2. T. Zaslavsky: Voltage-graphic geometry and the forest lattice (chaired by N. Robertson)

1. We begin with a theorem that provides a focal point for the general theory. Let $\Gamma = (N,E)$ be a graph, $n = |N|$, f_k = the number of k-tree spanning forests in Γ, and $t(\Delta)$ = the number of tree components of the graph Δ. Let $\mathfrak{F}$ = the set of forests of Γ, including the null graph, ordered in the following way: $F \leq F'$ if F' consists of some (or no) trees of F plus (optionally) additional edges linking some of these trees.

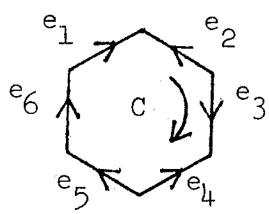
Forest Theorem. $\mathfrak{F}$ is a geometric lattice of rank n. Its rank function is $\operatorname{rk} F = n - t(F)$. Its characteristic polynomial (when Γ is finite) is

$$p_{\mathfrak{F}}(\lambda) = (-1)^n \sum_{k=0}^{n} (1-\lambda)^k f_k .$$

Some other facts about $\mathfrak{F}$: its 0 element is $(N,\emptyset)$, its 1 is $(\emptyset,\emptyset)$, its atoms are (N,e) for each link e and $(N\setminus\{v\}, \emptyset)$ for each vertex v.

The Forest Theorem can be proved directly, e.g. by deletion-contraction, but it is more interesting to derive it from the theory of voltage-graphic matroids.

2. A <u>voltage graph</u> is a pair (Γ, φ) consisting of a graph $\Gamma = (N,E)$ and a <u>voltage</u>, a mapping $\varphi\colon E \to \mathfrak{G}$ where $\mathfrak{G}$ is a group called the <u>voltage group</u>. The voltage on an edge depends on the sense in which the edge is traversed: if for e in one direction the voltage is $\varphi(e)$, then in the opposite direction it is $\varphi(e)^{-1}$. The voltage on a circle is the product of the edge voltages taken in order with consistent direction; if the product equals 1 the circle is called <u>balanced</u>. (While in general the starting point and orientation of C

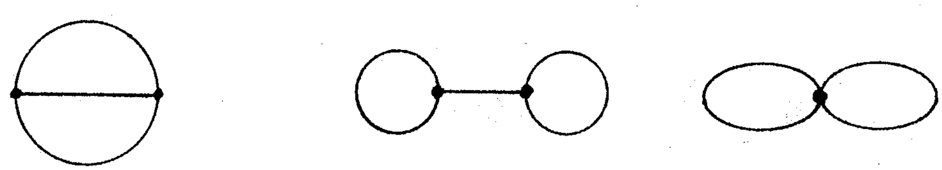


$$\varphi(C) = \varphi(e_1)\varphi(e_2)^{-1}\varphi(e_3)\varphi(e_4)^{-1}\varphi(e_5)\varphi(e_6)$$

influence its voltage, they have no effect on whether it is balanced.) A subgraph is balanced if every circle in it is balanced. For $S \subseteq E$, let $b(S)$ = the number of balanced components of (N,S).

<u>Matroid Theorem</u>. The function $\operatorname{rk} S = n - b(S)$ is the rank function of a matroid $G(\Gamma, \varphi)$ on the set E. A set $A \subseteq E$ is closed iff every edge $e \notin A$ has an endpoint in a balanced component of (N,A) but does not combine with edges in A to form a balanced circle. A set is a circuit iff it is a balanced circle or a bicircular graph containing no balanced circle.

Bicircular graphs



Theta graphs

Handcuffs

We call $G(\Gamma,\varphi)$ a voltage-graphic matroid. In case it is a simple matroid it is a subgeometry of the Dowling lattice $Q_n(\mathfrak{G})$ (see [1]).

Example 1. $\varphi \equiv 1$. Then $G(\Gamma,\varphi) = G(\Gamma)$, the usual graphic matroid.

Example 2. $\mathfrak{G} = \{\pm 1\}$. Then (Γ,φ) is a signed graph.

Example 2a. Same, with $\varphi \equiv -1$. Then $G(\Gamma,\varphi)$ is the even-circle matroid of Γ (see [2] for references).

Example 3. No balanced circles. Then $G(\Gamma,\varphi) = B(\Gamma)$, the bicircular matroid of Γ (see [4] for references). The balanced sets are the spanning forests. The closed sets correspond to the forests $F = (X, E(F))$ such that the subgraph of Γ induced on X^c has no trees. The circuits are the bicircular graphs (whence the name). The rank function is $\operatorname{rk} S = n - t(S)$.

The first parts of the Forest Theorem follow from these observations, the Matroid Theorem, and :

Lemma. $\mathfrak{F} \cong$ the lattice of flats of $B(\Gamma^o)$, where Γ^o denotes Γ with a loop at every node.

3. Now let Γ be finite and let $\mathfrak{G}$ have finite order g. A proper μ-coloring of (Γ,φ) is a mapping

$$\kappa : N \longrightarrow \{0\} \cup (\{1,\ldots,\mu\} \times \mathfrak{G})$$

such that, for any edge e from v to w (including loops), we have $\kappa(v) \neq 0$ or $\kappa(w) \neq 0$ and also

$$\kappa_1(v) \neq \kappa_1(w) \quad \text{or} \quad \kappa_2(w) \neq \kappa_2(v)\varphi(e) \qquad \text{if } \kappa(v), \kappa(w) \neq 0,$$

where κ_1 and κ_2 are the numerical and group parts of κ. Let $\chi(\mu g + 1)$ = the number of proper μ-colorings of (Γ, φ) and let $\chi^b(\mu g)$ = the number which do not take the value 0.

Chromatic Polynomial Theorem. $\chi(\mu g + 1)$ is a polynomial in μ. Indeed $\chi(\lambda) = \lambda^{b(E)} p(\lambda)$, where $p(\lambda)$ is the characteristic polynomial of $G(\Gamma, \varphi)$.

Balanced Chromatic Polynomial Theorem. $\chi^b(\mu g)$ is a polynomial in μ. Indeed $\chi^b(\lambda) = \Sigma_S \lambda^{b(S)}(-1)^{|S|}$, summed over balanced $S \subseteq E$.

Fundamental Theorem. Let $\chi^b_X(\lambda)$ denote the balanced chromatic polynomial of the induced voltage graph on $X \subseteq N$. Then

$$\chi(\lambda) = \sum_{X \text{ stable}} \chi^b_X(\lambda - 1) .$$

In particular for the forest lattice we look at $B(\Gamma^o)$. The necessary finite voltage group may be, for instance, the power set $\mathcal{P}(E)$ with symetric difference, with voltage $\varphi(e) = \{e\}$. Then the latter two theorems quickly yield the characteristic polynomial of $\mathfrak{F}$.

[1] T. A. Dowling, "A class of geometric lattices based on finite groups," J. Combinatorial Theory Ser. B, 14 (1973), 61-86. MR 46 #7066. Erratum, ibid. 15 (1973), 211. MR 47 #8369.

[2] T. Zaslavsky, "Signed graphs", submitted. Contains indications of proof of the Matroid Theorem.

[3] T. Zaslavsky, "Signed graph coloring" and "Chromatic invariants of signed graphs", submitted. Contain proofs of the coloring results.

[4] T. Zaslavsky, "Bicircular geometry and the lattice of forests of a graph", submitted. Has more detail, references, and applications to geometry.