Vector Valued Switching in Signed Graphs

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Abstract

We introduce the concept of vector valued switching function in signed graphs, which extends the concept of switching to higher dimensions. Using this concept, we define balancing dimension and strong balancing dimension for a signed graph, which can be used for a new classification of unbalanced signed graphs. We also calculate the balancing and strong balancing dimensions for some classes of signed graphs, and provide a bound for these dimensions.

Keywords: Signed graph, vector valued switching, balancing dimension.

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1 Motivational Background and Introduction

Given a signed graph $\Sigma = (G, \sigma)$ where $G = (V, E)$ is the underlying graph (which we assume is simple) and $\sigma : E \to \{-1, 1\}$ is the signing function, by switching Σ to a signed graph $\Sigma^\zeta = (G, \sigma^\zeta)$ using a switching function $\zeta : V \to \{-1, 1\}$, we mean the edge signing of Σ^ζ satisfies the condition $\sigma^\zeta(uv) = \sigma(uv)\zeta(u)\zeta(v)$. Switching does not change the signs of cycles. We say two signed graphs Σ_1 and Σ_2 are switching equivalent if one of them can be switched from the other.

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Given a cycle C in a signed graph, the sign of this cycle $\sigma(C)$ is defined as the product of the edge signs on it. If $\sigma(C) = 1$, we say that the cycle C is positive. A signed graph is said to be balanced if all cycles in it are positive. There are various characterizations of balanced signed graphs; one of them is by switching (e.g., see [2]), as follows.

Theorem 1.1. *A signed graph $\Sigma = (G, \sigma)$ is balanced if and only if it can be switched to an all positive signed graph.*

An undirected graph G can be considered as an all positive signed graph. This is a more restrictive property than balance because no switching is required to make all edges positive, but balance is still quite restrictive because it requires that all cycle signs be positive. Indeed, balanced signed graphs are the signed graphs that are the most like unsigned graphs.

If $\Sigma = (G, \sigma)$ is a signed graph, then $-\Sigma = (G, -\sigma)$ is the same signed graph with all signs reversed. For example, $-G$ means G with all negative edges. We say Σ is antibalanced when $-\Sigma$ is balanced. It is easy to see that $-(\Sigma^\zeta) = (-\Sigma^\zeta)$ (Theorem 1.4), so by Theorem 1.1 Σ is antibalanced if and only if it switches to all negative signs.

Motivated from the above theorem, as the product $\zeta(u)\zeta(v)$ can be viewed as the inner product of $\zeta(u)$ and $\zeta(v)$ on $\mathbb{R}$, we frame the following definitions to classify unbalanced signed graphs extending the concept of switching to a higher dimension. In what follows, $\Omega = \{-1, 0, 1\}$ and the inner product used is the same as that on $\mathbb{R}^k$ restricted to Ω^k.

Definition 1.2 (Vector Valued Switching or k-Switching)**.** Let $\Sigma = (G, \sigma)$ be a given signed graph where $G = (V, E)$. Let $\zeta : V \to \Omega^k \subset \mathbb{R}^k$ be the vector valued switching function such that

(i) (non-orthogonality) $\langle \zeta(u), \zeta(v) \rangle \neq 0$ for all edges $uv \in E$ and

(ii) the switched signed graph $\Sigma^\zeta = (G, \sigma^\zeta)$ has the signing

$$\sigma^\zeta(uv) = \sigma(uv)\,\mathrm{sgn}(\langle \zeta(u), \zeta(v)\rangle).$$

So the switching considered so far in the literature, from now onwards will be referred to as 1-switching and if $k > 1$, the generalized switching introduced now will be mentioned as k-switching.

Remark 1.3. The zero vector is a possible value of $\zeta(v)$, but only if v is an isolated vertex, because the inner product $\langle \zeta(u), \zeta(v)\rangle$ must not be 0 if there is an edge uv. For this reason, although an isolated vertex in 1-switching can take the value 0 so the usual switching is not precisely the same as 1-switching, the difference is not important.

The product of a 1-switching function η and a k-switching function ζ is defined by $(\eta\zeta)(v) = \eta(v)\zeta(v)$, that is, η acts on ζ pointwise as a scalar multiplier.

Theorem 1.4. *Let $k, k' \geq 1$. A k-switching function ζ, a k'-switching function ζ', and a 1-switching function η satisfy $(\Sigma^\zeta)^{\zeta'} = (\Sigma^{\zeta'})^\zeta$, $\Sigma^{\eta\zeta} = (\Sigma^\eta)^\zeta = (\Sigma^\zeta)^\eta$, $\Sigma^{-\zeta} = \Sigma^\zeta$, $(\Sigma^\zeta)^\zeta = \Sigma$, and $-(\Sigma^\zeta) = (-\Sigma)^\zeta$.*

