

The 2026 Cubic Jacobian Counterexample through the Borisov–Gabber–Vasiu Framework

Normalization, boundary, cubic algebra, resolvent, and positive-characteristic consequences

July 23, 2026

Abstract

Let $F: \mathbf{A}^3 \rightarrow \mathbf{A}^3$ be the degree-three Keller map announced in July 2026. This note identifies its finite cubic completion with the normalization X_F used by Borisov, Gabber, and Vasiu, and then computes the resulting BGV geometry explicitly. The completion is a smooth finite flat cubic cover $\pi: X \rightarrow \mathbf{A}^3$. Its boundary is one irreducible ramification divisor $D \simeq \mathbf{A}^2$, its branch and nonproperness locus is the discriminant surface Σ , and the omitted locus is the triple-root curve $\Gamma \simeq \mathbf{G}_m$. The reduced pullback of Σ has a second component $E \simeq \mathbf{A}^2$. Both D and E normalize Σ , and they meet tangentially along Γ with multiplicity two. The map $D \rightarrow \Sigma$ has conductor equal to the ideal of Γ , and its normalization quotient is $\mathcal{O}_\Gamma$. In BGV notation one obtains

$$\deg(F) = 3, \quad \mathcal{F}(F) = \{0, 1\}, \quad \rho(F) = 1, \quad \rho_{\text{ét}}(F) = 0,$$

with $\nu(F) = \nu^{\text{n-ét}}(F) = 1$, weak type exactly 2, and $\text{Cl}(X) = \text{Pic}(X) \simeq \mathbf{Z}$.

The cubic algebra $\pi_*\mathcal{O}_X$ is written down with an explicit free basis and three quadratic relations. Its BGV binary cubic index form is literally the inverse binary cubic of the counterexample. This gives a closed embedding $X \hookrightarrow \mathbf{A}^5$ and, together with the nontrivial canonical class, proves $\text{edim}(X) = 5$ and $\text{ciedim}(X) = \infty$. The Galois closure is the ordered-root cover with group S_3 . The BGV quadratic resolvent is $Y = V(w^2 - \Delta)$, whose singular locus is the curve Γ and whose transverse singularity is of type A_2 . Globally,

$$\text{Pic}(Y) = 0, \quad \text{Cl}(Y) = \text{Pic}(Y_{\text{reg}}) \simeq \mathbf{Z}/3\mathbf{Z}.$$

The generator is the BGV eigensheaf of the cyclic cubic cover. Thus their order-three obstruction and their codimension bound are both attained exactly.

The integral formulas also have stronger positive-characteristic consequences than the original degree-three example alone. Explicit integral deformations give, in every odd characteristic ℓ , nonautomorphic Keller maps of every prescribed geometric degree $d \geq 3$. Consequently Adjmagbo’s prime-to- ℓ conjecture fails in every odd characteristic, including characteristic 3. The proposed nonexistence in BGV Problem P24 for degrees strictly between 2 and ℓ is false, and the same problem’s requested counterexamples of degree greater than ℓ exist in every odd characteristic. Finally, the characteristic-three specialization of the cubic example gives a smooth but ramified quadratic resolvent. This contradicts Proposition 26.1(2.j) of the January 14, 2025 BGV manuscript as that statement is presently written. The proof printed by BGV instead supports an étaleness conclusion for the cyclic cover $W_e \rightarrow Y_e$, and that corrected conclusion holds for the specialization.

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1 Introduction and statement of the results

The recent counterexample [2] is most naturally understood neither as an isolated polynomial identity nor as a mysterious failure of local-to-global invertibility. It is the restriction of a finite branched cubic cover after its ramification divisor has been removed. The remarkable point is that the complement of that divisor is again affine three-space.

This geometric construction sits directly inside the framework developed by Borisov, Gabber, and Vasiu in [1]. For a Keller map $e: \mathbf{A}^n \rightarrow \mathbf{A}^n$, BGV normalize the target in the function field of the source. Zariski's Main Theorem embeds the source as an open subset of the resulting finite normalization X_e . They then package the finite map and the open embedding into a self-map $\psi_e: X_e \rightarrow X_e$, called the associated Jacobian variety. Their invariants record the boundary of the open embedding, its ramification, the nonproperness locus, the sizes of exceptional fibers, and the complexity of presenting the normalization.

For the 2026 example, all of these objects can be computed by hand. Several parts of the degree-three theory in [1, Section 26] become concrete equalities. Other parts become sharp boundary cases. The same formulas also settle open questions stated in the introduction and final problem list of [1].

Throughout sections 2 to 9 the ground field is $\mathbf{C}$. The arguments work over any algebraically closed field of characteristic different from 2 with the modifications described later. Positive characteristic is treated systematically in sections 10 and 11.

Theorem 1.1 (Summary theorem). *Let $F: \mathbf{A}^3 \rightarrow \mathbf{A}^3$ be the map defined in equation (2). Let*

$$\Phi_{p,q,r}(S, T) = 2pS^3 - qS^2T + 2ST^2 - rT^3$$

and let

$$X = V(\Phi) \subset \mathbf{A}_{p,q,r}^3 \times \mathbf{P}_{S,T}^1.$$

Then the following statements hold.

- (a) *The projection $\pi: X \rightarrow \mathbf{A}^3$ is finite flat of degree 3. The simple-root locus $U \subset X$ is isomorphic to $\mathbf{A}^3$, and under this isomorphism F is $\pi|_U$.*
- (b) *The BGV normalization X_F is X . If $\iota: \mathbf{A}^3 \xrightarrow{\sim} U \subset X$ is the open embedding, then the associated BGV Jacobian variety is*

$$\psi_F = \iota \circ \pi: X \rightarrow X.$$

Its image and its étale locus are both U .

- (c) *The boundary $D = X \setminus U$ is the multiple-root locus and is isomorphic to $\mathbf{A}^2$. It is the unique ramification divisor of π . Its image is the irreducible discriminant surface*

$$\Sigma = V(\Delta), \quad \Delta = q^2 - 16p - q^3r + 18pqr - 27p^2r^2.$$

The reduced pullback of Σ has a second component $E \simeq \mathbf{A}^2$. Both $D \rightarrow \Sigma$ and $E \rightarrow \Sigma$ are the normalization. The two components meet tangentially along the triple-root curve, with intersection multiplicity two. The boundary deletes D , while $E \setminus \Gamma$ is the simple sheet which remains in the affine source.

Writing $\nu_D: D \rightarrow \Sigma$ for the normalization, its conductor is the ideal of Γ and there is an exact sequence

$$0 \rightarrow \mathcal{O}_\Sigma \rightarrow (\nu_D)_* \mathcal{O}_D \rightarrow \mathcal{O}_\Gamma \rightarrow 0.$$

(d) *The reduced singular locus of Σ is the smooth curve*

$$\Gamma = V(12p - q^2, 3qr - 4) \simeq \mathbf{G}_m.$$

The fibers of F have cardinalities 3, 1, and 0 on $\mathbf{A}^3 \setminus \Sigma$, $\Sigma \setminus \Gamma$, and Γ , respectively. Thus

$$F(\mathbf{A}^3) = \mathbf{A}^3 \setminus \Gamma,$$

while the nonproperness locus is the larger divisor Σ .

(e) *In BGV notation,*

$$\begin{aligned} \deg(F) &= 3, & \mathcal{F}(F) &= \{0, 1\}, & \phi(F) &= \infty, \\ \rho(F) &= 1, & \rho_{\text{ét}}(F) &= 0, & \nu(F) &= \nu^{\text{n-ét}}(F) = 1. \end{aligned}$$

The BGV inequalities relating ρ and ν are equalities at their lower bounds. The map has weak type exactly 2.

(f) *One has*

$$\text{Pic}(X) = \text{Cl}(X) = \mathbf{Z}[D], \quad K_X \simeq \mathcal{O}_X(D).$$

In particular K_X and T_X are nontrivial. Hence X is not an affine complete intersection in any affine space. In fact

$$\text{ciedim}(X) = \infty, \quad \text{edim}(X) = 5.$$

(g) *If $R = \mathbf{C}[p, q, r]$, then $\pi_*\mathcal{O}_X$ is free with basis $1, \omega, \theta$ and multiplication*

$$\omega^2 = -4p + q\omega + 2p\theta, \quad \omega\theta = 2pr, \quad \theta^2 = -qr + r\omega + 2\theta.$$

For $\xi = u\omega + v\theta$, the determinant of $1, \xi, \xi^2$ in this basis is

$$2pu^3 - qu^2v + 2uv^2 - rv^3 = \Phi_{p,q,r}(u, v).$$

Thus the BGV binary cubic index form is exactly the inverse binary cubic of F .

(h) *The cubic function-field extension is non-Galois and has S_3 Galois closure. The BGV quadratic resolvent is*

$$Y = V(w^2 - \Delta) \subset \mathbf{A}^4.$$

Its singular locus is $V(w) \times_{\mathbf{A}^3} \Gamma$, of dimension $1 = 3 - 2$. Transversely to this curve, Y has an A_2 singularity. Moreover,

$$\mathcal{O}(Y)^* = \mathbf{C}^*, \quad \text{Pic}(Y) = 0, \quad \text{Cl}(Y) = \text{Pic}(Y_{\text{reg}}) \simeq \mathbf{Z}/3\mathbf{Z}.$$

The order-three eigensheaf of the ordered-root cover is the generator. It is a line bundle on Y_{reg} which extends to a rank-one reflexive sheaf on Y but not to a line bundle on Y .

(i) *There are integral Keller maps $F_d: \mathbf{A}^3 \rightarrow \mathbf{A}^3$ of every geometric degree $d \geq 3$ whose Jacobian determinant is -2 and which retain the same three-point collision. After reduction modulo any odd prime ℓ , their geometric degree remains d . Therefore Adjmagbo's conjecture fails in every odd characteristic. The proposed nonexistence in BGV Problem P24 for $2 < \deg(e) < \ell$ is false, and prime-to- ℓ counterexamples with degree greater than ℓ also exist.*

- (j) *In characteristic 3, the cubic specialization is surjective and has only fibers of cardinality 3 and 1. Its discriminant is a smooth affine plane, its boundary maps isomorphically to the discriminant, and its BGV quadratic resolvent is itself affine three-space. The quadratic map to the target is nevertheless ramified. Hence BGV Proposition 26.1(2.j) is false as printed. The conclusion supported by its printed proof is that $W_e \rightarrow Y_e$ is étale, and that statement is satisfied here.*

The proofs occupy the rest of the note. The final characteristic-three calculation in [section 11](#) is logically independent of the characteristic-zero counterexample. It detects a specific problem in one statement of [\[1\]](#) rather than merely showing that a BGV conjecture is false.

2 The map and its finite cubic completion

2.1 The polynomial map

Following [\[2, Theorem 3.1\]](#), put

$$A = 1 + xy, \quad B = A^2z + y^2(4 + 3xy), \quad (1)$$

and define

$$P = AB, \quad Q = y + 3xB, \quad R = 2x - 3x^2y - x^3z, \quad F = (P, Q, R). \quad (2)$$

The basic identities behind the example are most transparent on the dense open set $A \neq 0$. Put

$$s = \frac{x}{A}.$$

Then

$$Q = y + 3Ps, \quad R = 2s - ys^2 - Ps^3. \quad (3)$$

The first identity follows directly from $P = AB$. The second is a polynomial identity after denominators are cleared.

