Purity for Barsotti–Tate groups in some mixed characteristic situations

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ABSTRACT. Let p be a prime. Let R be a regular local ring of dimension $d \geq 2$ whose completion is isomorphic to $C(k)[[x_1, \ldots, x_d]]/(h)$, with $C(k)$ a Cohen ring with the same residue field k as R and with $h \in C(k)[[x_1, \ldots, x_d]]$ such that its reduction modulo p does not belong to the ideal $(x_1^p, \ldots, x_d^p) + (x_1, \ldots, x_d)^{2p-2}$ of $k[[x_1, \ldots, x_d]]$. We extend a result of Vasiu–Zink (for $d = 2$) to show that each Barsotti–Tate group over $\mathrm{Frac}(R)$ which extends to every local ring of $\mathrm{Spec}(R)$ of dimension 1, extends uniquely to a Barsotti–Tate group over R. This result corrects in many cases several errors in the literature. As an application, we get that if Y is a regular integral scheme such that the completion of each local ring of Y of residue characteristic p is a formal power series ring over some complete discrete valuation ring of absolute ramification index $e \leq p - 1$, then each Barsotti–Tate group over the generic point of Y which extends to every local ring of Y of dimension 1, extends uniquely to a Barsotti–Tate group over Y.

KEY WORDS: Barsotti–Tate groups, regular rings, purity, projective varieties, vector bundles, formal schemes.

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1 Introduction

Let R be a local noetherian ring of residue field k. Let $X = \mathrm{Spec}(R)$ and let U be the punctured spectrum of R (i.e., the complement in X of the closed point of X). Let p be a prime number. The notation R, k, X, U and p is fixed throughout this article. We recall from [23], Def. 2 that if $\mathrm{depth}(R) \geq 2$, then R is called *p-quasi-healthy* if each Barsotti–Tate group (i.e., p-divisible group) over U extends to a Barsotti–Tate group over X; from [7], Exp. III, Cor. 3.5 applied to coherent sheaves defined by the structure sheaves of truncated Barsotti–Tate groups over X we get that such an extension is unique up to unique isomorphism.

If $p = 0$ in R and $\mathrm{depth}(R) \geq 2$, then one can check that R is not p-quasi-healthy using a variant of Moret-Bailly's counterexample in [4], p. 192.[1] In [23], Thm. 3 it is shown that there are large classes of p-quasi-healthy *regular* rings of dimension 2 and mixed characteristic $(0, p)$. However, no such examples are (correctly) proved to exist in the literature for dimensions at least 3; this is so due to the fact that all claims in the literature for dimensions at least 3 rely on the erroneous argument in [4], Ch. V, Sect. 6, end of p. 183 and top of p. 184. The difficulty encountered consists in being able to provide examples of dimension 3 as the Grothendieck–Messing deformation theory for Barsotti–Tate groups allows a relatively direct passage from dimension 3 to higher dimensions (see Lemma 5).

Our goal is to obtain purity results for Barsotti–Tate groups over regular schemes. More precisely, we will provide many examples of p-quasi-healthy regular local rings of dimension at least 3 and we will use them to get as well new examples of faithfully flat $\mathrm{Spec}(\mathbb{Z}_{(p)})$-schemes which are p-healthy regular in the sense of either [21], Def. 3.2.1 9) or [23], Def. 1. Our main result is the following theorem which extends [23], Thm. 3.

Theorem 1 *Let R be a regular local ring of dimension $d \geq 1$ and mixed characteristic $(0, p)$ which satisfies the following condition:*

($\natural$) *The completion of R is isomorphic to $C(k)[[x_1, \ldots, x_d]]/(h)$, where $C(k)$ is a Cohen ring of k and where $h \in C(k)[[x_1, \ldots, x_d]]$ is such that its reduction $\bar{h}$ modulo p does not belong to the ideal $(x_1^p, \ldots, x_d^p) + (x_1, \ldots, x_d)^{2p-2}$ of $k[[x_1, \ldots, x_d]]$.*

[1] We consider a homomorphisms $\boldsymbol{\alpha}_{\mathbf{p},X} \to \boldsymbol{\alpha}_{\mathbf{p},X}^{\dim(R)}$ defined by a system of parameters of R (it is a closed embedding over U but not over X) and an embedding of $\boldsymbol{\alpha}_{\mathbf{p},X}^{\dim(R)}$ as a closed subgroup scheme of an abelian scheme $\mathcal{A}$ over X. If the Barsotti–Tate group $\mathcal{D}_U = \mathcal{A}_U[p^\infty]/\boldsymbol{\alpha}_{\mathbf{p},U}$ extends to a Barsotti–Tate group $\mathcal{D}$ over X, then by the depth assumption on R the isogeny $\mathcal{A}_U[p^\infty] \to \mathcal{D}_U$ extends to a morphism $\mathcal{A}[p^\infty] \to \mathcal{D}$, which must be an isogeny (cf. [2], Prop. 3.3.8 and the last part of Ex. 3.3.10), so its kernel is a finite flat subgroup scheme of $\boldsymbol{\alpha}_{\mathbf{p},X}^{\dim(R)}$, which leads to a contradiction.

Then the following two properties hold:

(a) *If a Barsotti–Tate group over Frac(R) extends to each local ring of $X =$ Spec(R) of dimension* 1*, then it extends uniquely (up to unique isomorphism) to a Barsotti–Tate group over X.*

(b) *If $d \geq 2$, then the regular local ring R is p-quasi-healthy.*

Clearly (a) implies (b) and the fact that X is also p-healthy regular. Condition ($\natural$) is stable under generization, cf. Proposition 5. Based on this and on the classical purity theorem of Zariski, Nagata and Grothendieck (see [7], Exp. X, Thm. 3.4 (i)]), we get that in fact (a) and (b) are equivalent (see Section 7).

For $d = 2$, Theorem 1 (b) is proved in [23], Thm. 3. Theorem 1 (b) and [23], Prop. 23 (b) imply [23], Thm. 3 and that R for $d \geq 2$ is also a quasi-healthy regular ring in the sense of [23], Def. 2, which means that each abelian scheme over U extends (uniquely up to a unique isomorphism) to an abelian scheme over X. In this way we get precisely those examples of quasi-healthy regular rings one gets based only on [23], Thm. 3 and Lem. 24. This is so as based on Propositions 3 and 4 we easily get that for $d \geq 2$ the condition on R in [23], Thm. 3 is equivalent to ($\natural$).

For $d \geq 3$, we prove Theorem 1 (b) by induction on $d \geq 3$ (see Section 6) and the hard part is the case when $d = 3$ (see Subsection 6.1). The proof of Theorem 1 (b) for $d = 3$ involves a study of the blow up of X along its closed point and relies heavily on several important results and ideas.

Firstly, the proof of Theorem 1 (b) for $d = 3$ relies on an application (see Proposition 2) of Raynaud's complement to Tate's extension theorem of [20], Thm. 4 obtained in [18], Prop. 2.3.1 and reproved in Proposition 1 based on a refinement of [24], Cor. 4 proved in Lemma 1. Secondly, the proof of Theorem 1 (b) for $d = 3$ relies on the case $d = 2$ of Theorem 1 (b) proved in [23]. Thirdly, it relies on the particular case $(N, l', \mathcal{C}) = (2, l, \mathbb{P}^1_l)$ of the general theorem below which we think is of interest in its own.

Theorem 2 *Let l be a field of characteristic p. Let $N \in \mathbb{N}^*$ be an integer. For each $n \in \mathbb{N}^*$ let $\mathcal{D}_n$ be a truncated Barsotti–Tate group of level n over an open subscheme $\mathcal{U}_n$ of $\mathbb{P}^N_l$. We assume that we have a chain of inclusions $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \cdots \supset \mathcal{U}_n \supset \cdots$ and that for all $n \in \mathbb{N}^*$ we have an identification $\mathcal{D}_{n+1}[p^n] = \mathcal{D}_{n,\mathcal{U}_{n+1}}$. We also assume that there exists a field extension l' of l and a closed subscheme $\mathcal{C}$ of $\mathbb{P}^N_{l'}$ of dimension greater than* 0 *and contained in $\mathcal{U}_{n,l'}$ for all $n \in \mathbb{N}^*$ such that the inductive system $\mathcal{C} \times_{\mathcal{U}_n} \mathcal{D}_n$ is a constant*

Barsotti–Tate group over $\mathcal{C}$, i.e., it is isomorphic to the pullback to $\mathcal{C}$ of a Barsotti–Tate group over Spec(l'). Then for each $n \in \mathbb{N}^*$, $\mathcal{D}_n$ extends uniquely (up to unique isomorphism) to a constant truncated Barsotti–Tate group $\mathcal{D}_n^+$ of level n over $\mathbb{P}_l^N$ isomorphic to $G[p^n]_{\mathbb{P}_l^N}$ for a suitable Barsotti–Tate group G over Spec(l) and the identification $\mathcal{D}_{n+1}[p^n] = \mathcal{D}_{n,\mathcal{U}_{n+1}}$ extends to an identification $\mathcal{D}_{n+1}^+[p^n] = \mathcal{D}_n^+$ which is the pullback to $\mathbb{P}_l^N$ of the identification $G[p^{n+1}][p^n] = G[p^n]$ over Spec(l) (therefore the Barsotti–Tate group over the stable under generization, pro-constructible subset $\mathcal{U}_\infty = \cap_{n=1}^{\infty}\mathcal{U}_n$ of $\mathbb{P}_l^N$ induced naturally by the $\mathcal{D}_n$'s, extends uniquely up to unique isomorphism to a constant Barsotti–Tate group over $\mathbb{P}_l^N$ isomorphic to $G_{\mathbb{P}_l^N}$[2]).

The proof of Theorem 2 is presented in Section 5 and it relies on properties of rings of formal-rational functions established in [13]. Theorem 1 is proved in Sections 6 and 7. Section 2 presents a few basic extension results that are required in the proof and the applications of Theorem 1, including a descent lemma (see Lemma 3) for p-quasi-healthy regular rings and the proof of Raynaud's complement to it. Section 3 groups together a few basic properties related to condition ($\natural$). Section 4 contains an elementary formal algebraic geometry property which is a standard application of the results of [13] and which plays a key role in the proof of Theorem 2.

The following corollary of Theorem 1 and its proof and of Bondarko's boundedness results for truncated Barsotti–Tate groups over discrete valuation rings of mixed characteristic $(0, p)$ (see [1] and [24]) extends [23], Cor. 5 and it can be used to correct the argument used in [4], Ch. V, Sect. 6, end of p. 183 and top of p. 184 in many situations.

Corollary 1 *Let Y be a regular integral scheme flat over Spec$(\mathbb{Z})$. We assume that for each point $z \in Y$ of characteristic p, condition ($\natural$) holds for the local ring $\mathcal{O}_{Y,z}$ of z in Y; for instance, this holds if the completion of $\mathcal{O}_{Y,z}$ is formally smooth (e.g., is a formal power series ring) over a complete discrete valuation ring of absolute ramification index $e \in \{1, \ldots, p-1\}$. Then the following two properties hold:*

[2]Note that $\mathcal{U}_\infty$ is a locally ringed space, and one can define a Barsotti–Tate group over a locally ringed space $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ as a projective system of commutative and cocommutative Hopf $\mathcal{O}_\mathcal{Y}$-algebras $\mathcal{H}_n$ indexed by $n \in \mathbb{N}^*$, locally free of finite rank as $\mathcal{O}_\mathcal{Y}$-modules, such that for every point $z \in \mathcal{Y}$ the inductive system Spec$((\mathcal{H}_n)_z)$ of finite flat group schemes indexed by $n \in \mathbb{N}^*$ and defined by the stalks at z constitutes a Barsotti–Tate group over the local scheme Spec$(\mathcal{O}_{\mathcal{Y},z})$.

(a) *Each Barsotti–Tate group over the generic point of Y which extends to every local ring of Y of dimension* 1*, extends uniquely (up to unique isomorphism) to a Barsotti–Tate group over Y.*

(b) *Each Barsotti–Tate group over an open subscheme of Y that contains $Y[\frac{1}{p}]$ and all points of $Y_{\mathbb{F}_p}$ of codimension* 0 *in $Y_{\mathbb{F}_p}$ extends to a Barsotti–Tate group over Y (thus if Y is a faithfully flat Spec($\mathbb{Z}_{(p)}$)-scheme, then it is p-healthy regular).*

The case $e < p-1$ of Corollary 1 (b) is a special case of [4], Ch. V, Thm. 6.4' (which is incorrect already in dimension 2) and was claimed more recently in [17], Subsect. 3.6.1 and in [21], Rm. 3.2.2 3) and the last two lines of Subsubsect. 3.2.17. The case $e = 1$ of Corollary 1 (b) was also claimed in [22], Thm. 1.3. Unfortunately, the inductive passages from dimension 2 to dimensions 3 in all these four references relied on [4], Ch. V, Sect. 6, argument on pp. 183–184, which, in the language of this paper, would in particular imply that for each integer $n \geq 2$ all local noetherian rings $C(k)[[x_1, x_2]]/(x_2^n)$ are p-quasi-healthy. The counterexample of [23], Subsect. 5.1 shows that the last statement is false.