Proof. Let $\Sigma = (G, \sigma)$ be a given signed graph where $G = (V, E)$ and let $\zeta : V \to \Omega^k$, $\zeta' : V \to \Omega^{k'}$ be vector valued switching functions. Then, for any edge uv in Σ,

$$\begin{aligned}
(\sigma^{\zeta'})^\zeta(uv) &= \sigma^{\zeta'}(uv)\,\mathrm{sgn}(\langle \zeta(u), \zeta(v)\rangle) \\
&= \sigma(uv)\,\mathrm{sgn}(\langle \zeta'(u), \zeta'(v)\rangle)\,\mathrm{sgn}(\langle \zeta(u), \zeta(v)\rangle) \\
&= \sigma^\zeta(uv)\,\mathrm{sgn}(\langle \zeta'(u), \zeta'(v)\rangle) \\
&= (\sigma^\zeta)^{\zeta'}(uv)
\end{aligned}$$

so that $(\sigma^{\zeta'})^\zeta = (\sigma^\zeta)^{\zeta'}$. Thus, $(\Sigma^{\zeta'})^\zeta = (\Sigma^\zeta)^{\zeta'}$.

Replacing ζ' by η,

$$\begin{aligned}(\sigma^\eta)^\zeta(uv) &= \sigma(uv)\operatorname{sgn}(\langle\eta(u),\eta(v)\rangle)\operatorname{sgn}(\langle\zeta(u),\zeta(v)\rangle)\\ &= \sigma(uv)\eta(u)\eta(v)\operatorname{sgn}(\langle\zeta(u),\zeta(v)\rangle)\\ &= \sigma(uv)\operatorname{sgn}(\langle\eta(u)\zeta(u),\eta(v)\zeta(v)\rangle)\\ &= \sigma^{\eta\zeta}(uv)\end{aligned}$$

so that $(\sigma^\eta)^\zeta = \sigma^{\eta\zeta}$ and $(\Sigma^\eta)^\zeta = \Sigma^{\eta\zeta} = (\Sigma^\zeta)^\eta$ by the first formula.

Setting $\eta(v) = -1$ for all v, we obtain $\Sigma^{-\zeta} = \Sigma^\zeta$.

Setting $\zeta' = \zeta$, we have

$$(\sigma^\zeta)^\zeta(uv) = \sigma(uv)\operatorname{sgn}(\langle\zeta(u),\zeta(v)\rangle)\operatorname{sgn}(\langle\zeta(u),\zeta(v)\rangle) = \sigma(uv)$$

so that $(\Sigma^\zeta)^\zeta = \Sigma$.

Finally, negating all edge signs in Σ does not affect the switching formula, so $-(\Sigma^\zeta) = (-\Sigma)^\zeta$. □

2 Balancing and Strong Balancing Dimensions

2.1 Definition and elementary properties

Definition 2.1 (Balancing Dimension)**.** Let $\Sigma = (G,\sigma)$ be a given signed graph where $G = (V,E)$. We say that the balancing dimension of Σ is k and write it as $\operatorname{bdim}(\Sigma)$, if $k \geq 1$ is the least integer such that a vector valued switching function $\zeta : V \to \Omega^k \subset \mathbb{R}^k$ switches Σ to an all positive signed graph. For example, $\operatorname{bdim}(\Sigma) = 1$ if and only if Σ is balanced. We call such a k-switching function ζ a positive k-switching function (briefly a k-positive function) for Σ.

The existence of a k-positive function implies existence in all higher dimensions.

Lemma 2.2. *A signed graph Σ has a k-positive function for every $k \geq$* $\operatorname{bdim}(\Sigma)$.

Proof. Let $j < k$ and let $\zeta : V \to \Omega^j$ be a j-positive function for Σ. Define $\zeta'(v) = (\zeta_1(v), \ldots, \zeta_j(v), 0, \ldots, 0) \in \Omega^k$. Then ζ' is a k-positive function for Σ. In particular, take $j = \text{bdim}(\Sigma)$. □

Definition 2.3 (Strong Balancing Dimension)**.** Let $\Sigma = (G, \sigma)$ be a given signed graph where $G = (V, E)$. We say that the strong balancing dimension of Σ is k and write it as $\text{sbdim}(\Sigma)$, if $k \geq 1$ is the least integer such that there is an injective vector valued switching function $\zeta : V \to \Omega^k$ which switches Σ to an all positive signed graph. However, in case Σ is all positive, we define $\text{sbdim}(\Sigma) = 1$.

We call such a k-switching function ζ an injective positive k-switching function (briefly, a strongly k-positive function) for Σ.

We chose to study injectivity because by allowing higher dimensional switching we open the door to new variations on the definition of a switching function, and injectivity seemed an interesting and attractive such variation.

Theorem 2.4. $\text{bdim}(\Sigma) = 1$ *if and only if* Σ *is balanced.*

Proof. Proof easily follows from Theorem 1.1. □

Theorem 2.5. $\text{sbdim}(\Sigma) = 1$ *if and only if* $\Sigma = K_1$, $K_1 \cup K_1$, $K_1 \cup K_1 \cup K_1$, $-K_2$, *or* $-K_2 \cup K_1$.