The changes of variables $(x, y, z) \mapsto (P, y, s)$ and $(P, y, s) \mapsto (P, Q, R)$ have Jacobian determinants

$$\det \frac{\partial(P, y, s)}{\partial(x, y, z)} = -A, \quad \det \frac{\partial(P, Q, R)}{\partial(P, y, s)} = 2(1 - ys) = \frac{2}{A}.$$

Their product is -2 . Since both sides extend polynomially across $A = 0$, one obtains

$$\det \text{Jac}(F) = -2. \quad (4)$$

The map is not injective. Indeed,

$$\begin{aligned} F(0, 0, -1/4) &= F(1, -3/2, 13/2) \\ &= F(-1, 3/2, 13/2) = (-1/4, 0, 0). \end{aligned} \quad (5)$$

This already disproves polynomial invertibility. The remaining structure comes from solving the inverse problem globally.

2.2 The inverse binary cubic

Let (p, q, r) be coordinates on the target. Eliminating $y = q - 3ps$ from [equation \(3\)](#) gives

$$2ps^3 - qs^2 + 2s - r = 0. \quad (6)$$

The correct global object, introduced in [\[2, Section 4\]](#), is its homogenization

$$\Phi_{p,q,r}(S, T) = 2pS^3 - qS^2T + 2ST^2 - rT^3. \quad (7)$$

Define the incidence threefold

$$X = V(\Phi) \subset \mathbf{A}_{p,q,r}^3 \times \mathbf{P}_{S,T}^1 \quad (8)$$

and let $\pi: X \rightarrow \mathbf{A}^3$ be projection to (p, q, r) .

Proposition 2.1. *The map π is finite flat of degree 3. The variety X is smooth and affine.*

Proof. For every (p, q, r) , the section $\Phi_{p,q,r}$ of $\mathcal{O}_{\mathbf{P}^1}(3)$ is nonzero because the coefficient of ST^2 is 2. Thus X is a relative effective Cartier divisor of degree 3 in $\mathbf{P}_{\mathbf{A}^3}^1$. Its projection is proper and has zero-dimensional fibers of length 3, so it is finite. The exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}_{\mathbf{A}^3}^1}(-3) \longrightarrow \mathcal{O}_{\mathbf{P}_{\mathbf{A}^3}^1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

shows that $\pi_*\mathcal{O}_X$ is locally free of rank 3. Hence π is finite flat of degree 3.

On the chart $T = 1$, the defining equation has derivative -1 with respect to r . On the chart $S = 1$, it has derivative 2 with respect to p . These two charts cover X , so X is smooth. A finite morphism to an affine scheme has affine source, hence X is affine. $\square$

Let $U = X^{\text{simp}}$ be the open locus at which $[S : T]$ is a simple root of $\Phi_{p,q,r}$.

Proposition 2.2 (The affine chart theorem). *The morphism*

$$\iota: \mathbf{A}_{x,y,z}^3 \longrightarrow X, \quad (x, y, z) \longmapsto (F(x, y, z), [x : 1 + xy]) \quad (9)$$

is an isomorphism from $\mathbf{A}^3$ onto U . Under this identification,

$$F = \pi|_U.$$

Proof. The pair $[x : A]$ is defined everywhere because x and $A = 1 + xy$ cannot vanish simultaneously. Homogenizing [equation \(3\)](#) gives

$$2Px^3 - Qx^2A + 2xA^2 - RA^3 = 0,$$

so ι takes values in X .

On the chart $T \neq 0$, write $s = S/T$ and set

$$D_s = 1 - qs + 3ps^2 = \frac{1}{2} \frac{\partial}{\partial s} \Phi_{p,q,r}(s, 1). \quad (10)$$

A finite root is simple exactly when $D_s \neq 0$. The inverse formulas are

$$y = q - 3ps, \quad x = \frac{s}{D_s}, \quad z = pD_s^3 - y^2(4 - sy)D_s. \quad (11)$$

Indeed, these formulas give $A = D_s^{-1}$ and $B = pD_s$, after which $P = p$, $Q = q$, and $R = r$ follow.

On the chart $S \neq 0$, put $u = T/S$ and

$$h(u) = 2p - qu + 2u^2 - ru^3.$$

On $h(u) = 0$ define

$$y = -\frac{q}{2} + 3u - \frac{3}{2}ru^2, \quad E = u - y = -\frac{1}{2}h'(u), \quad x = E^{-1}, \quad z = 2E^2 - 3yE - rE^3. \quad (12)$$

The root is simple exactly when $E \neq 0$. On the overlap these formulas agree with [equation \(11\)](#). At $u = 0$ they give $p = 0$, $x = 2/q$, $y = -q/2$, and $z = (5x - r)/x^3$. Thus the inverse is regular on all of U , and the two constructions are inverse to one another. $\square$

Remark 2.3. *The projective root coordinate is essential. The affine coordinate $s = S/T$ sees the generic field extension, but it misses the simple root at infinity which occurs when $p = 0$ and $q \neq 0$. This projective feature will reappear algebraically as nonmonogenicity of the cubic algebra after localization away from the discriminant.*

3 The BGV normalization and Jacobian variety

We recall the precise BGV construction because there are two related objects which should not be conflated. Given a Keller map $e: \mathbf{A}_s^n \rightarrow \mathbf{A}_t^n$, let X_e be the normalization of $\mathbf{A}_t^n$ in the function field of $\mathbf{A}_s^n$. Zariski's Main Theorem gives an open embedding

$$\iota_e: \mathbf{A}_s^n \hookrightarrow X_e,$$

and the normalization map gives a finite morphism

$$\pi_e: X_e \rightarrow \mathbf{A}_t^n.$$

After identifying the two affine spaces by their coordinate names, BGV define the associated Jacobian variety to be the endomorphism

$$\psi_e = \iota_e \circ \pi_e: X_e \longrightarrow X_e. \quad (13)$$

Thus the normal variety X_e is the finite completion, while the phrase “Jacobian variety” in [\[1\]](#) refers to the self-map ψ_e together with its source.

Proposition 3.1. *For the map F , the BGV normalization is the incidence variety X of [equation \(8\)](#). Its open embedding is [equation \(9\)](#), its finite map is π , and its associated Jacobian variety is*

$$\psi_F = \iota \circ \pi: X \longrightarrow X. \quad (14)$$

Explicitly,

$$\psi_F((p, q, r), [S : T]) = (F(p, q, r), [p : 1 + pq]).$$

Moreover,

$$\mathrm{Im}(\psi_F) = U, \quad X_F^{\mathrm{\acute{e}t}} = U,$$

and the restriction $\psi_F|_U$ is conjugate to F through ι .

Proof. By [theorem 2.2](#), $U \simeq \mathbf{A}^3$ has the same function field as X , and π is finite. Since X is smooth, hence normal, it is the normalization of the target in that function field. This identifies X with X_F and proves the formula for ψ_F .

The image of ψ_F is the image of ι , namely U . Since ι is an open immersion, ψ_F is étale exactly where π is étale. A finite root cover is étale exactly at simple roots, so the étale locus is U . Finally,

$$\iota^{-1} \circ \psi_F \circ \iota = \iota^{-1} \circ \iota \circ \pi \circ \iota = F.$$

□

The entire construction is summarized by the commutative diagram

$$\begin{array}{ccccc} \mathbf{A}_s^3 & \xrightarrow{\iota} & X & \xrightarrow{\psi_F = \iota\pi} & X \\ & \searrow F & \downarrow \pi & \nearrow \iota & \\ & & \mathbf{A}_t^3 & & \end{array}$$

where the second copy of ι uses the standard identification $\mathbf{A}_t^3 = \mathbf{A}_s^3$.

4 Boundary, discriminant, and fiber stratification

4.1 The boundary is an affine plane

Let

$$D = X \setminus U.$$

This is the locus where the selected projective root is multiple. A multiple root cannot be $[0 : 1]$, because the derivative of $\Phi(s, 1)$ at $s = 0$ is 2. It is therefore convenient to use the chart $S = 1$ and the coordinate $t = T/S$. Put

$$g(t) = \Phi(1, t) = 2p - qt + 2t^2 - rt^3.$$

The equations $g(t) = g'(t) = 0$ are equivalent to

$$q = 4t - 3rt^2, \quad p = t^2 - rt^3. \tag{15}$$

Proposition 4.1. *The map*

$$\mathbf{A}_{t,r}^2 \longrightarrow D, \quad (t, r) \longmapsto (t^2 - rt^3, 4t - 3rt^2, r, [1 : t])$$

is an isomorphism. In particular, D is smooth and irreducible.

Proof. The equations $g = g' = 0$ give [equation \(15\)](#) uniquely. Conversely, substituting [equation \(15\)](#) makes both equations vanish. The coordinate t is recovered from the selected root and r is a target coordinate, so the parametrization has a regular inverse. □

The finite map π is ramified along D . At its generic point the selected root has multiplicity exactly two, so the generic ramification index is two. There are no other ramification points because U is the simple-root locus.

4.2 The discriminant and its normalization

The normalized discriminant computed in [2, Theorem 4.2] is

$$\Delta = q^2 - 16p - q^3r + 18pqr - 27p^2r^2. \quad (16)$$

The usual cubic discriminant is 4Δ . Set

$$\Sigma = V(\Delta), \quad \Gamma = V(12p - q^2, 3qr - 4). \quad (17)$$

Proposition 4.2. *The image of D under π is Σ . The induced map*

$$\nu_D: D \longrightarrow \Sigma$$

is finite and birational, hence it is the normalization of Σ . The surface Σ is irreducible.

Proof. The image consists exactly of cubics with a multiple root, so it is the discriminant hypersurface. The map is finite because it is the restriction of π . A general cubic on Σ has one double root and one simple root, hence the double root is unique. Therefore $D \rightarrow \Sigma$ is generically one-to-one. Since $D \simeq \mathbf{A}^2$ is normal, it is the normalization. Its image is irreducible. $\square$

A point of D maps to a triple-root cubic precisely when $g''(t) = 0$. Since

$$g''(t) = 4 - 6rt,$$

one obtains $rt = 2/3$, $q = 2t$, and $p = t^2/3$. Eliminating t gives the equations of Γ .

Proposition 4.3. *The curve Γ is smooth and isomorphic to $\mathbf{G}_m$. It is the reduced singular locus of Σ . More precisely, the Jacobian singular ideal of Σ can be written*

$$J_\Delta = (16p - q^3r, (3qr - 4)^2), \quad \sqrt{J_\Delta} = (12p - q^2, 3qr - 4). \quad (18)$$

Proof. The parametrization

$$t \longmapsto \left(\frac{t^2}{3}, 2t, \frac{2}{3t} \right)$$

identifies $\mathbf{G}_m$ with Γ . Direct differentiation of [equation \(16\)](#), followed by elementary elimination, gives [equation \(18\)](#). The radical is the ideal of the displayed smooth curve. $\square$

4.3 The conductor and the transverse cusp

The normalization map contains more information than its exceptional locus. Put

$$A_\Sigma = \mathbf{C}[p, q, r]/(\Delta), \quad B_D = \mathbf{C}[t, r].$$

The inclusion $A_\Sigma \subset B_D$ is obtained from [equation \(15\)](#). Define

$$g = 12p - q^2, \quad h = 3qr - 4, \quad \tau = 3rt - 2, \quad e = 2t - q.$$

On D one has

$$h = -\tau^2, \quad g = -t^2\tau^2, \quad e = t\tau. \quad (19)$$

In particular $e^2 = -g$, and $t = (e + q)/2$. Hence

$$B_D = A_\Sigma[e]. \quad (20)$$

Two further identities are

$$\begin{aligned} he &= 18pr + 2q - 3rq^2, \\ ge &= -54p^2r + 18pq - q^3. \end{aligned} \quad (21)$$

If $b = 18pr + 2q - 3rq^2$, then

$$2b = 3rg - qh. \quad (22)$$

Thus both g and h annihilate the class of e in B_D/A_Σ .