If Y is as in Corollary 1 and faithfully flat over $\mathrm{Spec}(\mathbb{Z}_{(p)})$, then Y is also a healthy regular scheme in the sense of [21], Def. 3.2.1 2), i.e., each abelian scheme over an open subscheme of Y that contains $Y_{\mathbb{Q}}$ and all points of $Y_{\mathbb{F}_p}$ of codimension 0 in $Y_{\mathbb{F}_p}$ extends to an abelian scheme over Y (cf. [22], Prop. 4.1). From this and [23], Lem. 29 one gets a second proof of the uniqueness of integral canonical models of Shimura varieties defined in [21], Def. 3.2.3 6) over discrete valuation rings of mixed characteristic $(0, p)$ and absolute ramification index at most $p-1$ which was first proved in [23], Cor. 30.

The examples of [23], Thm. 28 (i) and (ii) for $d = 2$ can be easily adapted to provide many examples of regular local rings R of arbitrary dimension $d \geq 2$ which are of mixed characteristic $(0, p)$ and are not p-quasi-healthy. For instance, the regular complete local rings $C(k)[[x_1, \ldots, x_d]]/(p - ax_1^p)$ and $C(k)[[x_1, \ldots, x_d]]/(p - a\prod_{i=1}^{d} x_i^{p-1})$ are not p-quasi-healthy, where $a \in C(k)[[x_1, \ldots, x_d]]$. See [5] for details, generalizations, and the fact that if R is regular henselian of dimension 2, then condition ($\natural$) holds if and only if R is p-quasi-healthy.

Complements to Corollary 1 and to Lemma 8 used in its proof are included in Section 8.

Let $\mathcal{O}_\flat$ be the structure sheaf of a scheme $\flat$. For each local noetherian ring R, let $\widehat{R}$ be its completion. If R is an integral domain, let $K = \mathrm{Frac}(R)$;

so if R is a discrete valuation ring, then $U = \mathrm{Spec}(K)$. If k is perfect of characteristic p, then $C(k)$ is the ring $W(k)$ of p-typical Witt vectors with coefficients in k. We think of a Barsotti–Tate group over a scheme as an inductive system or alternatively as its limit fppf sheaf, see [2], Subsect. 3.3.2.

2 Basic Extension Properties

In this section we gather some basic extension properties of different nature that will be often used in what follows. We begin by recalling some standard properties of algebraic geometry.

2.1 Reflexive sheaves

The proof of the following elementary fact is left to the reader.

Fact 1 *Let $\mathcal{N}$ be a normal noetherian scheme. Let $\mathcal{U}$ be an open dense subscheme of $\mathcal{N}$ which contains all codimension 1 points (i.e., points whose local rings are discrete valuation rings). Let $\mathcal{E}$ and $\mathcal{H}$ be two finite flat group schemes over $\mathcal{N}$. Let $\mathcal{V}$ be a reflexive coherent $\mathcal{O}_{\mathcal{N}}$-module. Then the following four properties hold:*

(a) *If $\mathcal{N}$ is regular of dimension at most 2, then the restriction from $\mathcal{N}$ to $\mathcal{U}$ induces an equivalence of categories from the category of coherent locally free $\mathcal{O}_{\mathcal{N}}$-modules to the category of coherent locally free $\mathcal{O}_{\mathcal{U}}$-modules.*

(b) *If $\mathcal{N}$ is regular of dimension at most 2, then the restriction from $\mathcal{N}$ to $\mathcal{U}$ induces an equivalence of categories from the category of finite flat group schemes over $\mathcal{N}$ to the category of finite flat group schemes over $\mathcal{U}$.*

(c) *The restriction homomorphisms $Hom(\mathcal{E},\mathcal{H}) \to Hom(\mathcal{U}\times_{\mathcal{N}}\mathcal{E},\mathcal{U}\times_{\mathcal{N}}\mathcal{H})$ and $H^0(\mathcal{N},\mathcal{V}) \to H^0(\mathcal{U},\mathcal{V})$ are isomorphisms.*

(d) *Let $f : \mathcal{E} \to \mathcal{H}$ be a homomorphism. Let $\mathcal{W}$ be the largest open subscheme of $\mathcal{N}$ such that the restriction $f_{\mathcal{W}} : \mathcal{E}_{\mathcal{W}} \to \mathcal{H}_{\mathcal{W}}$ of f to a homomorphism over $\mathcal{W}$ is an isomorphism. Then $\mathcal{W}$ is the set of all points $z \in \mathcal{N}$ at which the fiber f_z of f is an isomorphism. Moreover, $\mathcal{W}$ contains all points of codimension at most 1 in $\mathcal{N}$ if and only if we have $\mathcal{W} = \mathcal{N}$ (i.e., f is an isomorphism if and only if its fibers over points of codimension at most 1 in $\mathcal{N}$ are isomorphisms).*

For the second part of (c) above we refer to [11], Prop. 1.6.

2.2 A refinement of a result of [24]

In this subsection we assume that R is a discrete valuation ring of mixed characteristic $(0,p)$. Thus $K = \mathrm{Frac}(R)$ and $U = \mathrm{Spec}(K)$. Let $s \in \mathbb{N}$ be such that [24], Thm. 1 holds for R; it has upper bounds in terms only on the absolute ramification index e of R (the best upper bound $s = [\log_p(\frac{pe}{p-1})]$ is obtained in [1], Thm. A). If H is a finite flat commutative group scheme of p power order over X, for $n \in \mathbb{N}^*$ let $H\lceil p^n\rceil$ be the schematic closure of $H_K[p^n]$ in H.

The following lemma refines [24], Cor. 4 which worked in a context in which H' is a Barsotti–Tate group of level n over X.

Lemma 1 *With R, K and s as above, let $n > 2s$ be an integer. Let H and H' be finite flat group scheme over X such that their generic fibers H_K and H'_K are Barsotti–Tate groups of level n over $U = Spec(K)$. For $i \in \{0, \ldots, n\}$ let $H_i = H\lceil p^i\rceil$ and $H'_i = H'\lceil p^i\rceil$. Then the following three properties hold:*

(a) *If $f : H \to H'$ is a homomorphism such that $f_K : H_K \to H'_K$ is an isomorphism, then the homomorphism $H_{n-s}/H_s \to H'_{n-s}/H'_s$ induced by f is an isomorphism.*

(b) *If $H_K \to H'_K$ is a homomorphism (resp. is an isomorphism), then the homomorphism $H_K[p^{n-s}]/H_K[p^s] \to H'_K[p^{n-s}]/H'_K[p^s]$ induced by it extends to a homomorphism (resp. to an isomorphism) $H_{n-s}/H_s \to H'_{n-s}/H'_s$.*

(c) *If $n \geq 2s+2$, then H_{s+t}/H_s is a Barsotti–Tate group of level t over X for all natural numbers $t \leq n-2s$.*

Proof: For $i \in \{0, \ldots, n-1\}$, let $\mathrm{gr}_{i+1}(H) = H_{i+1}/H_i$ and $\mathrm{gr}_{i+1}(H') = H'_{i+1}/H'_i$. To prove (a), we consider the commutative diagram

$$\begin{array}{ccc} H/H_{n-s-1} & \xrightarrow{p^{n-s-1}} & H_{s+1} \\ \downarrow & & \downarrow \\ H'/H'_{n-s-1} & \xrightarrow{p^{n-s-1}} & H'_{s+1}, \end{array}$$

whose vertical arrows are induced by f. Let $\alpha : H/H_{n-s-1} \to H'_{s+1}$ be the diagonal homomorphism of the diagram and let $\beta : H'_{s+1} \to H/H_{n-s-1}$ be a homomorphism such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are both the multiplication by p^s, cf. [24], Thm. 1. The homomorphism β induces a homomorphism $\mathrm{gr}_{s+1}(H') \to$

$\mathrm{gr}_{n-s}(H)$ which is an inverse of the homomorphism $\mathrm{gr}_{n-s}(H) \to \mathrm{gr}_{s+1}(H')$ induced by α. Factoring the isomorphism $\mathrm{gr}_{n-s}(H) \to \mathrm{gr}_{s+1}(H')$ we get that for $i \in \{s+1, \ldots, n-s\}$ the homomorphisms $\mathrm{gr}_{i+1}(H) \to \mathrm{gr}_{i+1}(H')$ induced by f are isomorphisms. This implies that (a) holds.

Using the schematic closure $\tilde{H}$ of the graph of $H_K \to H'_K$ in $H \times_R H'$, the two projections $\tilde{H} \to H$ and $\tilde{H} \to H'$ induce homomorphisms $\tilde{H}_{n-s}/\tilde{H}_s \to H_{n-s}/H_s$ and $\tilde{H}_{n-s}/\tilde{H}_s \to H'_{n-s}/H'_s$, the first one being an isomorphism (resp. both of them being isomorphisms) (cf. (a)). From this (b) follows.

To prove (c), from the proof of (a) applied to the identity automorphism of H, we get that $\mathrm{gr}_{n-s}(H) \to \mathrm{gr}_{s+1}(H)$ is an isomorphism, which implies that for $i \in \{s+1, \ldots, n-s-1\}$ all successive homomorphisms $\mathrm{gr}_{i+1}(H) \to \mathrm{gr}_i(H)$ are isomorphisms. Thus H_{n-s}/H_s is an fppf sheaf flat over $\mathbb{Z}/p^{n-2s}\mathbb{Z}$. From [16], Ch. 1, Def. 1.2 we get that (c) holds for $t = n-2s$. If $0 \le t \le n-2s-1$, then $(H_{n-s}/H_s)[p^t]$ is a Barsotti–Tate group of level t over X and is equal to the flat closed subgroup scheme H_{s+t}/H_s of H_{n-s}/H_s as this is so over U. $\square$

2.3 On Raynaud's complement to Tate's extension theorem

We recall Raynaud's complement to Tate's extension theorem (see [18], Prop. 2.3.1) and include a new proof of it based on Lemma 1.

Proposition 1 *We assume R is a discrete valuation ring of mixed characteristic $(0,p)$. Let D_K be a Barsotti–Tate group over $K = Frac(R)$. If for each $n \in \mathbb{N}^*$, $D_K[p^n]$ extends to a finite flat group scheme E_n over X, then D_K extends to a Barsotti–Tate group over X.*

Proof: If $t \ge 2$ is an integer, then for $n = 2s+t \ge 2s+2$, the quotient group scheme $F_t = E_n[p^{n-s}]/E_n[p^s]$ is a Barsotti–Tate group of level $t = n-2s$ which extends $D_K[p^t]$ and does not depend on E_n (see Lemma 1 (b) and (c)). Taking $E_n = E_{n+1}[p^n]$, we get that $F_t = F_{t+1}[p^t]$. Thus $F_1 = F_t[p]$ does not depend on $t \ge 2$ and the inductive system F_m indexed by $m \in \mathbb{N}^*$ is a Barsotti–Tate group over X which extends D_K. $\square$

In the proof of Theorem 1 (b) we will need the following application of Proposition 1.

Proposition 2 *Let R be a regular local ring of mixed characteristic $(0,p)$ and dimension $d \ge 2$. Let D_U be a Barsotti–Tate group over U (the punctured* *spectrum of R). Let O be the discrete valuation ring which dominates R, which has the same field of fractions K as R, and which is a local ring of the blow up of X along its closed point (thus the residue field of O is a field of rational functions in $d-1$ variables over k). Then the generic fiber D_K of D_U over Spec(K) extends uniquely (up to unique isomorphism) to a Barsotti–Tate group over Spec(O).*

Proof: For each $n \in \mathbb{N}^*$, from Lemma 2 below applied with $(H_K, H_U) = (D_K[p^n], D_U[p^n])$ we get that the truncated Barsotti–Tate group $D_K[p^n]$ of level n over $\mathrm{Spec}(K)$ extends to a finite flat group scheme over $\mathrm{Spec}(O)$. Based on this, the proposition follows from Proposition 1 applied to O. $\square$

Lemma 2 *Let R, d, U, O, K, and X be as in Proposition 2. Let H_K be a finite group scheme over Spec(K) which extends to a finite flat group scheme H_U over U. Then H_K extends to a finite flat group scheme over Spec(O).*

Proof: Let R_1 be a regular local ring of dimension 2 which dominates R, which is dominated by O, and whose punctured spectrum U_1 is $X_1 \times_X U$, where $X_1 = \mathrm{Spec}(R_1)$. For instance, we can take R_1 to be a local ring of the blow up of X along a regular closed subscheme of it of dimension 1. From Fact 1 (b) we get that $U_1 \times_U H_U$ extends uniquely (up to unique isomorphism) to a finite flat group scheme H_1 over X_1. The pullback of H_1 to $\mathrm{Spec}(O)$ is a finite flat group scheme over $\mathrm{Spec}(O)$ that extends H_K. $\square$

2.4 A descent lemma

In what follows we will often use the following general descent lemma.