Proof. It follows from the injectivity requirement of the 1-switching function. If $\text{sbdim}(\Sigma) = 1$, then Σ has at most 3 vertices and if it has 3, the one with $\zeta(v) = 0$ must be isolated. If there are two non-isolated vertices, they must be negatively adjacent. □

We note that, since 1-switching allows $\zeta(v) = 0$ when v is an isolated vertex, this theorem has a few more examples than would exist if 1-switching were identical to ordinary switching.

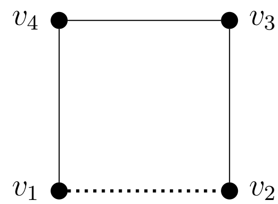

Figure 1: The negative cycle C_4^-

Example 2.6. $\operatorname{bdim}(C_4^-) = \operatorname{sbdim}(C_4^-) = 2$. Take $\zeta(v_1) = (-1, 0)$, $\zeta(v_2) = (1, -1)$, $\zeta(v_3) = (0, -1)$ and $\zeta(v_4) = (-1, -1)$ to see that both dimensions are 2.

Theorem 2.7. bdim *is* 1-*switching invariant.*

Proof. Let Σ be 1-switched to Σ^η and let $\operatorname{bdim}(\Sigma) = k$. Let ζ be a k-positive function for Σ. By Theorem 1.4, $(\Sigma^\eta)^{\eta\zeta} = ((\Sigma^\eta)^\eta)^\zeta = \Sigma^\zeta$, which is all positive, so $\eta\zeta$ is a k-positive function forΣ^η. This shows that $\operatorname{bdim}(\Sigma^\eta) \leq k = \operatorname{bdim}(\Sigma)$. But since $(\Sigma^\eta)^\eta = \Sigma$, that proves the theorem. □

It should be noted that the above theorem is not true for the general k-switching when $k > 1$. We will provide an example in the later part of this paper.

Remark 2.8. Strong balancing dimension need not be 1-switching invariant. Consider the signed graph shown in Figure 2.

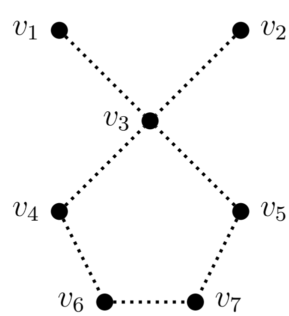

Figure 2: A signed graph Σ with $\operatorname{sbdim}(\Sigma) = 3$

For any non-zero $\alpha \in \Omega^2$, the cardinality of the set $\{\beta \in \Omega^2 : \langle \alpha, \beta \rangle < 0\}$ is 3. Thus, there does not exist an injective switching function from $V(\Sigma)$ to Ω^2 that switches Σ to all positive. Consequently, $\mathrm{sbdim}(\Sigma) > 2$. Now, the switching function $\zeta : V(\Sigma) \to \Omega^3$ defined by $\zeta(v_1) = (1,0,0), \zeta(v_2) = (0,0,1), \zeta(v_3) = (-1,-1,-1), \zeta(v_4) = (0,1,0), \zeta(v_5) = (-1,1,1), \zeta(v_6) = (1,-1,1)$ and $\zeta(v_7) = (1,1,-1)$ is injective, and switches Σ to all positive. Hence, $\mathrm{sbdim}(\Sigma) = 3$.

Let η be the 1-switching function defined on $V(\Sigma)$ as follows: $\eta(v_1) = \eta(v_2) = -1$ and $\eta(v_3) = \eta(v_4) = \eta(v_5) = \eta(v_6) = \eta(v_7) = 1$. The corresponding switched signed graph Σ^η is shown in Figure 3.

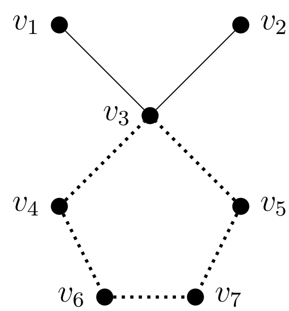

Figure 3: The 1-switched signed graph Σ^η with $\mathrm{sbdim}(\Sigma^\eta) = 2$

A simple computation shows that the injective switching function $\zeta' : V(\Sigma^\mu) \to \Omega^2$ defined by $\zeta'(v_1) = (1,1), \zeta'(v_2) = (1,-1), \zeta'(v_3) = (1,0), \zeta'(v_4) = (-1,1), \zeta'(v_5) = (-1,-1), \zeta'(v_6) = (0,-1)$ and $\zeta'(v_7) = (0,1)$. Consequently, $\mathrm{sbdim}(\Sigma^\mu)$ is 2 and hence sbdim is not 1-switching invariant.

Theorem 2.9. *For any subgraph* Σ' *of* Σ, $\mathrm{bdim}(\Sigma') \leq \mathrm{bdim}(\Sigma)$ *and* $\mathrm{sbdim}(\Sigma') \leq \mathrm{sbdim}(\Sigma)$.

Proof. The balancing dimension of Σ' cannot exceed $\mathrm{bdim}(\Sigma)$ since the switching function for Σ, restricted to $V(\Sigma')$ switches Σ' to all positive. The same proves the result for strong balancing dimension. □

Our next theorem shows how the balancing dimension of a disconnected signed graph depends on its connected components.