Theorem 4.4. *The conductor of the normalization $A_\Sigma \subset B_D$ is*

$$\mathfrak{c} = (12p - q^2, 3qr - 4) = I_\Gamma. \quad (23)$$

Moreover, multiplication by the class of $e = 2t - q$ gives an isomorphism

$$A_\Sigma/I_\Gamma \xrightarrow{\sim} B_D/A_\Sigma. \quad (24)$$

Equivalently, there is an exact sequence on Σ

$$0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow (\nu_D)_* \mathcal{O}_D \longrightarrow \mathcal{O}_\Gamma \longrightarrow 0. \quad (25)$$

At every point of Γ , the surface Σ has an ordinary cusp transverse to Γ .

Proof. By equation (20), the quotient B_D/A_Σ is generated by the class of e . The relations equation (21) show that I_Γ annihilates this class. It remains to compute the annihilator locally along Γ .

Every point of Γ has $r \neq 0$ and $t \neq 0$. In the completed local ring of D , use r as a parameter along Γ and $\tau = 3rt - 2$ as a transverse parameter. Solving for t and substituting in equation (15) gives

$$t = \frac{\tau + 2}{3r}, \quad q = \frac{4 - \tau^2}{3r}, \quad p = \frac{4 - 3\tau^2 - \tau^3}{27r^2}. \quad (26)$$

Conversely,

$$\tau^2 = 4 - 3rq, \quad \tau^3 = 4 - 3\tau^2 - 27r^2p,$$

so the inclusions determined by equation (26) are equalities. If z is a local parameter on Γ , these formulas identify the completed local rings with

$$\hat{\mathcal{O}}_{D,\tilde{\gamma}} \simeq \mathbf{C}[[z, \tau]], \quad \hat{\mathcal{O}}_{\Sigma,\gamma} \simeq \mathbf{C}[[z, \tau^2, \tau^3]]. \quad (27)$$

The conductor of

$$\mathbf{C}[[z, \tau^2, \tau^3]] \subset \mathbf{C}[[z, \tau]]$$

is (τ^2, τ^3) . Under equations (19) and (22), this ideal is generated in A_Σ by g and h . The quotient of the normalization by the cusp ring is freely generated over $\mathbf{C}[[z]]$ by the class of τ , and the class of $e = t\tau$ is also a generator because t is a unit along Γ .

Away from Γ , the normalization is an isomorphism. The local calculations therefore globalize to equations (23) to (25). The completed ring in equation (27) is the product of a smooth curve with the plane cusp $\mathbf{C}[[\tau^2, \tau^3]]$, which proves the final assertion. $\square$

4.4 Fibers, image, and nonproperness

Theorem 4.5. *For $a = (p, q, r) \in \mathbf{A}^3$, the fiber of F has cardinality*

$$\#F^{-1}(a) = \begin{cases} 3, & a \notin \Sigma, \\ 1, & a \in \Sigma \setminus \Gamma, \\ 0, & a \in \Gamma. \end{cases}$$

Consequently,

$$F(\mathbf{A}^3) = \mathbf{A}^3 \setminus \Gamma.$$

The nonproperness locus of F is Σ .

Proof. By theorem 2.2, points of $F^{-1}(a)$ are exactly the simple projective roots of Φ_a . Away from Σ , all three roots are simple. On $\Sigma \setminus \Gamma$, there is one double root and one simple root, so only the simple root contributes a point of U . On Γ , the cubic has one triple root, which lies in D , so the fiber in U is empty.

Since π is finite, F is proper over $\mathbf{A}^3 \setminus \Sigma$. At a general point of Σ , the ramified point lies on D and can be approached by points of U escaping every compact subset under the identification $U \simeq \mathbf{A}^3$. Thus every point of Σ is a nonproper value. This also follows from the general fact that the nonproperness locus is the image of the boundary in the finite normalization. Hence the nonproperness locus is exactly $\pi(D) = \Sigma$. $\square$

There are three distinct loci which play different roles:

$$D = X \setminus U, \quad \Sigma = \pi(D), \quad \Gamma = \mathbf{A}^3 \setminus F(\mathbf{A}^3).$$

The first is a divisor in the normalization, the second is a divisor in the target, and the third has codimension two. The distinction between Σ and Γ is central in the BGV formalism.

4.5 The full discriminant divisor upstairs

The principal divisor of $\Delta \circ \pi$ on X has two components. Besides the ramification divisor D , there is a divisor E which generically selects the simple root of a cubic with one double root.

Proposition 4.6. *As Weil divisors on the smooth threefold X ,*

$$\operatorname{div}_X(\Delta \circ \pi) = 2D + E. \tag{28}$$

The component E is generically étale over Σ . Its intersection with D lies above the triple-root curve Γ .

Proof. Near a general point of the double-root sheet, the finite map has the local form $u = t^2$, so the discriminant parameter vanishes to order two on the ramification divisor. The remaining simple sheet maps isomorphically to the discriminant divisor at its generic point, so the vanishing order there is one. These are all points over the generic point of Σ . The two selected roots coincide exactly for a triple-root cubic, which gives the last assertion. $\square$

4.6 The surviving discriminant sheet

Let E be the reduced closure in X of the simple-root sheet above $\Sigma \setminus \Gamma$. Thus E is the component in [equation \(28\)](#) which is not the ramification divisor.

Theorem 4.7. *The second discriminant component is another copy of the normalization:*

$$E \simeq \mathbf{A}_{t,r}^2, \quad \pi|_E = \nu_D.$$

An explicit isomorphism is

$$\begin{aligned} j: \mathbf{A}_{t,r}^2 &\longrightarrow E, \\ (t, r) &\longmapsto (t^2 - rt^3, 4t - 3rt^2, r, [r : 2(1 - rt)]). \end{aligned} \tag{29}$$

The reduced intersection $D \cap E$ is the curve Γ . On either smooth component, the scheme-theoretic intersection is twice this curve:

$$E|_D = 2\Gamma, \quad D|_E = 2\Gamma. \tag{30}$$

Consequently D and E are tangent along Γ . Moreover,

$$E \cap U = E \setminus \Gamma \xrightarrow{\sim} \Sigma \setminus \Gamma, \quad F^{-1}(\Sigma) \simeq \mathbf{A}^2 \setminus V(3rt - 2). \tag{31}$$

Proof. For a point of D with coordinates (t, r) , substitute [equation \(15\)](#) into the cubic on the chart $S = 1$. One obtains the factorization

$$2p - qu + 2u^2 - ru^3 = -(u - t)^2(ru + 2rt - 2). \tag{32}$$

The double root is $u = t$. The remaining simple root is

$$[1 : u] = [r : 2(1 - rt)],$$

where the expression on the right also covers the root at infinity when $r = 0$. This proves that [equation \(29\)](#) takes values in E and that its composition with π equals the normalization map ν_D .

The residual double root of a point of E cannot be $[0 : 1]$, because the coefficient of ST^2 in Φ is the unit 2. It therefore has a unique coordinate $t = T/S$, while r is already a target coordinate. Formula [equation \(32\)](#) then reconstructs the selected simple root. Thus (t, r) is a regular inverse to j , proving $E \simeq \mathbf{A}^2$.

The selected root and the double root coincide exactly when

$$3rt - 2 = 0.$$

This is the triple-root curve, so the reduced intersection is Γ . Locally along this curve, r is a unit. The discriminant of the residual quadratic on D is, up to a unit, the square of the difference between the simple root and the double root. By [equation \(32\)](#), that difference is $(2 - 3rt)/r$. Hence a local equation of E restricts to a unit times $(3rt - 2)^2$ on D . The same calculation with the roles of the selected and residual roots reversed gives the second equality in [equation \(30\)](#).

Away from Γ , both normalization maps are isomorphisms. The component E selects the unique simple root and is contained in U , while the component D selects the deleted double root. This proves [equation \(31\)](#). $\square$

The divisor D is the BGV boundary, while E is not. This is the exact geometric reason that a general point of the nonproperness divisor still has one affine preimage. At Γ the two normalization sheets coalesce tangentially, so the surviving sheet also reaches the deleted boundary and the fiber becomes empty.

5 The BGV invariants

We now translate the geometry of [section 4](#) into the notation of [\[1\]](#). Their invariant $\rho(e)$ is the number of irreducible components of the boundary $X_e \setminus \iota_e(\mathbf{A}^n)$. The invariant $\rho_{\text{ét}}(e)$ counts those boundary components whose generic points lie in the étale locus of ψ_e . The nonproper locus is denoted N_e , and $\nu(e)$ is its number of irreducible components. The corresponding nonétale image and its number of irreducible components are denoted $N_e^{\text{n-ét}}$ and $\nu^{\text{n-ét}}(e)$.

The non-generic fiber invariant is

$$\mathcal{F}(e) = \{j \in \{0, \dots, \deg(e) - 1\} : \text{some geometric fiber has } j \text{ reduced points}\}.$$

The gap invariant $\phi(e)$ is the cardinality of the complement of the image when that complement is finite, and is ∞ otherwise.

Theorem 5.1. *For the cubic counterexample F one has*

$$\begin{aligned} \deg(F) &= 3, & \mathcal{F}(F) &= \{0, 1\}, & \phi(F) &= \infty, \\ \rho(F) &= 1, & \rho_{\text{ét}}(F) &= 0, & \nu(F) &= 1, & \nu^{\text{n-ét}}(F) &= 1. \end{aligned} \tag{33}$$

The fiber strata of [\[1, Remark 10.2\]](#) are

$$N_0(F) = \Gamma, \quad N_1(F) = \Sigma \setminus \Gamma, \quad N_3(F) = \mathbf{A}^3 \setminus \Sigma. \tag{34}$$

The auxiliary BGV invariants satisfy

$$\varsigma_F = v(F) = 1.$$

Proof. The degree and fiber statements follow from [theorem 4.5](#). Since $\Gamma \simeq \mathbf{G}_m$ has infinitely many complex points, $\phi(F) = \infty$.

The boundary is the single irreducible divisor D , so $\rho(F) = 1$. Its generic point is a ramification point of π , hence also of ψ_F . Therefore $\rho_{\text{ét}}(F) = 0$. Both the nonproper locus and the image of the nonétale locus are the irreducible surface Σ , which gives $\nu(F) = \nu^{\text{n-ét}}(F) = 1$. Finally, the only positive exceptional fiber cardinality is 1, so the minimum and maximum positive elements of $\mathcal{F}(F)$ are both 1. $\square$

BGV prove the general inequalities

$$\nu(e) \leq \rho(e) \leq \nu(e)(\deg(e) - 1)$$

and

$$\nu^{\text{n-ét}}(e) \leq \rho(e) - \rho_{\text{ét}}(e) \leq \nu^{\text{n-ét}}(e)(\deg(e) - 1).$$

For F , both lower bounds are equalities:

$$1 = \nu(F) = \rho(F), \quad 1 = \nu^{\text{n-ét}}(F) = \rho(F) - \rho_{\text{ét}}(F). \tag{35}$$

Thus the counterexample has the smallest possible number of boundary components compatible with nonproperness and ramification.

5.1 Weak type

BGV define weak type by adjoining auxiliary functions to the graph of a Keller map. Their Lemma 22.4 says that every Keller map has weak type at most 2. Their Corollary 25.7 says that a nonautomorphism in characteristic zero has weak type exactly 2.

Corollary 5.2. *The map F has BGV weak type exactly 2.*

Proof. The map is a Keller map and is not an automorphism by [equation \(5\)](#). Apply [1, Corollary 25.7]. $\square$

This explains one precise sense in which the example lies immediately beyond the positive results of [1, Section 25]. Their weak-type-one theorem cannot apply, and the universal upper bound is attained.