Lemma 3 *Let $R \to R'$ be a faithfully flat extension of noetherian local rings with $\dim(R) = \dim(R')$ and depth $(R) \geq 2$. If R' is p-quasi-healthy, then R is p-quasi-healthy.*

Proof: Let D_U be a Barsotti–Tate group over U. We have to show that D_U extends (automatically uniquely up to unique isomorphism) to a Barsotti–Tate group D over $X = \mathrm{Spec}(R)$. Let $X' = \mathrm{Spec}(R')$ and let $f : X' \to X$ be the morphism induced by the monomorphism $R \to R'$. From [15], Thm. 15.1 we get that the closed fiber of f has dimension zero, and therefore $U' = f^{-1}(U)$ is the punctured spectrum of R'. Thus $D_{U'} = U' \times_U D_U$ extends to a Barsotti–Tate group $D_{X'}$ over X'.

For $n \in \mathbb{N}^*$, let $A_n = H^0(U, D_U[p^n])$. We have to show that the following two properties hold:

(i) for each $n \in \mathbb{N}^*$, A_n is a finitely generated projective R-module (so for $t \in \mathbb{N}$, the t-fold tensor power of A_n over R maps isomorphically to $\mathcal{O}(D_U[p^n]^t)$), and thus the group scheme structure of $D_U[p^n]$ endows A_n with a structure of commutative and cocommutative Hopf R-algebra that defines a finite flat commutative group scheme D_n over X that extends $D_U[p^n]$);

(ii) for all $n, m \in \mathbb{N}^*$ the complex $0 \to D_m \to D_{n+m} \to D_n \to 0$ that extends the short exact sequence $0 \to D_U[p^m] \to D_U[p^{n+m}] \to D_U[p^n] \to 0$, is a short exact sequence of commutative finite flat group schemes over X.

If these two properties hold, then we can take D to be the inductive system D_n. To check that properties (i) and (ii) hold we can work locally in the fpqc topology and therefore these two properties follow from the fact that for each $n \in \mathbb{N}^*$ we have $D_{X'}[p^n] = \mathrm{Spec}(R' \otimes_R A_n) = X' \times_X D_n$. $\square$

2.5 Extending vector bundles

The following criterion of extending vector bundles will be used repeatedly in connection to extending truncated Barsotti–Tate groups.

Lemma 4 *We assume that depth$(R) \geq 3$. Let $y \in R$ be such that (y, p) is a regular sequence of R (thus R is of mixed characteristic $(0, p)$). Let S be either R or $R/(y^n)$ for some $n \in \mathbb{N}^*$. Let $\mathcal{F}$ be a vector bundle over the punctured spectrum U_S of S. If $S = R$, let τ be y and if $S = R/(y^n)$, let τ be p. For each $t \in \mathbb{N}^*$, let $\iota_t : U_S \cap Spec(S/(\tau^t)) \to Spec(S/(\tau^t))$ be the restriction modulo τ^t of the open embedding $\iota : U_S \to Spec(S)$. We assume that the following two conditions hold:*

(i) *for each $t \in \mathbb{N}^*$, the direct image $\iota_{t,*}\mathcal{F}_t$ of the restriction $\mathcal{F}_t$ of $\mathcal{F}$ to $U_S \cap Spec(S/(\tau^t))$ is a vector bundle $\mathcal{F}_t^+$ over $Spec(S/(\tau^t))$;*

(ii) *if $S = R/(y^n)$, then depth$(R) \geq 4$.*

Then the direct image $\iota_*\mathcal{F}$ is a vector bundle over $Spec(S)$ whose restriction to $Spec(S/(\tau^t))$ maps isomorphically to $\mathcal{F}_t^+$ for all $t \in \mathbb{N}^*$.

Proof: If $S = R$ is regular, then the lemma is a particular case of [4], Ch. V, Sect. 6, Lem. 6.6 applied over $\widehat{S}$. If $S = R$ (resp. if $S = R/(y^n)$), then from the Auslander–Buchsbaum formula we get that depth(S) is depth$(R) - 1 \geq 2$

(resp. depth$(R) - 2 \geq 2$). Thus, regardless of what S is, the lemma is a particular case of [7], Exp. IX, Prop. 1.4 and Ex. 1.5 applied over $\widehat{S}$. $\square$

2.6 Extending Barsotti–Tate groups

The next lemma is a natural application of Lemma 4 and is the essence of the inductive step of the induction we will use to prove Theorem 1 (b).

Lemma 5 *Let R be a local noetherian ring of mixed characteristic $(0, p)$ such that depth$(R) \geq 4$. We assume there exists $y \in R$ such that (y, p) is a regular sequence of R. Let $S = R/(y)$. Let D_U be a Barsotti–Tate group over the punctured spectrum U of R. If the restriction of D_U to the punctured spectrum of S extends to Spec(S), then D_U extends to a Barsotti–Tate group over $X =$ Spec(R).*

Proof: Based on the proof of Lemma 3 we can assume that R is complete. For $t \in \mathbb{N}^*$ let

$$X_t = \mathrm{Spec}(R/(y)^t)$$

and $U_t = U \cap X_t$. By induction on $t \in \mathbb{N}^*$ we will show that the restriction D_{U_t} of D_U to U_t extends uniquely (up to unique isomorphism) to a Barsotti–Tate group D_{X_t} over X_t. We already know that this holds for $t = 1$. Let $\iota_{U_1} : D_{U_1} \to U_1 \times_{X_1} D_{X_1}$ be the canonical isomorphism.

The passage from t to $t+1$ goes as follows. Assuming that D_{X_t} exists, let $\iota_{U_t} : D_{U_t} \to U_t \times_{X_t} D_{X_t}$ be the canonical isomorphism. For $q \in \mathbb{N}^*$, let $D_{X_{t,q}}$ and $D_{U_{t,q}}$ be the reductions modulo p^q of D_{X_t} and D_{U_t} (respectively); they are Barsotti–Tate groups over the reductions $X_{t,q}$ and $U_{t,q}$ of X_t and U_t (respectively) modulo p^q. From the Grothendieck–Messing deformation theory (see [14], Thm. 4.4 and Cor. 4.7 and [16], Ch. V, Thm. 1.6) we get that the lifts of $D_{X_{t,q}}$ to Barsotti–Tate groups over $X_{t+1,q}$ are parametrized by the global sections of a torsor under the group of global sections of a coherent locally free $\mathcal{O}_{\mathrm{Spec}(S/p^qS)}$-module $\mathcal{F}_{t,q}$ (note that the ideal that defines the closed embedding $X_{t,q} \to X_{t+1,q}$ has a canonical trivial divided power structure and therefore this closed embedding has a canonical structure of a nilpotent divided power thickening). Similarly, the lifts of $D_{U_{t,q}}$ to Barsotti–Tate groups over $U_{t+1,q}$ are parametrized by a torsor under the group $H^0(U \cap \mathrm{Spec}(S/p^qS), \mathcal{F}_{t,q})$. As (y, p) is a regular sequence of R, we have $\mathrm{depth}_S(S/p^qS) = \mathrm{depth}(R) - 2 \geq 2$ (cf. Auslander–Buchsbaum formula) and thus we can identify $H^0(U \cap \mathrm{Spec}(S/p^qS), \mathcal{F}_{t,q}) = H^0(\mathrm{Spec}(S/p^qS), \mathcal{F}_{t,q})$.

This implies that there exists a unique (up to unique isomorphism) Barsotti–Tate group $D_{X_{t+1,q}}$ over $X_{t+1,q}$ that lifts compatibly both $D_{X_t,q}$ and $D_{U_{t+1},q}$. As $R/(y)^{t+1}$ is p-adically complete, there exists a unique (up to unique isomorphism) Barsotti–Tate group $D_{X_{t+1}}$ over X_{t+1} which lifts the system $D_{X_{t+1},q}$, $q \in \mathbb{N}^*$ (cf. [16], Ch. II, Lem. 4.16 which implies that the categories of Barsotti–Tate groups over X_{t+1} and over the formal scheme which is the p-adic completion of X_{t+1} are canonically equivalent).

For each $n \in \mathbb{N}^*$, from Lemma 4 applied with

$$\big(S, \tau, (\mathcal{F}_q, \mathcal{F}_q^+)_{q\in\mathbb{N}^*}\big) = \big(R/(y)^{t+1}, p, (\mathcal{O}_{D_{U_{t+1},q}[p^n]}, \mathcal{O}_{D_{X_{t+1},q}[p^n]})_{q\in\mathbb{N}^*}\big),$$

we get that $\mathcal{O}_{D_{U_{t+1}}[p^n]}$ is the restriction to U_{t+1} of a vector bundle over X_{t+1} whose restriction to $X_{t+1,q}$ is compatibly identified with $\mathcal{O}_{D_{X_{t+1},q}[p^n]}$ for all $q \in \mathbb{N}^*$, and thus (as X_{t+1} is p-adically complete) this vector bundle can be functorially identified with $\mathcal{O}_{D_{X_{t+1}}[p^n]}$. We deduce an isomorphism $\iota_{U_{t+1}}[p^n] : D_{U_{t+1}}[p^n] \to U_{t+1} \times_{X_{t+1}} D_{X_{t+1}}[p^n]$ of U_{t+1}-schemes, and it must be a group scheme isomorphism. The $\iota_{U_{t+1}}[p^n]$'s glue together to define an isomorphism $\iota_{U_{t+1}} : D_{U_{t+1}} \to U_{t+1} \times_{X_{t+1}} D_{X_{t+1}}$. Therefore the restriction of $D_{X_{t+1}}$ to U_{t+1} is canonically identified with $D_{U_{t+1}}$. This ends the induction.

From Lemma 4 applied with $(S, \tau) = (R, y)$ we similarly get that for each $n \in \mathbb{N}^*$ the locally free $\mathcal{O}_U$-module associated to $D_U[p^n]$ extends to a locally free $\mathcal{O}_X$-module whose reduction modulo each $(y)^t$ is the $\mathcal{O}_{X_t}$-module associated to $D_{X_t}[p^n]$ and which defines naturally a Barsotti–Tate group D_n of level n over X. The inductive system D_n is a Barsotti–Tate group over X which extends D_U (and thus our notation matches); we note that it is also the unique (up to unique isomorphism) Barsotti–Tate group over X which lifts compatibly D_{X_t} for all $t \in \mathbb{N}^*$ (again cf. [16], Ch. II, Lem. 4.16). Thus the lemma holds. $\square$

3 On the condition (♮)

In this section we assume that R is regular of dimension $d \geq 1$ and mixed characteristic $(0, p)$ and study the condition (♮) introduced in Theorem 1. Let $C(k)$ be a Cohen ring with residue field k which is a coefficient ring of $\widehat{R}$, as in the Cohen structure theorem for complete local noetherian rings of mixed characteristic (see [8], Ch. 0, Subsect. 19.8 or [15], Subsect. 29). Thus $C(k)$ is a subring of $\widehat{R}$. Let $y_1, \ldots, y_d$ be a regular system of parameters of

$\widehat{R}$. The natural $C(k)$-homomorphism $\varrho : C(k)[[x_1,\ldots,x_d]] \to \widehat{R}$ that maps x_i to y_i is onto. This implies that we can identify

$$\widehat{R} = C(k)[[x_1,\ldots,x_d]]/(h)$$

for some element $h \in C(k)[[x_1,\ldots,x_d]]$ whose reduction $\bar{h}$ modulo p is a non-zero element of the ideal $(x_1,\ldots,x_d)$ of $k[[x_1,\ldots,x_d]]$. The reduction h_p of h modulo the ideal $(x_1,\ldots,x_d)$ of $C(k)[[x_1,\ldots,x_d]]$ is an element of $C(k)$ with the property that $C(k)/(h_p) = \widehat{R}/(y_1,\ldots,y_d) = k$. Thus $(h_p) = (p) \subset C(k)$ and therefore h_p is p times a unit of $C(k)$.