Theorem 2.10. *The balancing dimension of a disconnected graph is the largest balancing dimension of its connected components.*

Proof. Let $\Sigma = (G, \sigma)$ be a signed graph having t components $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$. Let n be the largest balancing dimension of any component Σ_i. By Theorem 2.9, $\mathrm{bdim}(\Sigma) \geq \mathrm{bdim}(\Sigma_i)$ for all i. Thus, $\mathrm{bdim}(\Sigma) \geq n$. Since $n \geq \mathrm{bdim}(\Sigma_i)$ for every i, by Lemma 2.2 there exists an n-positive function ζ_i for every component Σ_i. Define $\zeta : V(\Sigma) \to \Omega^n$ by $\zeta(v) = \zeta_i(v)$ if the component that contains vertex v is Σ_i. Since each ζ_i switches Σ_i to all positive, ζ switches Σ to all positive. Thus, $\mathrm{bdim}(\Sigma) = n$. □

Remark 2.11. Because of the additional injectivity condition, the above result will not hold for strong balancing dimension. As an illustration, let us consider Σ as the signed graph that consists of 3 disjoint copies of the negative cycle C_3^-. Since Σ has 9 vertices, there does not exist an injective switching function $\zeta : V(\Sigma) \to \Omega^3$.

Theorem 2.12. *Adding pendant edges to a signed graph will not change its balancing dimension.*

Proof. Let Σ be a given signed graph with vertices $v_1, v_2, \ldots, v_n$, and let Σ' be the signed graph obtained by adding a pendant edge $v_i u$, where u is the pendant vertex. Suppose $\mathrm{bdim}(\Sigma) = k$ and let ζ be the corresponding k-positive function. We extend ζ to Σ' by defining $\zeta(u)$ as follows.

$$\zeta(u) = \begin{cases} \zeta(v_i) & \text{if } \sigma(v_i u) = +1, \\ -\zeta(v_i) & \text{if } \sigma(v_u) = -1. \end{cases}$$

This proves that $\mathrm{bdim}(\Sigma') = k$. Similarly, we can add any number of pendent edges to a signed graph without changing its balancing dimension.

2.2 Bounds for balancing dimensions

We begin with upper bounds. Let $\Sigma = (G, \sigma)$ be a signed graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges $e_1, e_2, \ldots, e_m$. For each edge $e_k = v_i v_j$, we define a vector $\mathbf{b}(e_k) = \begin{pmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{pmatrix} \in \mathbb{R}^{n\times 1}$, whose i^{th} and j^{th} entries are $b_{ik} = \pm 1$ and $b_{jk} = b_{ik}\sigma(e_k)$, respectively, and whose other entries are 0. We now define B as the $n \times m$ matrix whose k^{th} column is the column vector $\mathbf{b}(e_k)$; that is,

$$B = \begin{bmatrix} \mathbf{b}(e_1) & \mathbf{b}(e_2) & \cdots & \mathbf{b}(e_m) \end{bmatrix} = \begin{pmatrix} b_{ij} \end{pmatrix}_{n\times m}.$$

The matrix B is precisely the incidence matrix of the signed graph $-\Sigma$. We now define a switching function $\mu : V(\Sigma) \to \Omega^m$ by

$$\mu(v_i) = (b_{i1}, b_{i2}, \ldots, b_{im})$$

for $i = 1, 2, \ldots, n$. Then, the function μ satisfies the following properties:

Property (i): $\langle \mu(v_i), \mu(v_j) \rangle = \sigma(v_i v_j)$ for all edges $v_i v_j$ in Σ.

Property (ii): $\|\mu(v_i)\|^2 = d(v_i)$ for all vertices v_i in Σ.

By Property (i),

$$\begin{aligned} \sigma^\mu(v_i v_j) &= \sigma(v_i v_j)\,\mathrm{sgn}(\langle \mu(v_i), \mu(v_j) \rangle) \\ &= \sigma(v_i v_j)^2 = 1. \end{aligned}$$

Thus, for every signed graph Σ with m edges, there always exists a switching function $\mu : V(\Sigma) \to \Omega^m$ that switches Σ to all positive, but μ is not always injective. This is the first step towards the following theorem.

Theorem 2.13. *Let Σ be a signed graph. It satisfies the inequalities $1 \leq \mathrm{bdim}(\Sigma) \leq \mathrm{sbdim}(\Sigma)$ and $\mathrm{bdim}(\Sigma) \leq m$. Furthermore, $\mathrm{sbdim}(\Sigma) \leq m$ if Σ has at most one isolated vertex and no component that is a positive edge.*

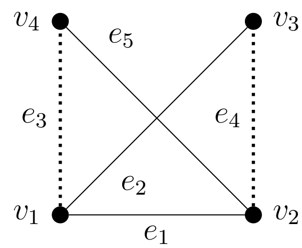

Figure 4: An illustratiion

Proof. The inequalities $1 \leq \mathrm{bdim}(\Sigma) \leq \mathrm{sbdim}(\Sigma)$ follow from the definitions. We have already shown that $\mathrm{bdim}(\Sigma) \leq m$.