5.2 Class group and maximality of the étale locus

BGV observe generally that

$$\mathrm{Cl}(X_e) \simeq \mathbf{Z}^{\rho(e)}.$$

Since X is smooth, its class group equals its Picard group. Therefore [theorem 5.1](#) already gives

$$\mathrm{Pic}(X) = \mathrm{Cl}(X) \simeq \mathbf{Z}. \quad (36)$$

The direct calculation in [section 6](#) identifies a preferred generator and computes the canonical class.

BGV Theorem 7.3 also gives a geometric maximality statement. For an affine regular open subset $V \subset X$ containing U , the tangent bundle T_V is trivial exactly when V is contained in the étale locus of ψ_F . Since the étale locus is already U , one obtains the following.

Corollary 5.3. *The open set $U \simeq \mathbf{A}^3$ is maximal among affine regular open subsets $V \subset X$ which contain U and have trivial tangent bundle.*

The statement is not tautological. The ambient X is smooth and affine, but adjoining any generic point of the boundary destroys triviality of the tangent bundle.

6 Projective compactification and divisor theory

The cubic completion has a useful projective model. Let

$$\overline{X} = \mathbf{P}^1 \times \mathrm{Sym}^2(\mathbf{P}^1) \simeq \mathbf{P}^1 \times \mathbf{P}^2.$$

A point of $\overline{X}$ consists of one selected linear factor and a residual quadratic factor. Multiplication gives a finite morphism

$$\overline{\pi}: \mathbf{P}^1 \times \mathrm{Sym}^2(\mathbf{P}^1) \longrightarrow \mathrm{Sym}^3(\mathbf{P}^1) \simeq \mathbf{P}^3. \quad (37)$$

Generically this map has degree 3, one choice for each root of the cubic.

Let $H \subset \mathbf{P}^3$ be the hyperplane on which the coefficient of ST^2 vanishes. Its complement is the affine chart in which that coefficient can be normalized to 2, hence

$$\mathbf{P}^3 \setminus H \simeq \mathbf{A}_{p,q,r}^3.$$

The incidence variety is the inverse image of this chart:

$$X = \pi^{-1}(\mathbf{P}^3 \setminus H) = \overline{X} \setminus H_\infty, \quad H_\infty = \pi^{-1}(H). \quad (38)$$

Let L and M denote the pullbacks of the hyperplane classes from $\mathbf{P}^1$ and $\mathbf{P}^2$. Since the coefficients of a product of a linear and a quadratic binary form are bilinear,

$$[H_\infty] = L + M. \quad (39)$$

The divisor H_∞ is irreducible. Indeed, over each point of $\mathbf{P}^1$ its equation is one nonzero linear equation in $\mathbf{P}^2$, so it is a $\mathbf{P}^1$ -bundle over $\mathbf{P}^1$.

Proposition 6.1. *The unit group, Picard group, and canonical class of X are*

$$\mathcal{O}(X)^* = \mathbf{C}^*, \quad \text{Pic}(X) \simeq \mathbf{Z}, \quad K_X \simeq \mathcal{O}_X(D). \quad (40)$$

The class of D is a generator of $\text{Pic}(X)$.

Proof. The localization sequence for the complement of the irreducible divisor H_∞ gives

$$\mathbf{Z}[H_\infty] \longrightarrow \text{Pic}(\overline{X}) = \mathbf{Z}L \oplus \mathbf{Z}M \longrightarrow \text{Pic}(X) \longrightarrow 0.$$

By [equation \(39\)](#), the first map sends 1 to $(1, 1)$. Hence

$$\text{Pic}(X) \simeq \mathbf{Z}^2 / \mathbf{Z}(1, 1) \simeq \mathbf{Z}.$$

The same localization sequence for units begins

$$1 \longrightarrow \mathbf{C}^* \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathbf{Z}[H_\infty] \longrightarrow \text{Pic}(\overline{X}).$$

The final map is injective, so $\mathcal{O}(X)^* = \mathbf{C}^*$.

The ramification divisor $\overline{D}$ of π satisfies Riemann–Hurwitz:

$$K_{\overline{X}} = \pi^* K_{\mathbf{P}^3} + \overline{D}.$$

Now $K_{\overline{X}} = -2L - 3M$, while $\pi^* \mathcal{O}_{\mathbf{P}^3}(1) = L + M$. Therefore

$$[\overline{D}] = (-2, -3) + 4(1, 1) = (2, 1). \quad (41)$$

Its restriction to X is the ramification divisor D . Modulo $(1, 1)$, the class $(2, 1)$ equals $(1, 0)$ and is a generator.

Finally,

$$[K_X] = [-2L - 3M] \bmod (L + M) = (1, 0) = [D].$$

This proves $K_X \simeq \mathcal{O}_X(D)$. □

The same canonical bundle calculation follows from adjunction in $\mathbf{A}^3 \times \mathbf{P}^1$. There X is a relative cubic divisor, so

$$K_X \simeq \mathcal{O}_X(-2 + 3) = \mathcal{O}_X(1).$$

The relative different identifies this line bundle with $\mathcal{O}_X(D)$.

Corollary 6.2. *The tangent bundle T_X is nontrivial. The variety X admits no complete-intersection embedding into any affine space. In BGV notation,*

$$\text{ciedim}(X) = \infty.$$

Proof. The determinant of the cotangent bundle is K_X , which is nontrivial by [theorem 6.1](#). Hence T_X is nontrivial. A smooth affine complete intersection has trivial canonical bundle by adjunction. Therefore no such embedding exists. This also recovers [1, Theorem 7.3(5)], since the associated Jacobian variety is regular but nonétale. $\square$

Remark 6.3. *The presentation $X = V(\Phi) \subset \mathbf{A}^3 \times \mathbf{P}^1$ is a hypersurface presentation in a nonaffine ambient space. It does not conflict with [theorem 6.2](#). The obstruction concerns complete-intersection embeddings into affine space.*

The divisor relation of [theorem 4.6](#) becomes particularly simple in $\text{Pic}(X)$:

$$[E] = -2[D]. \quad (42)$$

Indeed, $2D + E$ is principal. Thus the ramified and unramified components over the discriminant have unequal and opposite divisor classes.

7 The finite cubic algebra and the BGV index form

The finite morphism π can be described by a rank-three algebra over

$$R = \mathbf{C}[p, q, r].$$

The construction is clearest for a general binary cubic

$$f(S, T) = aS^3 + bS^2T + cST^2 + dT^3 \quad (43)$$

with coefficients in a ring R .

On the chart $T \neq 0$, let $\alpha = S/T$. Define

$$\omega = a\alpha, \quad \theta = a\alpha^2 + b\alpha + c. \quad (44)$$

On the overlap with the chart $S \neq 0$, where $t = T/S$, the equation $f(1, t) = 0$ gives

$$\omega = -b - ct - dt^2, \quad \theta = -dt. \quad (45)$$

Thus ω and θ extend across roots at infinity.

Proposition 7.1 (The cubic multiplication table). *Let $X_f = V(f) \subset \mathbf{P}_R^1$ and suppose $X_f \rightarrow \text{Spec } R$ is finite flat of degree 3. Then*

$$\Gamma(X_f, \mathcal{O}_{X_f}) = R \oplus R\omega \oplus R\theta$$

and multiplication is given by

$$\begin{aligned} \omega^2 &= -ac - b\omega + a\theta, \\ \omega\theta &= -ad, \\ \theta^2 &= -bd - d\omega + c\theta. \end{aligned} \quad (46)$$

Proof. The identities follow directly from $a\alpha^3 + b\alpha^2 + c\alpha + d = 0$. For example,

$$\omega^2 = a(a\alpha^2) = a\theta - b\omega - ac$$

and

$$\omega\theta = a\alpha(a\alpha^2 + b\alpha + c) = -ad.$$

Also $\alpha\theta = -d$, so

$$\theta^2 = (a\alpha^2 + b\alpha + c)\theta = -d\omega - bd + c\theta.$$

These identities reduce every product to the R -span of $1, \omega, \theta$. Fiberwise this span has dimension 3, so it is the full finite flat algebra and the three displayed elements form a basis. $\square$

For the counterexample,

$$a = 2p, \quad b = -q, \quad c = 2, \quad d = -r.$$

Hence the algebra $A_X = \Gamma(X, \mathcal{O}_X)$ is free over R with basis $1, \omega, \theta$ and

$$\begin{aligned} \omega^2 &= -4p + q\omega + 2p\theta, \\ \omega\theta &= 2pr, \\ \theta^2 &= -qr + r\omega + 2\theta. \end{aligned} \tag{47}$$

7.1 An explicit affine model and exact embedding dimension

The multiplication table gives a closed embedding into affine five-space.

Proposition 7.2. *There is an isomorphism*

$$X \simeq \operatorname{Spec} \frac{\mathbf{C}[p, q, r, \omega, \theta]}{(\omega^2 - q\omega - 2p\theta + 4p, \omega\theta - 2pr, \theta^2 - r\omega - 2\theta + qr)}. \tag{48}$$

The defining ideal is a height-two perfect ideal generated by the signed 2×2 minors of

$$\begin{pmatrix} -\theta & -r \\ \omega - q & \theta - 2 \\ -2p & -\omega \end{pmatrix}. \tag{49}$$

Proof. The relations are exactly [equation \(47\)](#). They reduce every polynomial in ω, θ to an R -linear combination of $1, \omega, \theta$. The induced map from the quotient in [equation \(48\)](#) to A_X is therefore a surjection between R -modules generated by the same three basis elements. Since their images are an R -basis of A_X , the map is an isomorphism.

A direct calculation of the minors of [equation \(49\)](#) gives the three displayed quadrics up to signs. Since the quotient is finite over the three-dimensional ring R , the ideal has height two. The Hilbert–Burch theorem then gives perfection. $\square$

This is the explicit form of the three-quadrics presentation predicted by [1, Proposition 26.1(9)].

Theorem 7.3. *The affine embedding dimension of X is exactly*

$$\operatorname{edim}(X) = 5.$$

Proof. The presentation [equation \(48\)](#) gives $\text{edim}(X) \leq 5$. Since $\dim X = 3$, an embedding into $\mathbf{A}^3$ would identify X with all of $\mathbf{A}^3$. This is impossible because $\text{Pic}(X) \simeq \mathbf{Z}$.

Suppose that X embedded into $\mathbf{A}^4$. Its prime ideal would have height one. Since a polynomial ring over a field is a unique factorization domain, that prime ideal would be principal. Thus X would be an affine hypersurface and hence a complete intersection. This contradicts [theorem 6.2](#). Therefore $\text{edim}(X) \geq 5$. $\square$

The conclusion is stronger than merely saying that the three quadrics in [equation \(48\)](#) cannot be replaced by two generators in the same ambient space. No affine embedding in any dimension can be a complete intersection, and no embedding in fewer than five coordinates exists.

7.2 The index form is the inverse cubic

Let

$$\xi = u\omega + v\theta$$

for u, v in an R -algebra. The element ξ generates the cubic algebra precisely when $1, \xi, \xi^2$ form a basis. The determinant of these three elements in the basis $1, \omega, \theta$ is the cubic index form.

Proposition 7.4. *For the general multiplication table [equation \(46\)](#),*

$$\det(1, \xi, \xi^2) = au^3 + bu^2v + cuv^2 + dv^3. \quad (50)$$

Consequently, for X the index form is

$$L(u, v) = 2pu^3 - qu^2v + 2uv^2 - rv^3 = \Phi_{p,q,r}(u, v). \quad (51)$$

Proof. From [equation \(46\)](#), the coefficients of ω and θ in ξ^2 are

$$-bu^2 - dv^2 \quad \text{and} \quad au^2 + cv^2.$$

The determinant of the coordinate columns of $1, \xi, \xi^2$ is therefore

$$u(au^2 + cv^2) - v(-bu^2 - dv^2) = au^3 + bu^2v + cuv^2 + dv^3.$$

Specializing the coefficients gives [equation \(51\)](#). $\square$

Thus the binary cubic introduced in the counterexample note to solve the inverse problem is exactly the binary cubic which BGV construct abstractly from a free rank-three algebra. There is no change of variables and no unspecified unit in the basis [equation \(44\)](#).