The ring R/pR is regular if and only if $\bar{h} \notin (x_1,\ldots,x_d)^2$ and if and only if $\widehat{R}$ is isomorphic to $C(k)[[x_1,\ldots,x_{d-1}]]$. If R/pR is regular, then there are $C(k)$-epimorphisms $\theta : C(k)[[x_1,\ldots,x_d]] \to \widehat{R}$ under which the images of $x_1,\ldots,x_d$ in $\widehat{R}$ do not form a regular system of parameters of $\widehat{R}$; but each such $C(k)$-epimorphism has a kernel generated by an element of $C(k)[[x_1,\ldots,x_d]]$ whose reduction modulo p belongs to a regular system of parameters of $k[[x_1,\ldots,x_d]]$ and therefore differs from $\bar{h}$ by a k-automorphism of $k[[x_1,\ldots,x_d]]$. If R/pR is not regular, then there exists no $C(k)$-epimorphism $\theta : C(k)[[x_1,\ldots,x_d]] \to \widehat{R}$ under which the images of $x_1,\ldots,x_d$ in $\widehat{R}$ do not form a regular system of parameters of $\widehat{R}$.

If $C'(k)$ is another Cohen ring which is a coefficient ring of $\widehat{R}$ and if $\varrho' : C'(k)[[x_1,\ldots,x_d]] \to \widehat{R}$ is a $C'(k)$-epimorphism such that the images of $x_1,\ldots,x_d$ form a regular system of parameters of $\widehat{R}$, then there exists an isomorphism $\nu : C(k)[[x_1,\ldots,x_d]] \to C'(k)[[x_1,\ldots,x_d]]$ such that we have $\varrho' = \varrho \circ \nu$. To check this, up to a $C'(k)$-automorphism of $C'(k)[[x_1,\ldots,x_d]]$ preserving the ideal $(x_1,\ldots,x_d)$, we can assume that $\varrho'(x_i) = y_i$. As the homomorphism $\mathbb{Z}_p \to C(k)$ is formally smooth, there exists a homomorphism $\nu_p : C(k) \to C'(k)[[x_1,\ldots,x_d]]$ whose composite with ϱ' is the inclusion $C(k) \to \widehat{R}$. Thus we can take ν such that it extends ν_p and maps x_i into x_i for all $i \in \{1,\ldots,d\}$.

From the last two paragraphs we get that the ideal $(\bar{h})$ of $k[[x_1,\ldots,x_d]]$ coming from a presentation of $\widehat{R}$ as $C(k)[[x_1,\ldots,x_d]]/(h)$ is uniquely determined by R up to automorphisms of $k[[x_1,\ldots,x_d]]$ inducing the identity on the residue field k, and $\bar{h}$ is uniquely determined up to units and such automorphisms. We say that $\bar{h}$ is a Cohen element of R. Thus condition ($\natural$) for R means that one Cohen element of R does not belong to the ideal $(x_1^p,\ldots,x_d^p) + (x_1,\ldots,x_d)^{2p-2}$ of $k[[x_1,\ldots,x_d]]$, equivalently all Cohen elements of R have this property.

Recall ([15], Exerc. 14.5) that the multiplicity of R/pR is the order of vanishing of p, i.e., it is the positive integer e such that $p \in \mathfrak{m}^e \setminus \mathfrak{m}^{e+1}$, where $\mathfrak{m}$ is the maximal ideal of R. As $\widehat{R}/p\widehat{R} \simeq k[[x_1, \ldots, x_d]]/(\bar{h})$, e is also the order of vanishing of $\bar{h}$. In particular, we have:

(i) if $e < p$, then condition ($\natural$) holds;

(ii) if condition ($\natural$) holds, then $e \leq 2p-3$ for $d \geq 2$ and $e \leq p-1$ for $d = 1$.

Fact 2 *Suppose $\widehat{R}$ is presented as $C(k)[[x_1, \ldots, x_d]]/(h)$. Let l be a field extension of k and let $C(k) \to C(l)$ be a monomorphism between Cohen rings which modulo p is the inclusion $k \to l$, cf. [3], Cor. 1. Then the R-algebra $R' = C(l)[[x_1, \ldots, x_d]]/(h)$ is faithfully flat, R' is regular, and condition ($\natural$) holds for R if and only if it holds for R'.*

Proof: As the $C(k)$-algebra $C(l)$ is faithfully flat, the $C(k)[x_1, \ldots, x_d]$-algebra $C(l)[x_1, \ldots, x_d]$ is also faithfully flat. Taking completions of localizations with respect to maximal ideals $(p, x_1, \ldots, x_d)$ we get that the $C(k)[[x_1, \ldots, x_d]]$-algebra $C(l)[[x_1, \ldots, x_d]]$ is flat, see [15], Thm. 22.4 (i). Hence $\widehat{R} \to R'$ is flat; being local, it is faithfully flat. The same holds for $R \to R'$, and R' is regular by [15], Thm. 23.7. The last assertion is clear. $\square$

3.1 Taking slices

In this subsection we study for $d \geq 3$ the behavior of the condition ($\natural$) under restrictions to quotient rings of $\widehat{R}$ which are regular of dimension $d-1$. We have the following general and abstract lemma on homogeneous polynomials which also holds for $d = 2$.

Lemma 6 *Let l be a field of characteristic p. Let $d \geq 2$ be an integer and let $c \in \{1, \ldots, (d-1)(p-1)\}$. Let $\mathbb{I}_c$ be the subset of $\{0, \ldots, p-1\}^d$ formed by all those d-tuples whose entries sum up to c. We consider a non-zero homogeneous polynomial*

$$\bar{h}_c(x_1, \ldots, x_d) = \sum_{(i_1, \ldots, i_d) \in \mathbb{I}_c} \delta_{i_1, \ldots, i_d} \prod_{t=1}^{d} x_t^{i_t} \in l[x_1, \ldots, x_d]$$

of degree c. Then there exists an open dense subscheme $\mathbb{O}$ of the affine scheme $\mathbb{A}_l^{d-1}$ such that for each $d-1$-tuples $(v_1, \ldots, v_{d-1}) \in \mathbb{O}(l) \subset \mathbb{A}_l^{d-1}(l) = l^{d-1}$

the following homogeneous polynomial

$$\bar{q}_c(x_1,\ldots,x_{d-1}) = \bar{h}_c(x_1,\ldots,x_{d-1},\sum_{t=1}^{d-1} v_t x_t)$$

has a non-zero image in $l[x_1,\ldots,x_{d-1}]/(x_1^p,\ldots,x_{d-1}^p)$.

Proof: For $q \in \mathbb{N}\cap\{c-p+1,\ldots,c\}$ let $\mathbb{J}_q^{d-1}$ be the subset of $\{0,\ldots,p-1\}^{d-1}$ formed by all those $d-1$-tuples whose entries sum up to q. For $(j_1,\ldots,j_{d-1}) \in \mathbb{J}_c^{d-1}$, the coefficient of $\prod_{t=1}^{d-1} x_t^{j_t}$ in $\bar{q}_c$ is a sum of the form

$$\sum_{q\in\mathbb{N}\cap\{c-p+1,\ldots,c\}} \sum_{(i_1,\ldots,i_{d-1})\in\mathbb{J}_q^{d-1}, i_1\le j_1,\ldots,i_{d-1}\le j_{d-1}} \gamma_{i_1,\ldots,i_{d-1}}^{j_1,\ldots,j_{d-1}} \delta_{i_1,\ldots,i_{d-1},c-q} \prod_{t=1}^{d-1} v_t^{j_t-i_t},$$

where each $\gamma_{i_1,\ldots,i_{d-1}}^{j_1,\ldots,j_{d-1}}$ is a multinomial coefficient that divides $(c-q)!$ and thus is a non-zero element of l.

If $\bar{h}_c$ is a polynomial only in the variables $x_1,\ldots,x_{d-1}$, then $\bar{q}_c = \bar{h}_c$ does not depend on the $d-1$-tuple $(v_1,\ldots,v_{d-1})$ and has a non-zero image in $l[x_1,\ldots,x_{d-1}]/(x_1^p,\ldots,x_{d-1}^p)$; thus in this case we can take $\mathbb{O} = \mathbb{A}_l^{d-1}$.

Therefore we can assume that $\bar{h}_c$ is not a polynomial only in the variables $x_1,\ldots,x_{d-1}$, i.e., there exists a non-zero coefficient $\delta_{i_1,\ldots,i_d}$ of $\bar{h}_c$ with $i_d \ge 1$ or equivalently with $\sum_{t=1}^{d-1} i_t < c$. As $c \le (d-1)(p-1)$, if for each $t \in \{1,\ldots,d-1\}$ we consider an element $j_t \in \{i_t,\ldots,p-1\}$ such that we have $\sum_{t=1}^{d-1} j_t = c$, then the coefficient of $\prod_{t=1}^{d-1} x_t^{j_t}$ in $\bar{q}_c$ is a non-zero polynomial $\bar{q}_{c,j_1,\ldots,j_{d-1}}(v_1,\ldots,v_{d-1}) \in l[v_1,\ldots,v_{d-1}]$. We consider the open dense subscheme $\mathbb{O}$ of $\mathbb{A}_l^{d-1}$ such that for a $d-1$-tuple $(v_1,\ldots,v_{d-1}) \in \mathbb{A}_l^{d-1}(l) = l^{d-1}$ we have $\bar{q}_{c,j_1,\ldots,j_{d-1}}(v_1,\ldots,v_{d-1}) \neq 0$ if and only if $(v_1,\ldots,v_{d-1}) \in \mathbb{O}(l)$. Thus if $(v_1,\ldots,v_{d-1}) \in \mathbb{O}(l)$, the coefficient of $\prod_{t=1}^{d-1} x_t^{j_t}$ in $\bar{q}_c$ is non-zero and therefore the image of $\bar{q}_c$ in $l[x_1,\ldots,x_{d-1}]/(x_1^p,\ldots,x_{d-1}^p)$ is non-zero. $\square$

Proposition 3 *We assume that $d \ge 3$, that k is an infinite field, that $R = \widehat{R}$ is complete, and that condition ($\natural$) holds for R. Then for each $d_1 \in \{2,\ldots,d-1\}$ there exist closed regular subschemes $Spec(S)$ of $X = Spec(R)$ of dimension d_1 and mixed characteristic $(0,p)$ such that condition ($\natural$) holds for S.*

Proof: Proceeding by induction on $d \ge 3$, it suffices to consider the case when $d_1 = d-1$. We write $\bar{h} = \bar{h}_0 + \sum_{i=1}^{2p-3} \bar{h}_i$, where $\bar{h}_0$ belongs to the ideal $(x_1^p, \ldots, x_d^p) + (x_1, \ldots, x_d)^{2p-2}$ of $k[[x_1, \ldots, x_d]]$ and where each $\bar{h}_i$ is either 0 or a homogeneous polynomial in the variables $x_1, \ldots, x_d$ of degree i. Let $c \in \{1, \ldots, 2p-3\}$ be the smallest integer such that $\bar{h}_c$ is non-zero (it exists as condition ($\natural$) holds for R). The field k is infinite and we have $c \leq 2p-3 < (d-1)(p-1)$ as $d \geq 3$. Thus, from Lemma 6 applied with $l = k$ we get that there exists a $d-1$-tuple $(v_1, \ldots, v_{d-1}) \in \mathbb{O}(k) \subset k^{d-1}$ such that the homogeneous polynomial $\bar{h}_c(x_1, \ldots, x_{d-1}, \sum_{t=1}^{d-1} v_t x_t)$ of degree c has a non-zero-image in $k[[x_1, \ldots, x_{d-1}]]/(x_1^p, \ldots, x_{d-1}^p)$. This implies that $\bar{h}(x_1, \ldots, x_{d-1}, \sum_{t=1}^{d-1} v_t x_t)$ does not belong to the ideal $(x_1^p, \ldots, x_{d-1}^p) + (x_1, \ldots, x_{d-1})^{2p-2}$ of $k[[x_1, \ldots, x_{d-1}]]$ and therefore condition ($\natural$) holds for the regular local ring

$$S = R/(x_d - \sum_{t=1}^{d-1} w_i x_i) = C(k)[[x_1, \ldots, x_{d-1}]]/(h(x_1, \ldots, x_{d-1}, \sum_{t=1}^{d-1} w_t x_t)$$

of dimension $d-1$ and mixed characteristic $(0, p)$, where $(w_1, \ldots, w_{d-1}) \in C(k)^{d-1}$ is such that its reduction modulo p is $(v_1, \ldots, v_{d-1}) \in k^{d-1}$. $\square$