In order to bound $\mathrm{sbdim}(\Sigma)$ we need to know when μ is not injective. There are two ways μ can fail to be injective. First, since $\mu(v) = \mathbf{0}$ if v is isolated, $\mu(v) = \mu(w)$ if both v and w are isolated. Second, if G has a component K_2, then $\mu(u) = \sigma(uv)\mu(v) =$ the vector with 1 in the position of edge uv and 0 in all other positions, so $\mu(u) = \mu(v)$ if uv is a positive edge. In all other cases, every vertex has a different set of incident edges so all vectors $\mu(v)$ are distinct. This proves the third inequality. □

Example 2.14. Let Σ be the unbalanced signed graph shown in Figure 4. Then,

$$B = \begin{bmatrix} \mathbf{b}(e_1) & \mathbf{b}(e_2) & \mathbf{b}(e_3) & \mathbf{b}(e_4) & \mathbf{b}(e_5) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ \sigma(e_1) & 0 & 0 & 1 & 1 \\ 0 & \sigma(e_2) & 0 & \sigma(e_4) & 0 \\ 0 & 0 & \sigma(e_3) & 0 & \sigma(e_5) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Define $\mu : V(\Sigma) \to \Omega^5$ as follows:

$$\mu(v_1) = (1, 1, 1, 0, 0),$$
$$\mu(v_2) = (1, 0, 0, 1, 1),$$
$$\mu(v_2) = (1, 0, 0, 1, 1),$$
$$\mu(v_3) = (0, 1, 0, -1, 0),$$
$$\mu(v_4) = (0, 0, -1, 0, 1).$$

Then μ is injective and satisfies $\langle \mu(v_i), \mu(v_j) \rangle = \sigma(v_i v_j)$ for all edges $v_i v_j$ in Σ. Thus μ switches Σ to all positive. Also, we can observe that $\|\mu(v_i)\|^2 = d(v_i)$ for all vertices v_i in Σ.

A lower bound exists in terms of the structure of the underlying graph G of Σ. The clique number $\omega(G)$ denotes the largest order of a clique in G. Let $\lambda(k)$ denote the largest number of pairwise non-orthogonal lines generated by the vectors in Ω^k. For instance, $\lambda(2) = 2$. The largest number of pairwise non-orthogonal vectors in Ω^k equals $2\lambda(k)$. Computation of λ appears to be hard, but λ does give a lower bound on balancing dimension.

Theorem 2.15. *We have* $\operatorname{bdim}(\Sigma) \geq \lambda^{-1}(\frac{1}{2}\omega(G))$.

Proof. Let ζ be a positive k-switching function for Σ. In a clique of order p all the vectors $\zeta(v)$ for the vertices of the clique must be non-orthogonal. Therefore, $p \geq 2\lambda(k)$, equivalently $k \geq \lambda^{-1}(p/2)$. Considering a clique of maximum order gives the theorem. □

Negative triangles are important.

Theorem 2.16. *If* Σ *contains a negative triangle, then* $\operatorname{bdim}(\Sigma) \geq 3$.

Proof. Let C_3^- be a negative triangle in Σ. First we prove that $\operatorname{bdim}(C_3^-) \geq 3$. Suppose ζ is a 2-switching function that makes C_3^- all positive. All vectors $\zeta(v)$ are non-orthogonal because all vertices are adjacent. There are exactly 4 lines generated by Ω^2 and only two of them can be chosen to be non-orthogonal. Without loss of generality, let the lines be $x_2 = 0$ and $x_1 = x_2$. The corresponding vectors are $(1, 0)$, $(1, 1)$, $(-1, 0)$, and $(-1, -1)$. The first pair have positive inner product and the second pair have positive inner product, but any one of the first pair has negative inner product with each of the second pair. Therefore, the signs generated by ζ are the same as the signs generated by the 1-switching function μ given by

$$\mu(v) = \begin{cases} +1 & \text{if } \zeta(v) \in \{(1,0),\ (1,1)\}, \\ -1 & \text{if } \zeta(v) \in \{(-1,0),\ (-1,-1)\}. \end{cases}$$

Thus, μ is a 1-switching function that makes C_3^- all positive, hence $\operatorname{bdim}(C_3^-) = 1$, i.e., C_3^- is balanced, contrary to assumption.

Then $\operatorname{bdim}(\Sigma) \geq \operatorname{bdim}(C_3^-) \geq 3$ by Theorem 2.9 and the fact that Σ is unbalanced. □

There is a simple lower bound on strong balancing dimension.

Theorem 2.17. *For a signed graph with n vertices, none of them being isolated,* $\operatorname{sbdim}(\Sigma) \geq \log_3(n + 1)$*.*

Proof. All vectors $\zeta(v)$ must be distinct and non-zero. In Ω^k there are $3^k - 1$ distinct non-zero vectors. Therefore, $n \leq 3^k - 1$, from which the result follows. □

It would be interesting to know whether there are many signed graphs for which the lower bound is attained, i.e., $\operatorname{sbdim}(\Sigma) = \lceil \log_3(n + 1) \rceil$.