Corollary 7.5. *For any R -algebra C , the algebra $A_X \otimes_R C$ is generated by one element over C if and only if there are $u, v \in C$ such that*

$$\Phi_{p,q,r}(u, v) \in C^*.$$

Proof. An element $\xi = u\omega + v\theta$ generates a free cubic algebra exactly when $1, \xi, \xi^2$ are a basis. By [theorem 7.4](#), this happens exactly when the displayed determinant is a unit. $\square$

For the BGV degree-three notation, both the nonétale divisor and the nonproper divisor are defined by Δ . Thus one may take

$$h_0 = h_1 = h = \Delta.$$

The extra irreducibility condition $(\perp_h)$ in [1, Proposition 26.1] is then vacuous because $h/h_0 = 1$. Their clause (7) gives the following strong form of nonmonogenicity.

Corollary 7.6. *The localized cubic algebra*

$$(A_X)_\Delta$$

is not generated by one element over R_Δ .

Proof. If it were monogenic, [1, Proposition 26.1(7)] would force the characteristic to be 3. The ground field is $\mathbf{C}$. $\square$

The function field is generated by the affine root s , but the finite étale algebra over R_Δ is not globally generated by one regular element. The root at infinity when $p = 0$ is the visible geometric reason. A projective root coordinate $[S : T]$ is not an optional compactification device. It is forced by the global algebra.

8 Galois closure and the BGV quadratic resolvent

On the dense chart $A \neq 0$, the function fields satisfy

$$\mathbf{C}(x, y, z) = \mathbf{C}(p, q, r)(s), \quad 2ps^3 - qs^2 + 2s - r = 0. \quad (52)$$

The extension has degree 3 by [theorem 4.5](#), so the displayed cubic is irreducible over $\mathbf{C}(p, q, r)$.

Proposition 8.1. *The cubic extension [equation \(52\)](#) is non-Galois. Its Galois closure has group S_3 .*

Proof. The discriminant is 4Δ . It is enough to show that Δ is not a square in $\mathbf{C}(p, q, r)$. Regard Δ as a quadratic polynomial in p over $K = \mathbf{C}(q, r)$. Its discriminant with respect to p is

$$\text{disc}_p(\Delta) = -4(3qr - 4)^3, \quad (53)$$

which is nonzero in K . Thus Δ , as an element of $K[p]$, has two distinct roots over an algebraic closure of K . If an element of $K[p]$ is a square in $K(p)$, then every irreducible factor has even valuation, so every root over an algebraic closure has even multiplicity. Hence Δ is not a square in $K(p) = \mathbf{C}(p, q, r)$. A separable irreducible cubic has Galois group A_3 exactly when its discriminant is a square. Therefore the Galois closure has group S_3 . $\square$

This is exactly the non-Galois case (S_3) of [1, Proposition 26.1].

8.1 The ordered-root cover

Let

$$\overline{W} = (\mathbf{P}^1)^3$$

and let

$$\sigma: \overline{W} \longrightarrow \mathrm{Sym}^3(\mathbf{P}^1) \simeq \mathbf{P}^3$$

be the quotient map which forgets the ordering of the three roots. Put

$$W = \sigma^{-1}(\mathbf{P}^3 \setminus H).$$

Then W is a smooth affine threefold with an S_3 action, and $W \rightarrow \mathbf{A}^3$ is the normalization of the target in the splitting field of [equation \(52\)](#).

Let $S_2 \subset S_3$ be the stabilizer of the first root and let $A_3 \subset S_3$ be the alternating subgroup. There are quotient identifications

$$X = W/S_2, \quad Y = W/A_3, \quad \mathbf{A}^3 = W/S_3. \quad (54)$$

The quotient tower is

$$\begin{array}{ccccc}
 & & W & & \\
 & \swarrow 2 & \downarrow 3 & \searrow 6 & \\
 X = W/S_2 & & Y = W/A_3 & & \mathbf{A}^3 = W/S_3 \\
 & \searrow 3 & \downarrow 2 & & \\
 & & \mathbf{A}^3 & &
 \end{array}$$

where the degree labels refer to generic degrees. The diagonal arrow $W \rightarrow \mathbf{A}^3$ is the full S_3 cover.

8.2 The discriminant double cover

The unique quadratic subfield of the S_3 splitting field is obtained by adjoining a square root of the discriminant. Thus the BGV quadratic resolvent is

$$Y = \mathrm{Spec} \frac{\mathbf{C}[p, q, r, w]}{(w^2 - \Delta)}. \quad (55)$$

The finite map $\varphi: Y \rightarrow \mathbf{A}^3$ has degree 2 and is branched along Σ .

Proposition 8.2. *The singular locus of Y is*

$$\mathrm{Sing}(Y) = V(w, 12p - q^2, 3qr - 4) \simeq \Gamma. \quad (56)$$

Hence

$$\dim \mathrm{Sing}(Y) = 1 = 3 - 2.$$

At the generic point of this curve, and analytically at every complex point of it, the transverse singularity is of type A_2 .

Proof. The hypersurface equation is $w^2 - \Delta = 0$. Its gradient vanishes exactly when $w = 0$ and $d\Delta = 0$. By [theorem 4.3](#), this gives [equation \(56\)](#).

There is also a quotient-theoretic proof of the local type. The nonidentity elements of A_3 act on W by cyclically permuting the three roots. Their fixed locus is the small diagonal, where all three roots coincide. This diagonal maps to Γ . Along it, the tangent direction of the diagonal is fixed, while the normal two-dimensional representation has eigencharacters ζ and ζ^2 for a primitive cube root ζ . The quotient of the normal plane is

$$\text{Spec } \mathbf{C}[a, b, c]/(ab - c^3),$$

which is the A_2 surface singularity. The tangent direction along Γ contributes a smooth factor. $\square$

Equivalently, after completing at a point of Γ , one can translate the triple root and put the cubic into depressed form. The discriminant then has the local form $u^3 + v^2$ up to a unit. The resolvent equation becomes

$$w^2 = u^3 + v^2.$$

After a linear change in w, v , this is $ab = u^3$, again the A_2 equation times the smooth parameter along Γ .

Corollary 8.3. *The resolvent satisfies*

$$\text{edim}(Y) = \text{ciedim}(Y) = 4.$$

In particular the estimate $\text{ciedim}(Y) \leq n + 1$ in [1, Proposition 26.1(2.i)] is sharp for this example.

Proof. The equation [equation \(55\)](#) embeds Y as a hypersurface in $\mathbf{A}^4$. It cannot embed into $\mathbf{A}^3$, since a closed irreducible three-dimensional subvariety of $\mathbf{A}^3$ is all of $\mathbf{A}^3$, while Y is singular. Thus both dimensions equal 4. $\square$

BGV prove in general that $\dim \text{Sing}(Y_e) \leq n - 2$. The counterexample attains the largest possible value. Their stronger criterion in Proposition 26.1(2.j), which assumes dimension at most $n - 3$, does not apply in characteristic zero.

8.3 The order-three eigensheaf

Let

$$q: W \longrightarrow Y = W/A_3.$$

The action of $A_3 \simeq \mu_3$ is free away from the small diagonal. Hence

$$q: W|_{Y_{\text{reg}}} \longrightarrow Y_{\text{reg}}$$

is a finite étale cyclic cover of degree 3.

Choose a nontrivial character $\chi: \mu_3 \rightarrow \mathbf{G}_m$. The χ -eigensummand of $q_*\mathcal{O}_W$ is a rank-one reflexive sheaf $\mathcal{M}$ on Y . Its restriction to Y_{reg} is a line bundle with

$$\mathcal{M}^{\otimes 3}|_{Y_{\text{reg}}} \simeq \mathcal{O}_{Y_{\text{reg}}}. \tag{57}$$

Proposition 8.4. *The line bundle $\mathcal{M}|_{Y_{\text{reg}}}$ does not extend to a line bundle on Y . At the generic point of $\text{Sing}(Y)$, its reflexive class generates the local class group*

$$\text{Cl}(\hat{\mathcal{O}}_{Y,\eta}) \simeq \mathbf{Z}/3\mathbf{Z}.$$

Proof. The completed transverse local ring is the A_2 ring

$$K[[a, b, c]]/(ab - c^3),$$

where K is the function field of Γ . This is the invariant ring of $K[[x, y]]$ under μ_3 acting with weights 1 and 2. Its class group is $\mathbf{Z}/3\mathbf{Z}$, and the two nontrivial character modules are the two nonzero classes. The eigensheaf $\mathcal{M}$ is one of these modules. It is reflexive of rank one but is not free at the singular point. Therefore it cannot be the restriction of a line bundle on all of Y . $\square$

8.4 An explicit divisor representing the BGV eigensheaf

The order-three class can be written directly in the coordinates of the resolvent. Choose $\lambda \in \mathbf{C}$ with $\lambda^2 = -27$ and put

$$H = 3qr - 4, \quad \delta_0 = 27r^2p - 9qr + 8, \quad z = \lambda rw. \quad (58)$$

A direct expansion gives

$$z^2 = \delta_0^2 + H^3 \quad \text{in } \mathcal{O}(Y). \quad (59)$$

Proposition 8.5. *The ideals*

$$\mathfrak{p}_+ = (z - \delta_0, H), \quad \mathfrak{p}_- = (z + \delta_0, H) \quad (60)$$

are height-one prime ideals of $\mathcal{O}(Y)$. Their divisor classes are the two nonzero elements of $\text{Cl}(Y)$, and one of them is the BGV eigensheaf class $[\mathcal{M}]$. More precisely,

$$\text{div}_Y(z - \delta_0) = 3V(\mathfrak{p}_+), \quad \text{div}_Y(z + \delta_0) = 3V(\mathfrak{p}_-), \quad \text{div}_Y(H) = V(\mathfrak{p}_+) + V(\mathfrak{p}_-). \quad (61)$$

Thus

$$3[\mathfrak{p}_+] = 0, \quad [\mathfrak{p}_-] = -[\mathfrak{p}_+].$$

The corresponding maximal Cohen–Macaulay modules are described by the matrix factorization

$$\begin{pmatrix} z - \delta_0 & -H^2 \\ -H & z + \delta_0 \end{pmatrix} \begin{pmatrix} z + \delta_0 & H^2 \\ H & z - \delta_0 \end{pmatrix} = (z^2 - \delta_0^2 - H^3)I_2. \quad (62)$$

Proof. Identity equation (59) follows from

$$\delta_0^2 + H^3 = -27r^2\Delta$$

and $w^2 = \Delta$. Modulo $\mathfrak{p}_+$, the equation $H = 0$ makes r invertible and gives

$$q = \frac{4}{3r}, \quad z = \delta_0 = 27r^2p - 4.$$

The quotient is therefore isomorphic to $\mathbf{C}[r, r^{-1}, p]$, so $\mathfrak{p}_+$ is a height-one prime. The same argument applies to $\mathfrak{p}_-$.

Equation [equation \(59\)](#) factors as

$$(z - \delta_0)(z + \delta_0) = H^3.$$

At the generic point of $V(\mathfrak{p}_+)$, the factor $z + \delta_0$ is a unit. The relation then shows that

$$\mathfrak{p}_+ \mathcal{O}_{Y, \mathfrak{p}_+} = (H), \quad z - \delta_0 = H^3(z + \delta_0)^{-1}.$$

Since Y is normal, $\mathcal{O}_{Y, \mathfrak{p}_+}$ is a discrete valuation ring, and H is a uniformizer. The same argument with the signs reversed applies at $\mathfrak{p}_-$. It follows that $z - \delta_0$ and $z + \delta_0$ vanish to order 3 on the respective prime divisors, while H vanishes to order 1 on each. This proves the three divisor identities in [equation \(61\)](#).