3.2 Functorial properties

In this subsection we study the behavior of the condition ($\natural$) under local homomorphisms. Let $\varphi : R \to R'$ be a local homomorphism between regular local rings of mixed characteristic $(0, p)$. Let $d' \geq 1$ be the dimension of R'. Let k' be the residue field of R' and let $C(k')$ be a Cohen ring which is a coefficient ring of $\widehat{R}'$. Let $y'_1, \ldots, y'_{d'}$ be a regular system of parameters of $\widehat{R}'$. The natural $C(k')$-homomorphism $\varrho' : C(k')[[x'_1, \ldots, x'_{d'}]] \to \widehat{R}'$ that maps x'_i to y'_i is onto and we identify

$$\widehat{R}' = C(k')[[x'_1, \ldots, x'_{d'}]]/(h')$$

for some element $h' \in C(k')[[x'_1, \ldots, x'_{d'}]]$. As the homomorphism $\mathbb{Z}_p \to C(k)$ is formally smooth, the composite homomorphism $C(k) \to \widehat{R} \to \widehat{R}'$ admits a lift to a homomorphism $\Phi_p : C(k) \to C(k')[[x'_1, \ldots, x'_{d'}]]$. We consider a homomorphism $\Phi : C(k)[[x_1, \ldots, x_d]] \to C(k')[[x'_1, \ldots, x_{d'}]]$ that extends Φ_p, that maps each x_i into the ideal $(x'_1, \ldots, x'_{d'})$ of $C(k')[[x'_1, \ldots, x_{d'}]]$, and such that we have a commutative diagram:

$$\begin{tikzcd} C(k)[[x_1,\ldots,x_d]] \arrow[r, "\Phi"] \arrow[d, "\varrho"] & C(k')[[x'_1,\ldots,x_{d'}]] \arrow[d, "\varrho'"] \\ \widehat{R} \arrow[r, "\widehat{\varphi}"] & \widehat{R}'. \end{tikzcd}$$

Proposition 4 *With φ and Φ as above, the following three properties hold:*

(a) *We have an equality $(\Phi(h)) = (h')$ of ideals of $C(k')[[x'_1,\ldots,x'_{d'}]]$.*

(b) *If the condition $(\natural)$ holds for R', then it also holds for R.*

(c) *Let $X' = Spec(R')$. We assume that the morphism $X' \to X$ defined by φ is flat and its fiber over the closed point of X is regular (i.e., the local ring $R'/(\varphi(y_1),\ldots,\varphi(y_d))R'$ is regular). Then the condition $(\natural)$ holds for R if and only if it holds for R'.*

Proof: As Φ induces $\widehat{\varphi} : \widehat{R} \to \widehat{R}'$, we have $\Phi(h) \in (h')$. We consider the unique element $u \in C(k')[[x'_1,\ldots,x'_{d'}]]$ such that we have $\Phi(h) = h'u$. To prove (a) it suffices to show that u is a unit of $C(k')[[x'_1,\ldots,x'_{d'}]]$. If h'_p and u_p are the reductions modulo the ideal $(x'_1,\ldots,x'_{d'})$ of $C(k')[[x'_1,\ldots,x'_{d'}]]$, then the equality $\Phi(h) = h'u$ modulo the ideal $(x'_1,\ldots,x'_{d'})$ becomes an identity $\varphi_p(h_p) = h'_p u_p$ between elements of $C(k')$, where $\varphi_p : C(k) \to C(k') = C(k')[[x'_1,\ldots,x'_{d'}]]/(x'_1,\ldots,x'_{d'})$ is induced by Φ_p. We know that h_p and h'_p are p times units of $C(k)$ and $C(k')$ (respectively), see the beginning of Section 3. From the last two sentences we get that u_p is a unit of $C(k')$ and therefore u is a unit of $C(k')[[x'_1,\ldots,x'_{d'}]]$. Thus (a) holds.

To prove (b) let $\bar{h}'$ and $\bar{\Phi} : k[[x_1,\ldots,x_d] \to k'[[x'_1,\ldots,x'_{d'}]]$ be the reductions modulo p of h' and Φ (respectively). The elements $\bar{\Phi}(\bar{h})$ and $\bar{h}'$ differ by a unit, cf. (a). Thus if $\bar{h}$ belongs to the ideal $(x_1^p,\ldots,x_d^p) + (x_1,\ldots,x_d)^{2p-2}$ of $k[[x_1,\ldots,x_d]]$, then $\bar{\Phi}(\bar{h})$ and $\bar{h}'$ belong to the ideal of $k'[[x'_1,\ldots,x'_{d'}]]$ generated by $\bar{\Phi}((x_1^p,\ldots,x_d^p)) + \bar{\Phi}((x_1,\ldots,x_d)^{2p-2})$ and thus to the ideal $((x'_1)^p,\ldots,(x'_{d'})^p) + (x'_1,\ldots,x'_{d'})^{2p-2}$ of $k'[[x'_1,\ldots,x'_d]]$ and therefore the condition $(\natural)$ does not holds for R'. From this (b) follows.

Based on (b), to prove (c) it suffices to show that if the condition $(\natural)$ holds for R, then it also holds for R'. To ease notation we can assume that R and R' are complete; thus $\varphi = \widehat{\varphi}$, $y_1,\ldots,y_d \in R$, and $y'_1,\ldots,y'_{d'} \in R'$. Let c be the dimension of the regular local ring $\bar{R}' = R'/(\varphi(y_1),\ldots,\varphi(y_d))R'$ of characteristic p. As φ is flat, we have $d' = d+c$ (cf. [15], Thm. 15.1). We can assume that $y'_{d+1},\ldots,y'_{d+c}$ map into a regular system of parameters of $\bar{R}'$.

Let $R'_1 = R'/(y'_{d+1}, \ldots, y'_{d+c})$; it is a regular local ring of dimension $d = d' - c$. The composite homomorphism $\varphi_1 : R \to R'_1$ between local rings of dimension d is such that $R'_1/(\varphi(y_1), \ldots, \varphi(y_d)) = \mathrm{Spec}(k')$. This implies that φ_1 is flat, cf. [15], Thm. 23.1. Thus R'_1 has mixed characteristic $(0, p)$. If condition ($\natural$) holds for R'_1, then it also holds for R' (cf. (b)) and therefore by replacing R' with R'_1 we can assume that $d' = d$, that $c = 0$, and therefore that we have $y'_i = \varphi(y_i)$ for all $i \in \{1, \ldots, d\}$. We can also assume that we have $x'_i = \Phi(x_i)$ for all $i \in \{1, \ldots, d\}$, that $\bar{\Phi} : k[[x_1, \ldots, x_d] \to k'[[x'_1, \ldots, x'_{d'}]]$ can be identified with the canonical inclusion $k[[x_1, \ldots, x_d] \to k'[[x_1, \ldots, x_d]]$, and that under this identification we have $\bar{h} = \bar{h}'$. As $\bar{h}$ does not belong to the ideal $(x_1^p, \ldots, x_d^p) + (x_1, \ldots, x_d)^{2p-2}$ of $k[[x_1, \ldots, x_d]]$, it does not belong to the ideal $(x_1^p, \ldots, x_d^p) + (x_1, \ldots, x_d)^{2p-2}$ of $k'[[x_1, \ldots, x_d]]$ and thus the condition ($\natural$) holds for R'. Thus (c) holds. $\square$

3.3 A generization property

Proposition 5 *We assume that R is a regular local ring of dimension $d \geq 2$ and mixed characteristic $(0, p)$ for which condition ($\natural$) holds. Let R_1 be a local ring of $X = Spec(R)$ which is of mixed characteristic $(0, p)$ and dimension $d_1 \in \{1, \ldots, d-1\}$. Then condition ($\natural$) holds for R_1.*

Proof: Let k_1 be the residue field of R_1. Let $\mathfrak{p}_1$ be the prime ideal of R such that $R_1 = R_{\mathfrak{p}_1}$. We consider two cases as follows.

Case 1. We first consider the case when $R = \widehat{R}$ is complete. We fix an identification $R = C(k)[[x_1, \ldots, x_d]]/(h)$, where $h \in C(k)[[x_1, \ldots, x_d]]$ is such that its reduction $\bar{h}$ modulo p does not belong to the ideal $(x_1^p, \ldots, x_d^p) + (x_1, \ldots, x_d)^{2p-2}$ of $k[[x_1, \ldots, x_d]]$. Let $\mathfrak{p}$ be the inverse image of $\mathfrak{p}_1$ via the epimorphism $\varrho : C(k)[[x_1, \ldots, x_d]] \to R$. Let $\mathfrak{n} = \mathfrak{p}C(k)[[x_1, \ldots, x_d]]_{\mathfrak{p}}$ be the maximal ideal of $C(k)[[x_1, \ldots, x_d]]_{\mathfrak{p}}$ and let $\mathfrak{n}^{(p)}$ be the $\mathfrak{n}$-primary ideal of $C(k)[[x_1, \ldots, x_d]]_{\mathfrak{p}}$ generated by p and by p-th powers of elements of $\mathfrak{n}$. Let $\mathfrak{A} = C(k)[[x_1, \ldots, x_d]]_{\mathfrak{p}}/\mathfrak{n}^{(p)}$; it is a local artinian ring which contains k, which has $\mathfrak{j} = \mathfrak{n}/\mathfrak{n}^{(p)}$ as its maximal ideal, and which as a k-algebra is isomorphic to $k_1[y_1, \ldots, y_{d_1}]/(y_1^p, \ldots, y_{d_1}^p)$.

As $\bar{h}$ does not belong to the ideal $(x_1^p, \ldots, x_d^p) + (x_1, \ldots, x_d)^{2p-2}$ of the ring $k[[x_1, \ldots, x_d]]$, we can speak about the smallest integer $n \in \{1, \ldots, 2p-3\}$ such that $\bar{h}$ has in its writing a non-zero term of the form $v \prod_{j=1}^{d} x_j^{i_j}$ with each $i_j \in \{0, \ldots, p-1\}$ such that $n = \sum_{j=1}^{d} i_j$ and with $v \in k \setminus \{0\}$. This implies that there exist n derivations $\partial_1, \ldots, \partial_n$ of $C(k)[[x_1, \ldots, x_d]]$ which are $C(k)$-linear, which belong to the set $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\}$, and for which the element $(\partial_1 \circ \cdots \circ \partial_n)(h)$ is a unit of $C(k)[[x_1, \ldots, x_d]]$ (for instance, we can take $\partial_1 = \cdots = \partial_{i_1} = \frac{\partial}{\partial x_1}$, $\partial_{i_1+1} = \cdots = \partial_{i_1+i_2} = \frac{\partial}{\partial x_2}, \ldots, \partial_{i_1+\ldots+i_{d-1}+1} = \cdots = \partial_n = \frac{\partial}{\partial x_d}$). For $i \in \{1, \ldots, n\}$ the derivation of $C(k)[[x_1, \ldots, x_d]]_{\mathfrak{p}}$ that extends ∂_i induces a derivation $\bar{\partial}_i : \mathfrak{A} \to \mathfrak{A}$.

Let $\bar{h}_{(p)}$ be the image of $\bar{h}$ in $\mathfrak{A}$. As $(\partial_1 \circ \cdots \circ \partial_n)(h)$ is a unit of $C(k)[[x_1, \ldots, x_d]]$, the element $(\bar{\partial}_1 \circ \cdots \circ \bar{\partial}_n)(\bar{h}_{(p)})$ is a unit of $\mathfrak{A}$. We show that the assumption that $\bar{h}_{(p)} \in \mathfrak{j}^{2p-2}$ leads to a contradiction. By a decreasing induction on $i \in \{1, \ldots, n\}$ we get that $(\bar{\partial}_i \circ \cdots \circ \bar{\partial}_n)(\bar{h}_{(p)}) \in \mathfrak{j}^{2p-2-n+i-1}$. Thus the element $(\bar{\partial}_1 \circ \cdots \circ \bar{\partial}_n)(\bar{h}_{(p)})$ belongs to $\mathfrak{j}^{2p-2-n}$. From this and the inequality $2p - 2 - n \geq 1$ we get that $(\bar{\partial}_1 \circ \cdots \circ \bar{\partial}_n)(\bar{h}_{(p)}) \in \mathfrak{j}$ is not a unit of $\mathfrak{A}$. Contradiction. Thus $\bar{h}_{(p)} \notin \mathfrak{j}^{2p-2}$. This implies that h does not belong to the ideal $\mathfrak{n}^{(p)} + \mathfrak{n}^{2p-2}$ of $C(k)[[x_1, \ldots, x_d]]_{\mathfrak{p}}$.

Let $C(k_1)$ be a Cohen ring which is a coefficient ring of the completion $C(k)[[\widehat{x_1, \ldots}, x_d]]_{\mathfrak{p}}$. We fix a $C(k_1)$-isomorphism

$$C(k)[[\widehat{x_1, \ldots}, x_d]]_{\mathfrak{p}} \simeq C(k_1)[[y_1, \ldots, y_{d_1}]].$$

The completion $\widehat{R_1}$ of R_1 is canonically identified with $C(k)[[\widehat{x_1, \ldots}, x_d]]_{\mathfrak{p}}/(h)$ and thus with $C(k_1)[[y_1, \ldots, y_{d_1}]]/(h)$. From the last sentence of the previous paragraph we get that the image of $\bar{h}$ in $(y_1, \ldots, y_{d_1})$ does not belong to the ideal $(y_1^p, \ldots, y_{d_1}^p) + (y_1, \ldots, y_{d_1})^{2p-2}$ of $k_1[[y_1, \ldots, y_{d_1}]]$. Thus condition $(\natural)$ holds for R_1 if $R = \widehat{R}$ is complete.