2.3 Special m-switching

In Section 2.2, we have seen that for every signed graph Σ with m edges, there exists an injective function $\mu : V(\Sigma) \to \Omega^m$ that switches Σ to all positive. This function

μ, which induces a special kind of switching on Σ that we call special m-switching, satisfies some additional properties compared to the usual switching functions. More precisely, we define two types of special m-switching in signed graphs; namely, special m-switchings of the first kind and second kind.

Definition 2.18 (Special m-Switching of the First Kind)**.** Let $\Sigma = (G, \sigma)$ be a signed graph with n vertices and m edges. The function $\mu_1 : V(\Sigma) \to \Omega^m$ is called a special m-switching function of the first kind, if it satisfies the following properties:

(i) $\langle \mu_1(u), \mu_1(v) \rangle = \sigma(uv)$ for every edge uv in Σ.

(ii) $\|\mu_1(u)\|^2 = d(u)$ for every vertex u in Σ.

Definition 2.19 (Special m-Switching of the Second Kind)**.** Let $\Sigma = (G, \sigma)$ be a signed graph with n vertices and m edges. The function $\mu_2 : V(\Sigma) \to \Omega^m$ is called a special m-switching function of the second kind, if it satisfies the following properties:

(i) $\langle \mu_2(u), \mu_2(v) \rangle = -\sigma(uv)$ for every edge uv in Σ.

(ii) $\|\mu_2(u)\|^2 = d(u)$ for every vertex u in Σ.

We observe that a special m-switching function of Σ of the second kind is identical to a special m-switching function of $-\Sigma$ of the first kind.

The Laplacian matrix of $\Sigma = (G, \sigma)$ is defined as $L(\Sigma) = D(G) - A(\Sigma)$, where $A(\Sigma)$ is the adjacency matrix, an $n \times n$ matrix whose entries are $a_{ij} = \sigma(v_i v_j)$ if there is an edge $v_i v_j$ and 0 if not, and $D(G)$ is the diagonal degree matrix of G.

Theorem 2.20. *Let $\Sigma = (G, \sigma)$ be a signed graph with n vertices and m edges and $\mu_1, \mu_2 : V(\Sigma) \to \Omega^m$ be the special m-switching functions of the first and second kind respectively.*

(i) If B is the $n \times m$ matrix with i^{th} row given by $\mu_1(v_i)$, then $BB^T = L(-\Sigma)$.

(ii) If H is the $n \times m$ matrix with i^{th} row given by $\mu_2(v_i)$, then $HH^T = L(\Sigma)$.

Proof. The matrices B and H are the incidence matrices of $-\Sigma$ and Σ, respectively (see [2, Section 8B]). Hence, $BB^T = L(-\Sigma)$ and $HH^T = L(\Sigma)$. □

3 Some Classes of Signed Graphs

In this section, we compute the balancing and strong balancing dimensions of certain classes of unbalanced signed graphs.

3.1 Cycles and wheels

Lemma 3.1. *For any unbalanced cycle C_n^-,*

$$\text{bdim}(C_n^-) = \begin{cases} 3 & \text{if } n = 3, \\ 2 & \text{if } n > 3. \end{cases}$$

Proof. Since balancing dimension is 1-switching invariant, we have only to consider a signed cycle $C_n^- = v_1e_1v_2\cdots v_ne_nv_1$ where $\sigma(e_n) = -1$ and other edges e_i are all positive. If $n > 3$, define $\zeta : V(C_n^-) \to \Omega^2$ by $\zeta(v_1) = (1,0)$, $\zeta(v_2) = (1,1)$, $\zeta(v_n) = (-1,1)$ and for $i = 3,4,\ldots,n-1$, $\zeta(v_i) = (0,1)$. A simple computation proves that this is the required 2-switching for making $\text{bdim}(C_n^-) = 2$.

We now prove that $\text{bdim}(C_3^-) = 3$. We know $\text{bdim}(C_3^-) \geq 3$ by Theorem 2.16. Define $\zeta : V(C_3^-) \to \Omega^3$ by $\zeta(v_1) = (1,0,0)$, $\zeta(v_2) = (1,1,1)$ and $\zeta(v_3) = (-1,1,1)$. This 3-switching function shows that $\text{bdim}(C_3^-) \leq 3$. □

Remark 3.2. The switching function ζ defined for C_3^- in Lemma 3.1 is injective and hence $\text{sbdim}(C_3^-) = 3$. If C_3^- is all negative, then the injective switching function $\zeta : V(C_3^-) \to \Omega^3$ by $\zeta(v_1) = (-1,1,1), \zeta(v_2) = (1,-1,1)$ and $\zeta(v_3) = (1,1,-1)$

switches C_3^- to all positive. Thus, the unbalanced cycle C_3^- gives us an example of a signed graph for which the bound given in Theorem 2.13 is attained.