At the generic point of the singular curve, the completed transverse equation is precisely

$$(z - \delta_0)(z + \delta_0) = H^3.$$

The ideals $(z - \delta_0, H)$ and $(z + \delta_0, H)$ are the two standard nonfree rank-one reflexive modules of the A_2 singularity. Their local classes are nonzero and opposite. Since [theorem 8.6](#) below shows that the global class group has order three, they are its two nonzero classes, and one is $[\mathcal{M}]$. The matrix identity is a direct multiplication. $\square$

The local order-three class is the restriction of a global generator. This can be computed from the ordered-root compactification.

Theorem 8.6. *The unit, Picard, and divisor class groups of the quadratic resolvent are*

$$\mathcal{O}(Y)^* = \mathbf{C}^*, \quad \text{Pic}(Y) = 0, \quad \text{Cl}(Y) = \text{Pic}(Y_{\text{reg}}) \simeq \mathbf{Z}/3\mathbf{Z}. \quad (63)$$

The reflexive eigensheaf $\mathcal{M}$ is a generator of $\text{Cl}(Y)$. More precisely, the quotient map $q: W \rightarrow Y$ has an eigensheaf decomposition

$$q_* \mathcal{O}_W \simeq \mathcal{O}_Y \oplus \mathcal{M} \oplus \mathcal{M}^{[2]}, \quad (64)$$

where $\mathcal{M}^{[2]} = (\mathcal{M}^{\otimes 2})^{**}$ and $[\mathcal{M}]$ has order 3.

Proof. Let H_1, H_2, H_3 be the three hyperplane classes on

$$\overline{W} = (\mathbf{P}^1)^3.$$

The boundary $H_W = \overline{W} \setminus W$ is the inverse image of the hyperplane at infinity in $\text{Sym}^3(\mathbf{P}^1)$. It is an irreducible divisor of class

$$[H_W] = H_1 + H_2 + H_3.$$

For irreducibility, on the dense torus its equation is $x_1 + x_2 + x_3 = 0$, and its closure has no coordinate divisor as a component. The localization sequences for units and line bundles give

$$\mathcal{O}(W)^* = \mathbf{C}^*, \quad \text{Pic}(W) \simeq \mathbf{Z}^3 / \mathbf{Z}(1, 1, 1). \quad (65)$$

Let $\delta \subset W$ be the small diagonal and put $W^\circ = W \setminus \delta$. Since W is smooth and δ has codimension two, removal of δ changes neither the unit group nor the Picard group. The cyclic group A_3 acts freely on W° , and

$$W^\circ \longrightarrow Y_{\text{reg}}$$

is an A_3 -torsor. The action of a generator of A_3 cyclically permutes the three coordinates in equation (65). Its fixed subgroup is zero. Indeed, if the class of (a, b, c) is fixed, then

$$(b, c, a) - (a, b, c) = m(1, 1, 1)$$

for some integer m . Summing the three coordinates gives $3m = 0$, hence $m = 0$, and then $a = b = c$. The original class is therefore zero.

The low-degree descent sequence for the torsor now gives

$$\mathrm{Pic}(Y_{\mathrm{reg}}) \simeq H^1(A_3, \mathbf{C}^*) \simeq \mathrm{Hom}(A_3, \mathbf{C}^*) \simeq \mathbf{Z}/3\mathbf{Z}.$$

The two nonzero characters give the restrictions of $\mathcal{M}$ and $\mathcal{M}^{[2]}$.

The variety Y is normal and $Y \setminus Y_{\mathrm{reg}}$ has codimension two. Therefore restriction identifies

$$\mathrm{Cl}(Y) = \mathrm{Cl}(Y_{\mathrm{reg}}) = \mathrm{Pic}(Y_{\mathrm{reg}}).$$

By theorem 8.4, the generator $[\mathcal{M}]$ has nonzero image in the local class group at the generic point of $\mathrm{Sing}(Y)$. Its square does as well. Hence neither nonzero global divisor class is Cartier. Since $\mathrm{Pic}(Y)$ injects into $\mathrm{Cl}(Y)$, it follows that $\mathrm{Pic}(Y) = 0$.

Finally, a unit on Y pulls back to a unit on W . The first equality in equation (65) therefore implies $\mathcal{O}(Y)^* = \mathbf{C}^*$. The eigensheaf decomposition follows from the three characters of A_3 and reflexive extension across the codimension-two fixed locus. $\square$

This gives an explicit global realization of [1, Proposition 26.1(10)]. After localizing at Δ , the cyclic cover is represented by the restriction of the generator $\mathcal{M}$ to Y_Δ . The obstruction to extending it as a line bundle across the discriminant is concentrated on the codimension-two singular curve. The BGV threshold is sharp. If the singular locus had codimension at least three, their extension argument would apply. Here it has codimension exactly two, and the global class remains reflexive but non-Cartier.

9 How the example saturates the BGV degree-three theory

For the counterexample, the two BGV divisors coincide:

$$h_0 = h_1 = \Delta.$$

Here $h_0 = 0$ is the image of the nonétale locus of the finite completion, and $h_1 = 0$ is the nonproperness locus. The following table records how the principal clauses of [1, Proposition 26.1] appear in this example.

BGV statement	Explicit realization for F
(2.a)	The cubic extension is non-Galois and its Galois closure has group S_3 .
(2.b)	The Galois closure is the ordered-root cover W . The quadratic intermediate cover is $Y = W/A_3$.
(2.c)	The cyclic cover $W \rightarrow Y$ is Galois over Y_{reg} . The bound $\dim \mathrm{Sing}(Y) \leq n - 2$ is an equality.

BGV statement	Explicit realization for F
(2.d)	The nonétale locus of $Y \rightarrow \mathbf{A}^3$ is the preimage of $\Delta = 0$.
(2.e), (2.f)	The quadratic algebra is free with basis $1, w$ and equation $w^2 = \Delta$.
(2.i)	The resolvent is a hypersurface in $\mathbf{A}^4$, and in fact $\text{ciedim}(Y) = 4 = n + 1$.
(5)	The binary cubic index form is exactly $\Phi_{p,q,r}(u, v)$.
(7)	The cubic algebra over R_Δ is not monogenic in characteristic zero.
(9)	The cubic algebra has the explicit three-quadrics presentation equation (48) .
(10)	The cyclic cubic cover gives the generator of $\text{Cl}(Y) = \mathbf{Z}/3\mathbf{Z}$. Its restriction is a line bundle on Y_{reg} , while its reflexive extension is not locally free along Γ .

The example also realizes [1, Theorem 7.3(5)]: the finite normalization X is smooth, the associated Jacobian variety is not étale, and X is not an affine complete intersection. The compactification calculation strengthens this by identifying the obstruction as the nonzero canonical class $[K_X] = [D]$.

9.1 The two BGV conjectures answered by the cubic map

BGV Conjecture 1.1 states that every complex Keller map is surjective. The equality

$$F(\mathbf{A}^3) = \mathbf{A}^3 \setminus \Gamma$$

disproves it.

BGV also ask in their introduction whether their methods could prove or disprove Adjamagbo's positive-characteristic conjecture. The degree-three map answers this for every characteristic $\ell \geq 5$, and the integral higher-degree family of [section 10](#) answers it for characteristic 3 as well.

The relation is therefore stronger than a shared theme. BGV isolated the normalization, the boundary ranks, the discriminant cover, the cubic index form, the weak-type obstruction, and the order-three line-bundle obstruction. The 2026 example simultaneously realizes all of these structures in characteristic zero and turns two questions left open in [1] into explicit counterexamples.

9.2 What the relation is not

The explicit non-surjective examples constructed in [1] are mainly positive-characteristic maps built from Frobenius systems, Lang torsors, and related constructions. Their geometric degrees are generally divisible by the characteristic, which is why they do not by themselves disprove Adjamagbo's prime-to-characteristic conjecture.

The cubic example uses a different mechanism:

$$\text{finite binary-cubic root cover} \quad + \quad \text{deletion of its ramification divisor} \quad + \quad X \setminus D \simeq \mathbf{A}^3.$$

It is defined over $\mathbf{Z}[1/2]$ and does not depend on Frobenius. There is presently no documented derivation of the formula from a BGV example. The relation established here is structural and exact, not a claim of historical dependence.

10 Integral families and odd characteristic

The deformation below is an integral specialization of the arbitrary-degree construction in [2, Theorem 5.2 and Corollary 5.3]. The original map and its three collision points are defined over $\mathbf{Z}[1/2]$. Since the Jacobian identity is integral, reduction modulo every odd prime gives an étale noninjective endomorphism of $\mathbf{A}^3$. The degree remains 3. For primes $\ell \geq 5$, this already contradicts Adjmagbo's conjecture because $\ell \nmid 3$.

Characteristic 3 requires a degree prime to 3. The deformation family of the counterexample gives much more.

10.1 An integral map of every degree

Retain A, B, P, Q, R from equations (1) and (2). For $d \geq 4$, define

$$\begin{aligned} Q_d &= Q + dA^2x^{d-2}B^d(4P+1), \\ R_d &= R - (d-2)x^dB^d(4P+1), \\ F_d &= (P, Q_d, R_d). \end{aligned} \tag{66}$$

Set $F_3 = F$.

Theorem 10.1. *For every $d \geq 3$, the map F_d has integer coefficients and*

$$\det \text{Jac}(F_d) = -2.$$

It sends the three points in equation (5) to the same target. Over every algebraically closed field k of odd characteristic, its geometric degree is exactly d and it is not an automorphism.

Proof. The case $d = 3$ is the original map. Assume $d \geq 4$. On the dense open set $A \neq 0$, put $s = x/A$ and write $p = P$. Define

$$\begin{aligned} \phi_d(p, s) &= 3ps + dp^d(4p+1)s^{d-2}, \\ \theta_d(p, s) &= ps^3 + (d-2)p^d(4p+1)s^d. \end{aligned} \tag{67}$$

The polynomial formulas in equation (66) become

$$Q_d = y + \phi_d(P, s), \quad R_d = 2s - ys^2 - \theta_d(P, s). \tag{68}$$

Indeed,

$$A^2x^{d-2}B^d = P^ds^{d-2}, \quad x^dB^d = P^ds^d.$$

A direct differentiation gives the integral identity

$$\frac{\partial \theta_d}{\partial s} = s^2 \frac{\partial \phi_d}{\partial s}. \tag{69}$$

Consequently,

$$\det \frac{\partial(P, Q_d, R_d)}{\partial(P, y, s)} = 2 - 2ys - \frac{\partial \theta_d}{\partial s} + s^2 \frac{\partial \phi_d}{\partial s} = 2(1 - ys) = \frac{2}{A}.$$

Since

$$\det \frac{\partial(P, y, s)}{\partial(x, y, z)} = -A,$$

the Jacobian determinant is -2 on $A \neq 0$. This is an identity between polynomials with integer coefficients, so it holds everywhere and after reduction in every characteristic.

At each of the three points in [equation \(5\)](#), one has $P = -1/4$. The added terms in [equation \(66\)](#) therefore vanish because they contain $4P + 1$. The collision is unchanged.