Case 2. We consider the case when R is not complete. Let $\mathfrak{q}$ be a prime ideal of $\widehat{R}$ which is minimal over $\mathfrak{p}_1\widehat{R}$. We have a canonical homomorphism $R_1 = R_{\mathfrak{p}_1} \to \widehat{R}_{\mathfrak{q}}$ of local rings. From Case 1 applied to $\widehat{R}$ we get that condition $(\natural)$ holds for $\widehat{R}_{\mathfrak{q}}$. Thus condition $(\natural)$ also holds for R_1, by Proposition 4 (b).$\square$

4 On closed subschemes of projective spaces

Let l be a field. Let $N \in \mathbb{N}^*$. Let $\mathcal{C}$ be a closed connected subscheme (thus nonempty) of the projective space $\mathbb{P}_l^N$. Let $\widehat{\mathbb{P}_l^N}$ be the formal completion of $\mathbb{P}_l^N$ along $\mathcal{C}$. Let κ be the field of rational functions of $\mathbb{P}_l^N$ and let κ' be the ring of formal-rational functions on $\widehat{\mathbb{P}_l^N}$. We recall from [13], Def. (2.9.3) that $\mathcal{C}$ is said to be G3 in $\mathbb{P}_l^N$ if the homomorphism $\kappa \to \kappa'$ associated to the natural morphism $\chi : \widehat{\mathbb{P}^N_l} \to \mathbb{P}^N_l$ of locally ringed spaces is an isomorphism. For instance, from the proof of [13], Thm. (3.3) we get that if l is an algebraically closed field and if $\mathcal{C}$ is irreducible of positive dimension, then $\mathcal{C}$ is (universally) G3 in $\mathbb{P}^N_l$. In Section 5 we will need the following general result of formal algebraic geometry.

Lemma 7 *We assume that $\mathcal{C}$ is a geometrically connected scheme of positive dimension which is G3 in $\mathbb{P}^N_l$. Let $\mathcal{U}$ be an open subscheme of $\mathbb{P}^N_l$ which contains $\mathcal{C}$. Let $\mathcal{V}$ be a torsion free coherent sheaf on $\mathcal{U}$. Let $\widehat{\mathcal{V}} = \chi^*(\mathcal{V})$ be the coherent sheaf on $\widehat{\mathbb{P}^N_l}$ which is the natural pullback of $\mathcal{V}$. Then there exists an open subscheme $\mathcal{W}$ of $\mathcal{U}$ which contains $\mathcal{C}$ and such that the natural pullback homomorphism*

$$\rho_{\mathcal{W}} : H^0(\mathcal{W}, \mathcal{V}) \to H^0(\widehat{\mathbb{P}^N_l}, \widehat{\mathcal{V}})$$

between global sections is an isomorphism between finite dimensional l-vector spaces. Moreover, if $\mathcal{V}$ is a reflexive $\mathcal{O}_{\mathcal{U}}$-module, then we can take $\mathcal{W} = \mathcal{U}$.

Proof: Using a meromorphic basis of $\mathcal{V}$ and the assumption that $\kappa = \kappa'$, one sees that the finite dimensional κ-vector space V of meromorphic sections of $\mathcal{V}$ maps isomorphically to the space of meromorphic sections of $\widehat{\mathcal{V}}$. Thus, as $\mathcal{V}$ is torsion free, each homomorphism $\rho_{\mathcal{W}}$ is injective and we have natural inclusions $H^0(\mathcal{U}, \mathcal{V}) \subset H^0(\widehat{\mathbb{P}^N_l}, \widehat{\mathcal{V}}) \subset V$.

We will first show that for each $\xi \in H^0(\widehat{\mathbb{P}^N_l}, \widehat{\mathcal{V}}) \subset V$ there exists an open subscheme $\mathcal{W}_\xi$ of $\mathcal{U}$ which contains $\mathcal{C}$ and such that we have $\xi \in H^0(\mathcal{W}_\xi, \mathcal{V})$.

Let $\mathcal{O}$ be a local ring of $\mathbb{P}^N_l$ at a closed point z of $\mathcal{C}$; it is a subring of κ. Let $\widehat{\mathcal{O}}$ be the completion of $\mathcal{O}$ and let $\mathcal{M}$ be the $\mathcal{O}$-submodule of V defined by the global sections of $\mathcal{V}$ over $\mathrm{Spec}(\mathcal{O})$; we can identify $V = \kappa \otimes_{\mathcal{O}} \mathcal{M}$.

As the homomorphism $\mathcal{O} \to \widehat{\mathcal{O}}$ is faithfully flat, it is well known that we have $\kappa \cap \widehat{\mathcal{O}} = \mathcal{O}$ (the intersection being taken inside the field of fractions of $\widehat{\mathcal{O}}$). We recall the argument for this. For $a \in \kappa \cap \widehat{\mathcal{O}}$, we have inclusions of $\mathcal{O}$-modules $\mathcal{O} \subset \mathcal{O} + \mathcal{O}a \subset \kappa$. Tensoring with $\widehat{\mathcal{O}}$ over $\mathcal{O}$, we get inclusions $\widehat{\mathcal{O}} = \widehat{\mathcal{O}} + \widehat{\mathcal{O}}a \subset \widehat{\mathcal{O}} \otimes_{\mathcal{O}} \kappa$. As the homomorphism $\mathcal{O} \to \widehat{\mathcal{O}}$ is faithfully flat, it follows that the inclusion $\mathcal{O} \subset \mathcal{O} + \mathcal{O}a$ is as well an identity and therefore we have $a \in \mathcal{O}$; thus $\kappa \cap \widehat{\mathcal{O}} = \mathcal{O}$.

As $\mathcal{M}$ is a torsion free $\mathcal{O}$-module, the same argument shows that we have

$$\mathcal{M} = \mathcal{O} \otimes_{\mathcal{O}} \mathcal{M} = (\kappa \otimes_{\mathcal{O}} \mathcal{M}) \cap (\widehat{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{M}) = V \cap (\widehat{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{M}).$$

But as $\xi \in H^0(\widehat{\mathbb{P}_l^N}, \widehat{\mathcal{V}}) \subset V$, we also have $\xi \in V \cap (\widehat{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{M})$ and thus we have $\xi \in \mathcal{M}$. As z is an arbitrary closed point of $\mathcal{C}$, we get that there exists an open subscheme $\mathcal{W}_\xi$ of $\mathcal{U}$ which contains $\mathcal{C}$ and such that we have $\xi \in H^0(\mathcal{W}_\xi, \mathcal{V})$.

In this paragraph we consider the case when $\mathcal{V}$ is a reflexive $\mathcal{O}_{\mathcal{U}}$-module. As $\mathcal{C}$ has positive dimension, the complement of $\mathcal{W}_\xi$ in $\mathbb{P}_l^N$ (as it does not intersect $\mathcal{C}$) has codimension in $\mathbb{P}_l^N$ at least 2. This implies that $\xi \in H^0(\mathcal{W}_\xi, \mathcal{V}) = H^0(\mathcal{U}, \mathcal{V})$, cf. Fact 1 (c) for the equality part. Thus we can take $\mathcal{W}_\xi = \mathcal{U}$. This implies that $\rho_{\mathcal{U}}$ is surjective and therefore an isomorphism. It is well known that $H^0(\mathcal{U}, \mathcal{V})$ is a finite dimensional l-vector space (this can be checked by considering a coherent $\mathcal{O}_{\mathbb{P}_l^N}$-module that extends $\mathcal{V}$ and by recalling that $\mathbb{P}_l^N \setminus \mathcal{U}$ has codimension in $\mathbb{P}_l^N$ at least 2). We conclude that $H^0(\widehat{\mathbb{P}_l^N}, \widehat{\mathcal{V}})$ is a finite dimensional l-vector space when $\mathcal{V}$ is a reflexive $\mathcal{O}_{\mathcal{U}}$-module.

We now consider the general case when $\mathcal{V}$ is torsion free but not necessarily a reflexive $\mathcal{O}_{\mathcal{U}}$-module. As $\mathcal{V}$ is torsion free, it is a $\mathcal{O}_{\mathcal{U}}$-submodule of a coherent locally free $\mathcal{O}_{\mathcal{U}}$-module. From this and the previous paragraph we get that $H^0(\widehat{\mathbb{P}_l^N}, \widehat{\mathcal{V}})$ is a finite dimensional l-vector space. Let $\xi_1, \ldots, \xi_n \in H^0(\widehat{\mathbb{P}_l^N}, \widehat{\mathcal{V}}) \subset V$ be such that they generate $H^0(\widehat{\mathbb{P}_l^N}, \widehat{\mathcal{V}})$ as an l-vector space. For each $i \in \{1, \ldots, n\}$ let $\mathcal{W}_{\xi_i}$ be an open subscheme of $\mathcal{U}$ which contains $\mathcal{C}$ and such that we have $\xi_i \in H^0(\mathcal{W}_{\xi_i}, \mathcal{V})$ (see above). Then for the open subscheme $\mathcal{W} = \cap_{i=1}^n \mathcal{W}_i$ of $\mathcal{U}$, $\rho_{\mathcal{W}}$ is surjective and therefore also an isomorphism between finite dimensional l-vector spaces. $\square$

5 Proof of Theorem 2

In this section we prove Theorem 2. By enlarging l' and making $\mathcal{C}$ smaller if needed, to prove Theorem 2 we can assume that $\mathcal{C}$ is a geometrically integral curve and that its normalization $\mathcal{C}^{\mathrm{n}}$ is smooth over $\mathrm{Spec}(l')$. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{O}_{\mathbb{P}_{l'}^N}$ which defines $\mathcal{C}$. We have the following general constancy result.

Proposition 6 *In the context of Theorem 2, we assume that $l' = l$ and that $\mathcal{C}$ is a geometrically integral curve with smooth normalization. Then the Barsotti–Tate group $\mathcal{D}_{\widehat{\mathbb{P}_l^N}}$ over the formal completion $\widehat{\mathbb{P}_l^N}$ of $\mathbb{P}_l^N$ along $\mathcal{C}$*

induced naturally by $\varinjlim \mathcal{D}_{n,\mathcal{U}_\infty}$ is as well constant, i.e., it is isomorphic to $G_{\widehat{\mathbb{P}_l^N}}$ with G a Barsotti–Tate group over Spec(l).

Proof: We first consider the case when $\mathcal{C} = \mathcal{C}^{\mathrm{n}}$ is smooth over $\mathrm{Spec}(l)$. The conormal sheaf $\mathcal{I}/\mathcal{I}^2$ of $\mathcal{C}$ inside $\mathbb{P}_l^N$ is a subsheaf of $\mathcal{O}_\mathcal{C} \otimes_{\mathcal{O}_{\mathbb{P}_l^N}} \Omega^1_{\mathbb{P}_l^N}$ and thus (cf. Euler's sequence) also of $\mathcal{O}_\mathcal{C}(-1)^{N+1}$. Therefore for each $t \in \mathbb{N}^*$, its t-th symmetric power $S^t(\mathcal{I}/\mathcal{I}^2) = \mathcal{I}^t/\mathcal{I}^{t+1}$ has no non-zero global section.

We check by induction on $t \in \mathbb{N}$ that the Barsotti–Tate group $\varinjlim \mathcal{D}_{n,\mathcal{C}_t}$ over the t-th infinitesimal neighborhood $\mathcal{C}_t$ of $\mathcal{C}$ in $\mathbb{P}_l^N$ is a constant Barsotti–Tate group over $\mathcal{C}_t$. We recall the convention of [9], Def. 16.1.2 that $\mathcal{C}_t$ is defined by the ideal $\mathcal{I}^{t+1}$ of $\mathbb{P}_l^N$. As the inductive system $\mathcal{D}_{n,\mathcal{C}}$ is a constant Barsotti–Tate group over $\mathcal{C} = \mathcal{C}_0$ isomorphic to $G_\mathcal{C}$ for some Barsotti–Tate group G over $\mathrm{Spec}(l)$, the base of the induction holds for $t = 0$. As $G_{\mathcal{C}_{t+1}}$ is a canonical lift of $G_{\mathcal{C}_t}$ to $\mathcal{C}_{t+1}$, the lifts of $G_{\mathcal{C}_t}$ to Barsotti–Tate groups over $\mathcal{C}_{t+1}$ are parametrized by the global sections of an $\mathcal{O}_\mathcal{C}$-module isomorphic to $(\mathcal{I}^{t+1}/\mathcal{I}^{t+2})^{e_G}$, where e_G is the product of the dimension and the codimension of G. From this and the previous paragraph we get that $G_{\mathcal{C}_{t+1}}$ is the only lift of $G_{\mathcal{C}_t}$ to a Barsotti–Tate group over $\mathcal{C}_{t+1}$. Thus if the inductive system $\mathcal{D}_{n,\mathcal{C}_t}$ is a constant Barsotti–Tate group over $\mathcal{C}_t$ isomorphic to $G_{\mathcal{C}_t}$, then the inductive system $\mathcal{D}_{n,\mathcal{C}_{t+1}}$ is as well a constant Barsotti–Tate group over $\mathcal{C}_{t+1}$ isomorphic to $G_{\mathcal{C}_{t+1}}$. This ends the inductive step and thus also the induction.