Example 3.3. The balancing dimension of a unicyclic graph is that of the unique signed cycle in it. Let Σ be a signed graph with the unique cycle C. Suppose $\mathrm{bdim}(C) = k$ and let $\zeta : V(C) \to \Omega^k$ be the corresponding switching function. We can extend ζ to $V(\Sigma)$ by adding pendant edges. Hence, by Theorem 2.12, $\mathrm{bdim}(\Sigma) = \mathrm{bdim}(C) = k$.

We now show that the balancing dimension of an antibalanced signed wheel is 3. W_{n+1} denotes the wheel with n spokes.

Proposition 3.4. *For an antibalanced signed wheel W_{n+1}^- with $n \geq 3$, $\mathrm{bdim}(W_{n+1}^-) = 3$.*

Proof. Since balancing dimension is 1-switching invariant, we let $(W_{n+1}^-, \sigma) = (C_n \vee K_1, \sigma)$ with the sign function σ given by $\sigma(e) = -1$ if and only if $e \in E(C_n)$. Let $C_n = v_1 v_2 \cdots v_n$ and $v_{n+1} = K_1$. Define $\zeta : V(W_{n+1}^-) \to \Omega^3$ as follows: Choose $\zeta(v_{n+1}) = (1, 1, 1)$. If $n = 3k$ or $3k + 2$, assign $\zeta(v_1) = (-1, 1, 1)$ and for $i = 2, 3, \ldots, n$, $\zeta(v_i)$ is obtained by performing one left circular shift to $\zeta(v_{i-1})$. If $n = 3k + 1$, assign $\zeta(v_1) = (-1, 1, 1)$, $\zeta(v_n) = (1, 1, -1)$ and for $i = 2, 3, \ldots, n - 1$, $\zeta(v_i)$ is obtained by performing one left circular shift to $\zeta(v_{i-1})$. □

Remark 3.5. For the antibalanced signed wheel W_4^- defined above, $\mathrm{sbdim}(W_4^-) = 3$ since the switching function ζ defined in the proof of Proposition 3.4 is injective.

3.2 Complete graphs and antibalanced signed graphs

We now focus on the balancing dimension of unbalanced signed complete graphs. Since any unbalanced signed complete graph Σ contains C_3^- as a subgraph, $\mathrm{bdim}(\Sigma) \geq 3$. The following is an example in which the lower bound for balancing dimension is attained.

Example 3.6. Let Σ be a signed complete graph with n vertices $v_1, v_2, \ldots, v_n$ and having only one negative edge, say $v_1 v_n$. Then every 3-cycle containing the edge $v_1 v_n$ is negative and hence Σ is unbalanced. Define $\zeta : V(\Sigma) \to \Omega^3$ by, $\zeta(v_1) = (-1, 1, 1)$, $\zeta(v_n) = (1, 1, -1)$ and for $i = 2, 3, \ldots, n-1$, $\zeta(v_i) = (1, 1, 1)$. Then, ζ switches Σ to all positive and hence $\operatorname{bdim}(\Sigma) = 3$.

In the following proposition we provide a class of signed graphs in which the balancing dimension and strong balancing dimension coincide.

Proposition 3.7. *If Σ is an all negative signed complete graph, then* $\operatorname{bdim}(\Sigma) = \operatorname{sbdim}(\Sigma)$.

Proof. Let Σ be the all negative signed complete graph. Suppose $\operatorname{bdim}(\Sigma) = n$ and let $\zeta : V(\Sigma) \to \Omega^n$ be the corresponding switching function. If $\zeta(v_i) = \zeta(v_j)$ for some $i \neq j$, then $\operatorname{sgn}(\langle \zeta(v_i), \zeta(v_j) \rangle) = +1$ and hence $\sigma^{\zeta}(v_i v_j) = \sigma(v_i v_j) \operatorname{sgn}(\langle \zeta(v_i), \zeta(v_j) \rangle) = -1$, which is a contradiction. Thus ζ is injective and hence $\operatorname{bdim}(\Sigma) = \operatorname{sbdim}(\Sigma)$. □

Note that these are not the only signed graphs satisfying $\operatorname{bdim}(\Sigma) = \operatorname{sbdim}(\Sigma)$ (see Example 2.6).

Example 3.8. The relationship between balancing dimensions of Σ and $-\Sigma$ is an obvious question. We found that there exist signed graphs satisfying $\operatorname{bdim}(-\Sigma) = \operatorname{bdim}(\Sigma)$. Similarly, there exist signed graphs satisfying $\operatorname{bdim}(-\Sigma) \neq \operatorname{bdim}(\Sigma)$.

(i) Every bipartite signed graph Σ satisfies $\operatorname{bdim}(-\Sigma) = \operatorname{bdim}(\Sigma)$ since Σ can be 1-switched to $-\Sigma$ and bdim is 1-switching invariant.

The result doesn't hold for the sbdim. For example, if we consider Σ as the all positive tree with 3 vertices, then $\operatorname{sbdim}(\Sigma) = 1$ and $\operatorname{sbdim}(-\Sigma) = 2$.