Let k be an algebraically closed field of odd characteristic. Eliminating

$$y = q - \phi_d(p, s)$$

from [equation \(68\)](#) gives

$$\Omega_{p,q,r}^{(d)}(s) = 2ps^3 - qs^2 + 2s - r + 2p^d(4p + 1)s^d = 0. \quad (70)$$

Its derivative at a branch coming from the source is

$$\frac{\partial \Omega_{p,q,r}^{(d)}}{\partial s} = 2(1 - sy). \quad (71)$$

With

$$D = 1 - sy,$$

the source coordinates are recovered rationally by

$$x = \frac{s}{D}, \quad z = pD^3 - y^2(4 - sy)D. \quad (72)$$

It follows that

$$k(x, y, z) = k(p, q, r)(s). \quad (73)$$

For $d \geq 4$, write [equation \(70\)](#) as

$$r = f_{p,q}(s).$$

Over $K = k(p, q)$, the polynomial $f_{p,q}(S)$ has degree exactly d , since its leading coefficient is the nonzero element $2p^d(4p + 1)$. The polynomial

$$T - f_{p,q}(S)$$

generates a prime ideal in $K[S, T]$, because the quotient is $K[S]$. It is therefore irreducible. Gauss's lemma implies that $f_{p,q}(S) - r$ is irreducible in $K(r)[S]$. Together with [equation \(73\)](#), this proves that the geometric degree is d . The case $d = 3$ follows from the irreducible inverse cubic already treated in [section 2](#).

The determinant -2 is a unit in k , so F_d is étale. The three collision points remain distinct in every odd characteristic. Hence F_d is not injective and cannot be an automorphism. $\square$

Corollary 10.2. *Adjamagbo's conjecture, stated as BGV Conjecture 1.3 and Conjecture 2.10, fails over every algebraically closed field of odd characteristic.*

Proof. In characteristic $\ell \geq 5$, use F_3 , whose degree is 3 and is prime to ℓ . In characteristic 3, use F_4 , whose degree is 4. Each map is Keller and is not an automorphism by [theorem 10.1](#). $\square$

In fact, [theorem 10.1](#) proves the following uniform statement.

Corollary 10.3. *Let ℓ be an odd prime and let $d \geq 3$ satisfy $\ell \nmid d$. Then F_d modulo ℓ is a counterexample to Adjamagbo's conjecture of geometric degree d .*

For each such counterexample, [1, Corollary 25.7] gives weak type exactly 2.

10.2 BGV Problem P24

BGV Problem P24 asks, among other things, whether Keller maps with

$$2 < \deg(e) < p$$

can exist in characteristic p , and whether counterexamples to Adjamagbo's conjecture can exist with degree greater than p .

Corollary 10.4. *The proposed nonexistence in the range $2 < \deg(e) < p$ is false for every prime $p \geq 5$, already in dimension 3. Counterexamples to Adjamagbo's conjecture with degree greater than p exist for every odd prime p .*

Proof. For $p \geq 5$, the reduction of F_3 has degree 3, so $2 < 3 < p$. For every odd p , the reduction of F_{p+1} has degree $p+1 > p$ and $p \nmid p+1$. $\square$

This settles the two existence questions in P24 concerning degrees below and above the characteristic. It does not by itself decide the separate moduli-theoretic alternative in P24 involving the quantity $\deg_{p,n,q}$ under the inequality $q^n < p$.

10.3 Characteristic two

All maps in this integral family have Jacobian determinant -2 , which vanishes in characteristic 2. Therefore this construction gives no Keller map in characteristic 2. The characteristic-two case of Adjamagbo's conjecture is not addressed here.

11 Characteristic three and BGV Proposition 26.1(2.j)

The reduction of the original cubic map modulo 3 has a special geometry. It does not contradict Adjamagbo's conjecture because its degree is divisible by 3, but it creates a problem for one theorem in the current BGV manuscript.

Let k be an algebraic closure of $\mathbf{F}_3$. Multiplying the inverse cubic by the unit 2 gives

$$f(s) = ps^3 + qs^2 + s + r. \tag{74}$$

Its discriminant is

$$\Delta_3 = q^2 - p - q^3r. \tag{75}$$

The hypersurface $\Sigma_3 = V(\Delta_3)$ is smooth because

$$\frac{\partial \Delta_3}{\partial p} = -1.$$

It is in fact isomorphic to $\mathbf{A}_{q,r}^2$ by solving for p .

Proposition 11.1. *The reduction $F_{\mathbf{F}_3}$ is an étale, noninjective, surjective map of geometric degree 3. Its nonproperness locus is the smooth surface Σ_3 . Every fiber has either 3 or 1 reduced points.*

Proof. The determinant is $-2 = 1$, and the three collision points remain distinct. The function-field degree is 3 by the same irreducibility argument as before.

For a finite multiple root t , one has

$$f'(t) = 1 - qt = 0.$$

Thus $q \neq 0$ and $t = q^{-1}$. Since $f''(t) = -q \neq 0$, such a root has multiplicity exactly two. A multiple root at infinity is analyzed on the chart $S = 1$, where

$$h(u) = 2p - qu + 2u^2 - ru^3.$$

The root $u = 0$ is multiple exactly when $p = q = 0$, and then $h''(0) = 4 = 1$, so it also has multiplicity exactly two. There are no triple projective roots in this affine slice. Therefore a discriminant fiber has one double root and one simple root, and the simple root gives one affine preimage. Off the discriminant there are three. Hence the map is surjective and the nonproperness locus is Σ_3 . $\square$

Proposition 11.2. *The boundary map in characteristic 3 is an isomorphism*

$$D_3 \xrightarrow{\sim} \Sigma_3.$$

The second discriminant sheet $E_3 \rightarrow \Sigma_3$ is also an isomorphism, and $D_3 \cap E_3 = \emptyset$. The BGV invariants of the cubic specialization are

$$\mathcal{F}(F_{\mathbf{F}_3}) = \{1\}, \quad \phi(F_{\mathbf{F}_3}) = 0, \quad \rho = 1, \quad \rho_{\text{ét}} = 0, \quad \nu = \nu^{\text{n-ét}} = 1. \quad (76)$$

Its fiber strata are

$$N_1 = \Sigma_3, \quad N_3 = \mathbf{A}^3 \setminus \Sigma_3, \quad N_0 = \emptyset.$$

Proof. The boundary parametrization equation (15) reduces modulo 3 to

$$q = t, \quad p = t^2 - rt^3.$$

Since Σ_3 is the graph $p = q^2 - q^3r$, this identifies D_3 with Σ_3 . The second sheet specializes from theorem 4.7. Its intersection with D_3 would be defined by $3rt - 2 = 0$, which has no solution in characteristic 3. Thus $E_3 \rightarrow \Sigma_3$ is the disjoint simple sheet and is also an isomorphism. The unique boundary component is still a ramification divisor, so $\rho = 1$ and $\rho_{\text{ét}} = 0$. The remaining assertions follow from theorem 11.1 and the irreducibility of Σ_3 . $\square$

The change from characteristic zero to characteristic three can be summarized as follows.

Feature	Characteristic zero	Characteristic three
Discriminant	Cuspidal along Γ	Smooth affine plane
Discriminant sheets	Two normalizations tangent along Γ	Two disjoint copies of Σ_3
Exceptional fibers	Cardinalities 0 and 1	Cardinality 1 only
Image	$\mathbf{A}^3 \setminus \Gamma$	All of $\mathbf{A}^3$
Quadratic resolvent	Singular of transverse type A_2 along Γ	Smooth and isomorphic to $\mathbf{A}^3$

The discriminant Δ_3 is irreducible and not a square in $k(p, q, r)$. The cubic extension is separable and irreducible, so it is non-Galois and satisfies the BGV condition (S_3). Since the characteristic is not 2, the BGV quadratic resolvent is

$$Y_3 = \operatorname{Spec} \frac{k[p, q, r, w]}{(w^2 - \Delta_3)}. \quad (77)$$

But [equation \(75\)](#) allows one to eliminate p :

$$p = q^2 - q^3 r - w^2. \quad (78)$$

Therefore

$$Y_3 \simeq \mathbf{A}_{q,r,w}^3.$$

In particular, Y_3 is smooth.

Theorem 11.3 (Explicit Galois closure in characteristic three). *The BGV cyclic cubic cover is the Artin–Schreier torsor*

$$W_3 \simeq \operatorname{Spec} \frac{k[q, r, w, U]}{(U^3 - U - rw)} \longrightarrow Y_3 = \operatorname{Spec} k[q, r, w]. \quad (79)$$

It is finite étale of degree 3 and is nontrivial. The full S_3 action is generated by

$$\sigma(q, r, w, U) = (q, r, w, U + 1), \quad \tau(q, r, w, U) = (q, r, -w, -U). \quad (80)$$

Here σ has order 3, τ has order 2, and $\tau\sigma\tau = \sigma^{-1}$. The quotient by $\langle\sigma\rangle$ is Y_3 , and the full quotient has coordinates

$$p = q^2 - q^3 r - w^2, \quad q, \quad r.$$

Moreover,

$$\mathcal{O}(W_3)^* = k^*, \quad \operatorname{Pic}(W_3) \simeq \mathbf{Z}^2, \quad \operatorname{edim}(W_3) = \operatorname{ciedim}(W_3) = 4. \quad (81)$$

Proof. Work first over the dense open set $pq \neq 0$. Let α, β, γ be the roots of the monic cubic

$$s^3 + as^2 + bs + c, \quad a = \frac{q}{p}, \quad b = \frac{1}{p}, \quad c = \frac{r}{p},$$

and let σ cyclically permute them. Put

$$u = \frac{\gamma - \beta}{a}.$$

Since $\alpha + \beta + \gamma = -a$ and $-2 = 1$ in characteristic 3, one has $\sigma(u) = u - 1$. If

$$V = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha),$$

then subtraction of the equations for β and γ gives

$$u^3 - u = -\frac{V}{a^3}.$$

Choose the sign of w so that $w = p^2 V$. It follows that

$$u^3 - u = -\frac{pw}{q^3}.$$

Using $p = q^2 - q^3r - w^2$, the right hand side becomes

$$-\frac{pw}{q^3} = rw + \left(\frac{w}{q}\right)^3 - \frac{w}{q}.$$

Therefore $U = u - w/q$ satisfies $U^3 - U = rw$. The resulting Artin–Schreier extension is the cyclic degree-three subextension of the splitting field. The hypersurface in [equation \(79\)](#) is finite étale over Y_3 because its derivative with respect to U is -1 . It is normal and has the same function field as the BGV normalization W_3 , so the generic identification extends globally.

The transformations in [equation \(80\)](#) preserve the equation and satisfy the displayed group relations. Invariants of σ are q, r, w . Invariants of τ on $k[q, r, w]$ are generated by q, r, w^2 , and $w^2 = q^2 - q^3r - p$. This proves the quotient statements. The torsor is nontrivial because an equality $rw = g^3 - g$ with $g \in k[q, r, w]$ is impossible by total degree.

Write

$$S = \operatorname{Spec} k[r, w, U]/(rw - U^3 + U), \quad W_3 = \mathbf{A}_q^1 \times S.$$

The polynomial $U^3 - U$ has the three distinct roots $0, 1, -1$. After inverting r , the ring of S becomes $k[r, r^{-1}, U]$. The localization sequence for the three irreducible divisors

$$D_a = V(r, U - a), \quad a \in \{0, 1, -1\},$$

gives

$$\operatorname{Pic}(S) \simeq \mathbf{Z}[D_0] \oplus \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_{-1}]/\mathbf{Z}(D_0 + D_1 + D_{-1}) \simeq \mathbf{Z}^2.$$

The same sequence gives $\mathcal{O}(S)^* = k^*$. Homotopy invariance yields the unit and Picard statements for W_3 . Finally, [equation \(79\)](#) is a hypersurface embedding in $\mathbf{A}^4$. An embedding in $\mathbf{A}^3$ would make the three-dimensional irreducible variety W_3 isomorphic to $\mathbf{A}^3$, contradicting its nonzero Picard group. Hence both dimensions in [equation \(81\)](#) equal 4. $\square$

The cover in [equation \(79\)](#) has trivial tangent bundle because it is étale over $\mathbf{A}^3$. It is therefore an explicit smooth affine threefold with trivial tangent bundle and nonzero Picard group. This is the characteristic-three ordered-root variety predicted abstractly by the BGV construction.