From the previous paragraph we get that the Barsotti–Tate group $\mathcal{D}_{\widehat{\mathbb{P}_l^N}}$ over the formal completion $\widehat{\mathbb{P}_l^N}$ of $\mathbb{P}_l^N$ along $\mathcal{C}$ induced naturally by $\varinjlim \mathcal{D}_{n,\mathcal{U}_\infty}$ is as well constant isomorphic to $G_{\widehat{\mathbb{P}_l^N}}$.

We now consider the case when $\mathcal{C} \neq \mathcal{C}^{\mathrm{n}}$. Let $\varepsilon : \mathcal{C}^{\mathrm{n}} \to \mathbb{P}_l^N \times_{\mathrm{Spec}(l)} \mathbb{P}_l^3$ be a closed embedding whose projections are the composite morphism $\mathcal{C}^{\mathrm{n}} \to \mathcal{C} \to \mathbb{P}_l^N$ and a closed embedding $C^{\mathrm{n}} \to \mathbb{P}_l^3$. Let $\mathcal{J}$ be the ideal sheaf of $\mathcal{O}_{\mathbb{P}_l^N \times_{\mathrm{Spec}(l)} \mathbb{P}_l^3}$ which defines $\mathcal{C}^{\mathrm{n}}$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{n}} \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{nat}"] & \mathbb{P}_l^N \times_{\mathrm{Spec}(l)} \mathbb{P}_l^3 \arrow[d, "1^{\mathrm{st}}\ \mathrm{projection}"] \\
\mathcal{C} \arrow[r, "\mathrm{incl}"] & \mathbb{P}_l^N.
\end{tikzcd}$$

It is easy to see that for each $t \in \mathbb{N}^*$, the t-th symbolic power $\mathcal{I}^{(t)}$ of $\mathcal{I}$ (in the sense of [25], Ch. IV, Def. of Sect. 12) is the largest $\mathcal{O}_{\mathbb{P}_l^N}$-submodule of

$\mathcal{I}$ that maps naturally into $\mathcal{J}^t$. As above we argue that the t-th symmetric power $S^t(\mathcal{J}/\mathcal{J}^2) = \mathcal{J}^t/\mathcal{J}^{t+1}$ has no non-zero global section. From this and the natural inclusion $\mathcal{I}^{(t)}/\mathcal{I}^{(t+1)} \to \mathrm{nat}_*(\mathcal{J}^t/\mathcal{J}^{t+1})$ of $\mathcal{O}_{\mathcal{C}}$-modules, we get that also $\mathcal{I}^{(t)}/\mathcal{I}^{(t+1)}$ has no non-zero global section.[3]

As the inductive system $\mathcal{D}_{n,\mathcal{C}}$ is a constant Barsotti–Tate group over $\mathcal{C}$ isomorphic to $G_{\mathcal{C}}$, an induction on $t \in \mathbb{N}$ similar to the one above shows that the Barsotti–Tate group $\varinjlim \mathcal{D}_{n,\mathcal{C}_{(t)}}$ over the t-th symbolic infinitesimal neighborhood $\mathcal{C}_{(t)}$ of $\mathcal{C}$ in $\mathbb{P}^N_l$ (defined by the ideal sheaf $\mathcal{I}^{(t+1)}$ of $\mathbb{P}^N_l$), is a constant Barsotti–Tate group over $\mathcal{C}_{(t)}$ isomorphic to $G_{\mathcal{C}_{(t)}}$, compatibly for different t's.

By [26], Ch. VIII, Cor. 5 of Thm. 13 applied to inclusions $\mathcal{I}(U) \subset \mathcal{O}_{\mathbb{P}^N_l}(U)$ for finitely many affine open subschemes U of $\mathbb{P}^N_l$, we have that for each $t \in \mathbb{N}^*$ there exists an integer $q \geq t$ such that we have $\mathcal{I}^{(q)} \subset \mathcal{I}^t$. This implies that

$$\varprojlim_{t\in\mathbb{N}^*} \mathcal{O}_{\mathbb{P}^N_l}/\mathcal{I}^t = \varprojlim_{t\in\mathbb{N}^*} \mathcal{O}_{\mathbb{P}^N_l}/\mathcal{I}^{(t)}.$$

From this and the previous paragraph we get that the Barsotti–Tate group $\mathcal{D}_{\widehat{\mathbb{P}^N_l}}$ over the formal completion $\widehat{\mathbb{P}^N_l}$ of $\mathbb{P}^N_l$ along $\mathcal{C}$ induced naturally by $\varinjlim \mathcal{D}_{n,\mathcal{U}_\infty}$ is as well constant isomorphic to $G_{\widehat{\mathbb{P}^N_l}}$. $\square$

To continue the proof of Theorem 2, in this paragraph we consider the particular case when $l' = l$ is an algebraically closed field. This assumption implies that $\mathcal{C}$ is G3 in $\mathbb{P}^N_l$, cf. Section 4. From this and Lemma 7 applied with $(\mathcal{U}, \mathcal{V}) = (\mathcal{U}_n, \mathcal{V}_n)$, where $\mathcal{V}_n$ is the coherent locally free $\mathcal{O}_{\mathcal{U}_n}$-algebra that defines $\mathcal{D}_n$, we get that we have a natural ring identification $H^0(\mathcal{U}_n, \mathcal{V}_n) = H^0(\widehat{\mathbb{P}^N_l}, \widehat{\mathcal{V}_n})$, where $\widehat{\mathcal{V}_n} = \chi^*(\mathcal{V}_n)$ is the formal completion of $\mathcal{V}_n$ along $\mathcal{C}$. From Proposition 6 we get that $H^0(\widehat{\mathbb{P}^N_l}, \widehat{\mathcal{V}_n})$ is canonically identified with the ring of regular functions on $G[p^n]$. We have similar identifications for $\mathcal{V}_n \otimes_{\mathcal{O}_{\mathcal{U}_n}} \mathcal{V}_n$, $\widehat{\mathcal{V}_n} \otimes_{\mathcal{O}_{\widehat{\mathbb{P}^N_l}}} \widehat{\mathcal{V}_n}$ and $G[p^n] \times_{\mathrm{Spec}(l)} G[p^n]$, with certain compatibilities. Therefore

[3] The fact that $H^0(\mathcal{C}, \mathcal{J}^{(t)}/\mathcal{J}^{(t+1)}) = 0$ for $t > 0$ can be proved without using $\mathcal{C}^{\mathrm{n}}$ as follows. If $\mathcal{Z}$ is an integral closed scheme of a smooth k-scheme $\mathcal{X}$, $\mathcal{J}$ the ideal sheaf defining $\mathcal{Z}$ in $\mathcal{X}$, $\mathcal{J}^{(t)}$ the t-th symbolic power, Diff the sheaf of differential operators on $\mathcal{X}$ in the sense of [9], Sect. 16.8, filtered by $\mathrm{Diff}^n = \mathrm{Diff}^n_{\mathcal{X}/\mathrm{Spec}(k)}$ ($n \in \mathbb{N}$), then $\mathrm{Diff}^n(\mathcal{J}^{(t)}) \subset \mathcal{J}^{(t-n)}$ ($\mathcal{J}^{(m)} := \mathcal{O}_{\mathcal{X}}$ for $m \leq 0$) and this induces $\mathrm{Diff}^t/\mathrm{Diff}^{t-1} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{J}^{(t)}/\mathcal{J}^{(t+1)} = \underline{Hom}(\mathrm{Sym}^t(\Omega^1_{\mathcal{X}/k}), \mathcal{O}_{\mathcal{X}}) \to \mathcal{O}_{\mathcal{Z}}$, hence $\mathcal{J}^{(t)}/\mathcal{J}^{(t+1)} \to \mathrm{Sym}^t(\Omega^1_{\mathcal{X}/k}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{Z}}$ which is injective if $\mathcal{Z}$ is generically smooth. For $\mathcal{X}$ a projective space and $\mathcal{Z}$ of positive dimension and generically smooth, this proves that $H^0(\mathcal{Z}, \mathcal{J}^{(t)}/\mathcal{J}^{(t+1)}) = 0$ for $t > 0$.

we have a canonical homomorphism $\vartheta_n : \mathcal{D}_n \to G[p^n]_{\mathcal{U}_n}$ of finite flat group schemes whose pullback to the formal completion $\widehat{\mathbb{P}_l^N}$ is an isomorphism, being the truncation of level n of an isomorphism as in Proposition 6. From this and the first part of Fact 1 (d) we get that ϑ_n is an isomorphism over an open subscheme $\mathcal{W}_n$ of $\mathcal{U}_n$ which contains $\mathcal{C}$. As the complement of $\mathcal{W}_n$ in $\mathbb{P}_l^N$ has codimension in $\mathbb{P}_l^N$ at least 2, from the second part of Fact 1 (d) we get that in fact ϑ_n is an isomorphism. The isomorphisms ϑ_n's are compatible in the sense that for all $n, m \in \mathbb{N}^*$ we have $\vartheta_{n+m}[p^n] = \vartheta_{n,\mathcal{U}_{n+m}}$. Therefore $\mathcal{D}_n$ extends uniquely (up to unique isomorphism) to a constant truncated Barsotti–Tate group $\mathcal{D}_n^+$ of level n over $\mathbb{P}_l^N$ isomorphic to $G[p^n]_{\mathbb{P}_l^N}$. Due to the mentioned compatibility of ϑ_n's and the uniqueness of the extension $\mathcal{D}_n^+$ of $\mathcal{D}_n$, we have a canonical identification $\mathcal{D}_{n+1}^+[p^n] = \mathcal{D}_n^+$. Thus $\varinjlim \mathcal{D}_n^+$ is a constant Barsotti–Tate group over $\mathbb{P}_l^N$ isomorphic to $G_{\mathbb{P}_l^N}$. Moreover, $G[p^n]$ is the affine group scheme over $\mathrm{Spec}(l)$ defined by the Hopf l-algebra of global functions on $\mathcal{D}_n^+$ and thus also on either $\mathcal{D}_n$ or $\mathcal{D}_{n,\mathcal{U}_{n+m}}$, cf. Fact 1 (c) and the fact that the complement of $\mathcal{U}_n$ in $\mathbb{P}_l^N$ has codimension in $\mathbb{P}_l^N$ at least 2.

We now consider the general case. By enlarging l', we can assume that it is an algebraically closed field. Let $n, m \in \mathbb{N}^*$. If $\omega_n : \mathcal{D}_n \to \mathcal{O}_{\mathcal{U}_n}$ is the structure morphism, then the $\mathcal{O}_{U_n}$-linear map $\mathcal{O}_{\mathcal{U}_n} \otimes_l H^0(\mathcal{U}_n, \omega_{n,*}(\mathcal{O}_{\mathcal{D}_n})) \to \omega_{n,*}(\mathcal{O}_{\mathcal{D}_n})$ is an isomorphism (as the base change to $\mathcal{U}_{n,l'}$ is) and thus the Hopf $\mathcal{O}_{U_n}$-algebra structure on $\omega_{n,*}(\mathcal{O}_{\mathcal{D}_n})$ defines a commutative and cocommutative Hopf l-algebra structure on $H^0(\mathcal{U}_n, \omega_{n,*}(\mathcal{O}_{\mathcal{D}_n}))$ and hence a finite group scheme G_n over l and a canonical isomorphism $\mathcal{D}_n \to G_{n,\mathcal{U}_n}$. This implies that G_n is a truncated Barsotti–Tate group of level n over $\mathrm{Spec}(l)$. The canonical closed embedding homomorphisms $\mathcal{D}_{n,\mathcal{U}_{n+m}} \to \mathcal{D}_{n+m}$ and the canonical epimorphisms $\mathcal{D}_{n+m} \to \mathcal{D}_{n,\mathcal{U}_{n+m}}$ induce naturally homomorphisms $G_n \to G_{n+m}$ and $G_{n+m} \to G_n$ which are closed embeddings and epimorphisms (respectively) as their extensions to l' are so, cf. previous paragraph. These last homomorphisms define a Barsotti–Tate group G over $\mathrm{Spec}(l)$ such that for all $n \in \mathbb{N}^*$ we have $G_n = G[p^n]$. We conclude that for each $n \in \mathbb{N}^*$, $\mathcal{D}_n$ extends uniquely (up to unique isomorphism) to a constant truncated Barsotti–Tate group $\mathcal{D}_n^+$ of level n over $\mathbb{P}_l^N$ isomorphic to $G[p^n]_{\mathbb{P}_l^N}$ and the identification $\mathcal{D}_{n+1}[p^n] = \mathcal{D}_{n,\mathcal{U}_{n+1}}$ extends to an identification $\mathcal{D}_{n+1}^+[p^n] = \mathcal{D}_n^+$ which is the pullback to $\mathbb{P}_l^N$ of the identification $(G[p^{n+1}])[p^n] = G[p^n]$ over $\mathrm{Spec}(l)$. Thus Theorem 2 holds. $\square$

6 Proof of Theorem 1 (b)

If $d = 2$, then R admits a faithfully flat regular local extension of the form $W(l)[[x_1, x_2]]/(h)$ with l a perfect field which contains k and such that $\bar{h} \notin (x_1^p, x_2^p, x_1^{p-1}x_2^{p-1})$, cf. Fact 2. Thus R is p-quasi-healthy by [23], Thm. 3.