(ii) Let Σ be an odd unbalanced cycle. Then $-\Sigma$ is balanced and hence $\operatorname{bdim}(-\Sigma) = 1 < \operatorname{bdim}(\Sigma)$.

Definition 3.9 ([1])**.** Let W be a nonempty subset of a vector space over the field of real numbers. W is called a negative inner product (NIP) set if $\langle \alpha, \beta \rangle < 0$ for all α and β in W with $\alpha \neq \beta$.

Lemma 3.10 ([1])**.** *In a k-dimensional vector space, there are at most $k+1$ vectors in an NIP set.*

Definition 3.11. We define $\nu(k)$ to be the largest size of an NIP set in Ω^k. Thus, $\nu(k) \leq k+1$. It is easy to see that $\nu(2) = 2$ but $\nu(3)$ is not as easy to determine. We define $\bar{\nu}(n) = \min\{k : \nu(k) \geq n\}$.

Lemma 3.12. *$\nu(k) \geq n$ if and only if $k \geq \bar{\nu}(n)$. In particular, $\bar{\nu}(n) \geq n-1$.*

Proof. We restate the definition of $\bar{\nu}(n)$ as the minimum k such that there exists an NIP set of n elements in Ω^k. Thus, $k \geq \bar{\nu}(n)$ if and only if an NIP set of size n exists in Ω^k. This is equivalent to saying that $n \leq \nu(k)$.

Choosing $k = n-1$, we have $\nu(n-1) \leq n$ so, equivalently, $\bar{\nu}(n) \geq n-1$. □

Theorem 3.13. *Let Σ be an antibalanced signed complete graph on n vertices, where $n \geq 2$. Then* $\text{bdim}(\Sigma) = \bar{\nu}(n) \geq n-1$.

Proof. By 1-switching as necessary assume Σ is all negative. For switching Σ to all positive, we must assign each vertex of Σ one element from an NIP set with elements in Ω^k for some k. Thus, by Lemma 3.10, it is necessary and sufficient that $n \leq \nu(k)$; equivalently by Lemma 3.12, $\bar{\nu}(n) \leq k$. It follows that the smallest possible k is $\bar{\nu}(n) \geq n-1$. □

Example 3.14. Let Σ be the antibalanced signed complete graph on 5 vertices. Then by Theorem 3.13, we have $\text{bdim}(\Sigma) \geq 4$. Let us define a 3-switching function $\mu : V(K_5) \to \Omega^3$ as follows. $\mu(v_1) = \mu(v_2) = \mu(v_3) = (1,1,1)$, $\mu(v_4) = (-1,1,-1)$ and $\mu(v_5) = (-1,-1,1)$. We will show that the switched signed graph Σ^μ has balancing dimension 3. Since Σ^μ contains C_3^- as a subgraph, $\text{bdim}(\Sigma^\mu) \geq 3$. Now, the function $\zeta : V(\Sigma^\mu) \to \Omega^3$, defined by $\zeta(v_1) = (1, 1, -1)$, $\zeta(v_2) = (-1, 1, 1)$, $\zeta(v_3) = (1, -1, 1)$ and $\zeta(v_4) = \zeta(v_5) = (1, 1, 1)$ switches Σ^μ to all positive. Hence $\mathrm{bdim}(\Sigma^\mu) = 3$.

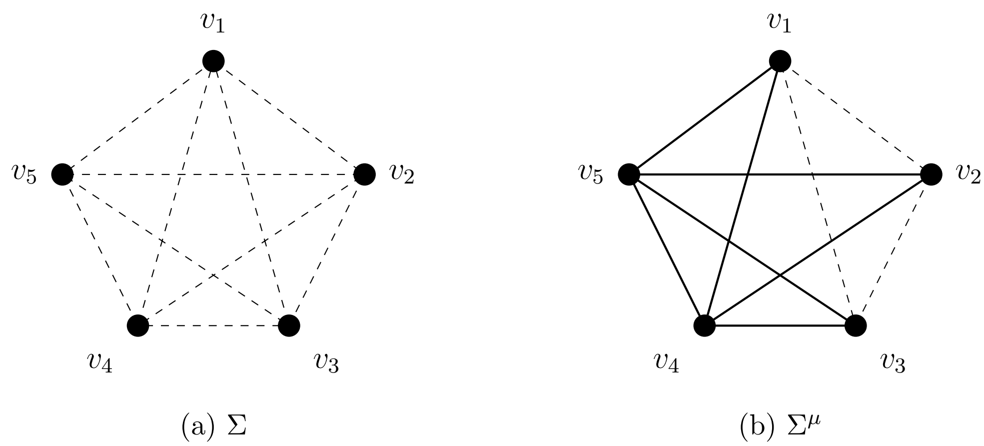

Figure 5: A 3-switching on $\Sigma = -K_5$

This example leads us to the following conclusions.

1. Though balancing dimension is 1-switching invariant, the same need not be true for a general k-switching, where $k \geq 2$.
2. If $\mathrm{bdim}(\Sigma) = n$, then for $k = 2, 3, \ldots, n-1$, a general k-switching need not leave the signs of all cycles unchanged.

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