Theorem 11.4. *For the characteristic-three reduction of F , the quadratic resolvent Y_3 is smooth, but the finite morphism*

$$\varphi_3: Y_3 \longrightarrow \mathbf{A}_{p,q,r}^3$$

is ramified along $w = 0$, equivalently along $\Delta_3 = 0$ in the target. Thus this example contradicts [1, Proposition 26.1(2.j)] as it is written in the January 14, 2025 manuscript currently posted by the authors.

Proof. Smoothness follows from [equation \(78\)](#). The map is the quadratic cover obtained by adjoining w with $w^2 = \Delta_3$. Its relative differential satisfies

$$2w \, dw = 0.$$

Since 2 is a unit in characteristic 3, the morphism is not unramified at points with $w = 0$. Hence it is not étale.

The hypotheses printed in [1, Proposition 26.1(2.j)] are satisfied. Here $n = 3$, the condition (S_3) holds, and

$$\operatorname{Sing}(Y_3) = \emptyset, \quad \dim \operatorname{Sing}(Y_3) \leq n - 3 = 0.$$

The conclusion of that clause says that φ_3 is étale, which is false. $\square$

Proposition 11.5 (A correction compatible with the BGV proof). *Under the hypotheses of [1, Proposition 26.1(2.j)], the conclusion supported by the printed proof is*

$$W_e \longrightarrow Y_e \quad \text{is finite étale,} \tag{82}$$

not that $\varphi: Y_e \rightarrow \mathbf{A}^n$ is étale. The characteristic-three specialization satisfies equation (82) even though φ_3 is ramified.

Proof. Clause (2.c) of the same proposition states that

$$W_e \times_{Y_e} \text{Reg}(Y_e) \longrightarrow \text{Reg}(Y_e)$$

is finite étale. If $\dim \text{Sing}(Y_e) \leq n - 3$, the complement of $\text{Reg}(Y_e)$ has codimension at least three. Clause (2.i), followed by the purity theorem cited in the printed proof of (2.j), extends this étale cover across that complement. This argument concerns the morphism $W_e \rightarrow Y_e$.

The subsequent proof of clause (10.c) confirms the same reading. BGV invoke (2.j) there in order to obtain a class in

$$H_{\text{ét}}^1(Y_e, (\mathbf{Z}/3\mathbf{Z})_K)$$

which defines the Galois cover $W_e \rightarrow Y_e$. That use requires equation (82). It does not follow from étaleness of the quadratic map $Y_e \rightarrow \mathbf{A}^n$.

For the specialization considered here, the corrected conclusion is also verified directly by theorem 11.3. The cyclic cover is the finite étale Artin–Schreier torsor $U^3 - U = rw$ over $Y_3 \simeq \mathbf{A}^3$. Equivalently, the generic inertia of the full S_3 cover along the discriminant is generated by a transposition, so it has trivial intersection with A_3 . $\square$

Remark 11.6. *This issue is independent of the failure of the Jacobian conjecture in characteristic zero. It is a direct characteristic-three counterexample to the statement printed in clause (2.j). The source of the failure is elementary: a double cover can have a smooth total space while remaining ramified over a smooth divisor. Replacing its conclusion by equation (82) makes the clause agree with both its printed proof and its use later in Proposition 26.1.*

12 Conclusions

The finite cubic completion is the organizing object. Once it is recognized, the apparent anomalies of the polynomial map become standard features of a branched cover.

The normalization X is smooth, finite flat of degree three over the target, and not isomorphic to affine space. Its ramification divisor D is an affine plane, yet deleting D produces affine three-space. The original Keller map is the finite projection restricted to this complement. The discriminant surface is the image of the deleted divisor. Its reduced inverse image has two components, the deleted ramification sheet D and the surviving simple sheet E . Both are affine planes and both normalize Σ . They meet tangentially with multiplicity two along Γ . The conductor of $D \rightarrow \Sigma$ is exactly the ideal of this triple-root curve. Thus the omitted curve is also the nonnormal locus of the discriminant, with normalization quotient $\mathcal{O}_\Gamma$. A generic point of Σ still has one affine preimage because the simple sheet remains. Exactly on Γ , the simple sheet merges with the deleted ramified sheet, and the target point is omitted.

This geometry explains the BGV invariants without approximation. There is one boundary component, it is nonétale, the nonproperness locus is one irreducible divisor, the exceptional fiber

sizes are 0 and 1, and weak type is exactly two. The binary cubic, quadratic resolvent, S_3 closure, three-quadratics algebra, and order-three eigensheaf are all visible in explicit coordinates. The eigensheaf generates the entire divisor class group of the resolvent:

$$\mathrm{Pic}(Y) = 0, \quad \mathrm{Cl}(Y) = \mathrm{Pic}(Y_{\mathrm{reg}}) = \mathbf{Z}/3\mathbf{Z}.$$

The example is also extremal within the BGV estimates. The class-rank inequalities are sharp at their lower bounds. The singular locus of the quadratic resolvent has the largest dimension allowed by BGV. The resolvent complete-intersection bound is an equality. The cyclic eigensheaf fails to become Cartier precisely because the singular locus has codimension two. The normalization has the smallest possible affine embedding dimension compatible with its nontrivial canonical class, namely five.

Finally, the integral deformation family changes the positive-characteristic conclusion from a sporadic observation to a uniform theorem. In every odd characteristic there are explicit nonautomorphic Keller maps of every degree at least three. This disproves Adjagbo's conjecture in every odd characteristic and answers the degree-existence parts of BGV Problem P24. The cubic specialization in characteristic three additionally disproves Proposition 26.1(2.j) as presently printed. Replacing its conclusion by étaleness of $W_e \rightarrow Y_e$ makes the statement agree with its printed proof and with its later use in the same proposition.

A A compact list of formulas

For reference, the principal objects are collected here.

The map

$$\begin{aligned} A &= 1 + xy, & B &= A^2z + y^2(4 + 3xy), \\ P &= AB, & Q &= y + 3xB, & R &= 2x - 3x^2y - x^3z. \end{aligned}$$

The inverse cubic and discriminant

$$\begin{aligned} \Phi_{p,q,r}(S, T) &= 2pS^3 - qS^2T + 2ST^2 - rT^3, \\ \Delta &= q^2 - 16p - q^3r + 18pqr - 27p^2r^2. \end{aligned}$$

Boundary normalization

$$D \simeq \mathbf{A}_{t,r}^2, \quad p = t^2 - rt^3, \quad q = 4t - 3rt^2.$$

Second discriminant sheet

$$\begin{aligned} E &\simeq \mathbf{A}_{t,r}^2, & (p, q, r, [S : T]) &= (t^2 - rt^3, 4t - 3rt^2, r, [r : 2(1 - rt)]), \\ D \cap E &= \Gamma, & E|_D &= D|_E = 2\Gamma. \end{aligned}$$

Triple-root curve

$$\Gamma = V(12p - q^2, 3qr - 4) = \left\{ \left(\frac{t^2}{3}, 2t, \frac{2}{3t} \right) : t \in \mathbf{C}^* \right\}.$$

Discriminant conductor

$$\begin{aligned} \mathfrak{c}_{D/\Sigma} &= I_\Gamma = (12p - q^2, 3qr - 4), \\ 0 &\longrightarrow \mathcal{O}_\Sigma \longrightarrow (\nu_D)_* \mathcal{O}_D \longrightarrow \mathcal{O}_\Gamma \longrightarrow 0. \end{aligned}$$

Cubic algebra

$$\begin{aligned} A_X &= R \oplus R\omega \oplus R\theta, \\ \omega^2 &= -4p + q\omega + 2p\theta, \quad \omega\theta = 2pr, \quad \theta^2 = -qr + r\omega + 2\theta. \end{aligned}$$

Resolvent

$$\begin{aligned} Y &= V(w^2 - \Delta), \quad \text{Sing}(Y) = V(w, 12p - q^2, 3qr - 4). \\ \text{Pic}(Y) &= 0, \quad \text{Cl}(Y) = \text{Pic}(Y_{\text{reg}}) \simeq \mathbf{Z}/3\mathbf{Z}. \end{aligned}$$

If $\lambda^2 = -27$, then a generator is represented by

$$(\lambda rw - 27r^2p + 9qr - 8, 3qr - 4),$$

and

$$(\lambda rw)^2 = (27r^2p - 9qr + 8)^2 + (3qr - 4)^3.$$

Characteristic-three Galois closure

$$\begin{aligned} Y_3 &\simeq \mathbf{A}_{q,r,w}^3, \quad p = q^2 - q^3r - w^2, \\ W_3 &= V(U^3 - U - rw) \subset \mathbf{A}_{q,r,w,U}^4. \end{aligned}$$

Integral degree- d family

For $d \geq 4$,

$$\begin{aligned} Q_d &= Q + dA^2x^{d-2}B^d(4P + 1), \\ R_d &= R - (d - 2)x^dB^d(4P + 1), \\ \Omega_{p,q,r}^{(d)}(s) &= 2ps^3 - qs^2 + 2s - r + 2p^d(4p + 1)s^d. \end{aligned}$$

B Verification of the discriminant singular ideal

Starting from

$$\Delta = q^2 - 16p - q^3r + 18pqr - 27p^2r^2,$$

write

$$\Delta_p = -16 + 18qr - 54pr^2,$$

$$\Delta_q = 2q - 3q^2r + 18pr,$$

$$\Delta_r = -q^3 + 18pq - 54p^2r.$$

Set

$$g = 16p - q^3r, \quad h = 3qr - 4.$$

The following identities show that the Jacobian ideal is contained in (g, h^2) :

$$8\Delta_p = -27r^2g - (3qr + 8)h^2,$$

$$8\Delta_q = 9rg + qh^2,$$

$$128\Delta_r = (-432pr - 27q^3r^2 + 144q)g - q^3(3qr + 8)h^2.$$

The polynomial is weighted homogeneous of weight 2 for

$$\text{wt}(p) = 2, \quad \text{wt}(q) = 1, \quad \text{wt}(r) = -1.$$

Thus

$$2p\Delta_p + q\Delta_q - r\Delta_r = 2\Delta,$$

which also places Δ in (g, h^2) . Hence

$$J_\Delta = (\Delta, \Delta_p, \Delta_q, \Delta_r) \subseteq (g, h^2).$$

For the reverse inclusion, direct addition gives

$$h^2 = -\Delta_p - 3r\Delta_q \in J_\Delta.$$

The identity for Δ_q then implies $rg \in J_\Delta$. Since $h^2g \in J_\Delta$ and

$$h^2g = (9q^2r^2 - 24qr + 16)g,$$

the first two terms on the right are multiples of rg . Therefore $16g \in J_\Delta$, and hence $g \in J_\Delta$. We have proved

$$J_\Delta = (16p - q^3r, (3qr - 4)^2).$$

To compute the radical, note that $3qr - 4 = 0$ implies $q, r \neq 0$, and then

$$16p = q^3r = \frac{4}{3}q^2,$$

which is equivalent to $12p = q^2$. Conversely, the equations $12p = q^2$ and $3qr = 4$ make both generators vanish. Hence

$$\sqrt{J_\Delta} = (12p - q^2, 3qr - 4).$$

The square in the second generator records the nonreduced Jacobian singular scheme along the triple-root curve.

C References

References

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