structure of the Tutte module.*

Define

$$\tau_e(M) := \begin{cases} d_e(M\backslash e) + c_e(M/e), & e \in E(M) \text{ not a loop or coloop,} \\ M, & \text{otherwise;} \end{cases}$$

$$\tau_{e_1\cdots e_k}(M) := \tau_{e_k}\cdots\tau_{e_1}(M).$$

Proposition 6. $\Delta = \big\{\tau_\sigma(M) - \tau_{\sigma'}(M) : \sigma, \sigma' \in \mathrm{Perm}(E(M))\big\}$.

How does Δ interact with $\tilde{A}\mathcal{D}$?
How does $\mathbf{w}$ compare with $\mathbf{W}$?

Comparison

Tutte module $\mathbf{w}(\mathcal{M})$ vs. Tutte algebra $\mathbf{W}(\mathcal{M})$:

An $\tilde{A}$-module homomorphism

$$\mathbf{w}(\mathcal{M}) \to \mathbf{W}(\mathcal{M}) \text{ extending } \mathcal{M} \to \mathbf{W}(\mathcal{M}).$$

(Proof 1. Obvious since $\mathbf{W}$ has more properties.)
(Proof 2. The extension exists by Theorem 4 because $\mathcal{M} \to \mathbf{W}$ is a Tutte function.)

Is $\mathbf{w}(\mathcal{M})$ *a submodule of* $\mathbf{W}(\mathcal{M})^+$*? I.e., is the mapping injective?*

Generally: No. (Counterexamples, based on a general property that prevents injectivity.)

Particularly: Sometimes. (Examples, e.g., $\mathcal{M}$ consisting of all 2-point matroids.)

Often? Interesting minor-closed classes? We don't know yet.
Esp., $\mathcal{M}$ closed under direct summation? (We guess "yes".)

Other Work

(1) Classification of types of multiplicativity of a Tutte function. (Done.)
- Unitary.
- Separator strong (Joanna A. Ellis-Monaghan and Lorenzo Traldi, Parametrized Tutte polynomials of graphs and matroids. *Combin. Prob. Comput.* **15** (2006), no. 6, 835–854.)
- Strict multiplicativity (excluding $\varnothing$ as a factor).

(2) Example minor-closed classes. (Partly done.)
- Small example: minors of a 3-point matroid.
- Closed under direct summation. (Important!)
- All minors of a fixed master matroid M_0. (Significant.)

(3) Use the structure of $\mathbf{w}(\mathcal{M})$ to classify all types of Tutte function, with

Recipes!

(Like the parametrized corank-nullity polynomial from Traldi 1989, a bit more generally in Zaslavsky 1992.)

Lorenzo Traldi,
A dichromatic polynomial for weighted graphs and link polynomials.
Proc. Amer. Math. Soc. **106** (1989), 279–286.

(4) What is the effect of choosing particular parameter values?
(Slightly done.)