In the rest of this section we will prove by induction on $d \geq 3$ that Theorem 1 (b) also holds for $d \geq 3$. The base of the induction (i.e., the case when $d = 3$) is checked in Subsection 6.1. The inductive step (i.e., the passage from $d - 1 \geq 3$ to $d \geq 4$) is checked in Subsection 6.2.

Let D_U be a Barsotti–Tate group over the punctured spectrum U of a regular local scheme $X = \mathrm{Spec}(R)$ of mixed characteristic $(0, p)$ and dimension $d \geq 3$. We have to show that if condition ($\natural$) holds for R, then D_U extends to a Barsotti–Tate group D over X. Based on Lemma 3 and Fact 2, by passing to a faithfully flat extension we can assume that R is also complete of the form $R = W(k)[[x_1, \ldots, x_d]]/(h)$, with k an algebraically closed field of positive characteristic p and with $h \in W(k)[[x_1, \ldots, x_d]]$ such that $\bar{h} \notin (x_1^p, \ldots, x_d^p) + (x_1, \ldots, x_d)^{2p-2}$. For $i \in \{1, \ldots, d\}$ let $y_i = x_i + (h) \in R$. We can also assume that the residue field k of R is uncountable and (cf. Section 3) that $y_1, \ldots, y_d$ is a regular system of parameters of R.

6.1 The base of the induction, i.e., the case $d = 3$

In this subsection we will assume that $d = 3$. Thus $R = W(k)[[x_1, x_2, x_3]]/(h)$ has y_1, y_2, y_3 as a regular system of parameters of R. Let Z be the blow up of X along its closed point; it is a regular scheme of dimension 3 which is projective over X and which is the union of the open subscheme U and of a closed subscheme $\mathbb{P}^2_k$. From Proposition 2 we get that the restriction of D_U to the generic point of Z (i.e., to $\mathrm{Spec}(\mathrm{Frac}(R))$) extends to a Barsotti–Tate group $D_{\mathrm{Spec}(O)}$ over the spectrum of the discrete valuation ring O which is the local ring in Z of the generic point of $\mathbb{P}^2_k$. From this and Fact 1 (b) we get that for each $n \in \mathbb{N}^*$ there exists a finite set $\mathcal{S}_n$ of closed points of $\mathbb{P}^2_k$ such that the truncated Barsotti–Tate group $D_U[p^n]$ of level n over U extends uniquely (up to unique isomorphism) to a finite flat group scheme E_n over $Z \setminus \mathcal{S}_n$ whose pullback to $\mathrm{Spec}(O)$ is $D_{\mathrm{Spec}(O)}[p^n]$.

Recall the standard fact that if $\triangle$ is a finite locally free commutative group scheme annihilated by p^n over a scheme Σ, then the set of points z of Σ such that the fiber of $\triangle$ at z is a truncated Barsotti–Tate group of level n over the residue field of z is open and $\triangle$ is a truncated Barsotti–Tate group of level n over the corresponding open subscheme. Thus, let $\mathcal{K}_n$ be the smallest reduced closed subscheme of $\mathbb{P}^2_k$ which contains $\mathcal{S}_n$ and such that the restriction of E_n to $Z \setminus \mathcal{K}_n$ is a truncated Barsotti–Tate group of level n; the dimension of $\mathcal{K}_n$ is at most 1.

We can assume we have a chain of inclusions $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_n \subset \cdots$. Thus we have a second chain of inclusions $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \subset \mathcal{K}_n \subset \cdots$ such that for each $n \in \mathbb{N}^*$ the truncated Barsotti–Tate groups $E_{n+1}[p^n]_{Z\setminus\mathcal{K}_{n+1}}$ and $E_{n,Z\setminus\mathcal{K}_{n+1}}$ of level n coincide over $Z \setminus \mathcal{K}_{n+1}$. The set $\mathcal{S}_\infty = \cup_{n\geq 1}\mathcal{S}_n$ is countable and the ind-constructible set $\mathcal{K}_\infty = \cup_{n\geq 1}\mathcal{K}_n$ has a countable number of maximal points, and is the union of their closures.[4] Moreover, D_U extends to a Barsotti–Tate group $D_{Z\setminus\mathcal{K}_\infty}$ over the stable under generization, pro-constructible subset $Z \setminus \mathcal{K}_\infty = \cap_{n=1}^{\infty}(Z \setminus \mathcal{K}_n)$ of Z (see footnote 2).

Claim 1 *For each $n \in \mathbb{N}^*$, $\mathcal{K}_n$ is a finite set and therefore (by enlarging $\mathcal{S}_n$) we can assume that $\mathcal{S}_n = \mathcal{K}_n$.*

We begin the proof of Claim 1 by introducing notation pertaining to h.

6.1.1 On $\bar{h} \in k[[x_1, x_2, x_3]]$

For each integer $i \in \{1, \ldots, 2p-3\}$ let $\mathbb{J}_i^3$ be the subset of $\{0, \ldots, p-1\}^3$ formed by those triples whose sum is i. We write

$$\bar{h}(x_1, x_2, x_3) = \bar{h}_0(x_1, x_2, x_3) + \sum_{i=1}^{2p-3} \bar{h}_i(x_1, x_2, x_3),$$

where $\bar{h}_0$ belongs to the ideal $(x_1^p, x_2^p, x_3^p) + (x_1, x_2, x_3)^{2p-2}$ of $k[[x_1, x_2, x_3]]$ and where each

$$\bar{h}_i(x_1, x_2, x_3) = \sum_{(i_1,i_2,i_3)\in\mathbb{J}_i^3} \delta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} \in k[x_1, x_2, x_3]$$

is either 0 or a homogeneous polynomial of degree i. As $\bar{h}$ does not belong to the ideal $(x_1^p, x_2^p, x_3^p) + (x_1, x_2, x_3)^{2p-2}$ of $k[[x_1, x_2, x_3]]$, there exists a smallest element $c \in \{1, \ldots, 2p-3\}$ such that we have $\bar{h}_c \neq 0$.

[4] Recall that a point of a topological space is said to be a maximal point if its closure is not strictly contained in the closure of another point. For sober spaces the maximal points are the generic points of the irreducible components.

6.1.2 Good regular closed subschemes of X of dimension 2

For each triple $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in W(k)^3$ such that its reduction $\bar{\zeta} = (\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3)$ modulo p is a non-zero element of k^3, we consider the regular closed subscheme $\mathrm{Spec}(S_\zeta)$ of X, where $S_\zeta = R/(\zeta_1 y_1 + \zeta_2 y_2 + \zeta_3 y_3)$. Let Z_ζ be the closed subscheme of Z which is the blow up of $\mathrm{Spec}(S_\zeta)$ along its closed point; it is the union of its open subscheme $U \cap \mathrm{Spec}(S_\zeta)$ and of a $\mathbb{P}^1_{k,\zeta}$ curve inside $\mathbb{P}^2_k$. It is easy to see that, with respect to the usual projective coordinates $[w_0, w_1, w_2]$ of $\mathbb{P}^2_k$, the curve $\mathbb{P}^1_{k,\zeta}$ of $\mathbb{P}^2_k$ is defined by the equation $\bar{\zeta}_1 w_0 + \bar{\zeta}_2 w_1 + \bar{\zeta}_3 w_2 = 0$. If $P = [\gamma_0, \gamma_1, \gamma_2] \in \mathcal{S}_\infty \subset \mathbb{P}^2_k$ with $\gamma_0, \gamma_1, \gamma_2 \in k$ not all zero, then the condition $P \notin \mathbb{P}^1_{k,\zeta}$ gets translated into the inequality $\bar{\zeta}_1\gamma_0 + \bar{\zeta}_2\gamma_1 + \bar{\zeta}_3\gamma_2 \neq 0$. Thus there exists a countable union $\mathcal{L}_\infty$ of lines in $\mathbb{P}^2_k$ such that we have $\mathbb{P}^1_{k,\zeta} \cap \mathcal{S}_\infty = \emptyset$ if and only if $[\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3] \notin \mathcal{L}_\infty(k)$.

Let $\mathcal{P}_\infty$ be the countable subset of $\mathbb{P}^2_k(k)$ such that $\mathbb{P}^1_{k,\zeta}$ is not contained in $\mathcal{K}_\infty$ if and only if $[\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3] \notin \mathcal{P}_\infty$.

From Lemma 6 applied with $(l, d) = (k, 3)$, we get that there exists an open dense subscheme $\mathbb{O}$ of $\mathbb{A}^2_k$ such that for each pair $(v_1, v_2) \in \mathbb{O}(k) \subset k^2$ the polynomial

$$\bar{h}_{c,\zeta}(x_1, x_2) = \bar{h}_c(x_1, x_2, v_1x_1 + v_2x_2)$$

does not belong to the ideal $(x_1^p, x_2^p, x_1^{p-1}x_2^{p-1})$ of $k[[x_1, x_2]]$. If $(\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3) = (-v_1, -v_2, 1)$ with $(v_1, v_2) \in \mathbb{O}(k)$, then the regular ring $S_\zeta = R/(\zeta_1 y_1 + \zeta_2 y_2 + \zeta_3 y_3)$ is isomorphic to $W(k)[[x_1, x_2]]/(h_\zeta)$, where

$$h_\zeta(x_1, x_2) = h(x_1, x_2, -\zeta_3^{-1}\zeta_1 x_1 - \zeta_3^{-1}\zeta_2 x_2) \in W(k)[[x_1, x_2]]$$

is such that its reduction $\bar{h}_\zeta$ modulo p has $\bar{h}_{c,\zeta}$ as its homogeneous component of degree c and therefore it does not belong to the ideal $(x_1^p, x_2^p, x_1^{p-1}x_2^{p-1})$ of $k[[x_1, x_2]]$; thus in such a case S_ζ is of mixed characteristic $(0, p)$ and is p-quasi-healthy (by [23], Thm. 3).

We identify $\mathbb{A}^2_k$ with an open subscheme of $\mathbb{P}^2_k$ via the embedding $(v_1, v_2) \to [-v_1, -v_2, 1]$. As k is uncountable, the set of all closed points of the open dense subscheme $\mathbb{O}$ of $\mathbb{A}^2_k \subset \mathbb{P}^2_k$ cannot be contained in $\mathcal{L}_\infty(k) \cup \mathcal{P}_\infty$, i.e., there exist pairs $(v_1, v_2) \in \mathbb{O}(k)$ such that $[-v_1, -v_2, 1] \notin \mathcal{L}_\infty(k) \cup \mathcal{P}_\infty$. If (v_1, v_2) is such a pair, then by choosing $(\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3) = (-v_1, -v_2, 1)$ we get that the regular closed subscheme $\mathrm{Spec}(S_\zeta)$ of X is good in the following sense:

($\sharp$) *the regular ring S_ζ is p-quasi-healthy and moreover the line $\mathbb{P}^1_{k,\zeta}$ neither intersects the countable subset $\mathcal{S}_\infty$ of $\mathbb{P}^2_k$ nor is contained in $\mathcal{K}_\infty$.*

6.1.3 Proof of the Claim 1

As S_ζ is p-quasi-healthy (cf. ($\sharp$)), the restriction of D_U to $U \cap \mathrm{Spec}(S_\zeta)$ extends uniquely (up to unique isomorphism) to a Barsotti–Tate group $D_{\mathrm{Spec}(S_\zeta)}$ over $\mathrm{Spec}(S_\zeta)$. As $\mathbb{P}^1_{k,\zeta}$ does not intersect $\mathcal{S}_n$ (cf. ($\sharp$)), for $n \in \mathbb{N}^*$ we can speak about the pullback $E_{n,\zeta}$ of E_n to Z_ζ. Let E_{n,O_ζ} be the restriction of $E_{n,\zeta}$ to the spectrum of the local ring O_ζ of Z_ζ which is a discrete valuation ring that dominates S_ζ. As $\mathbb{P}^1_{k,\zeta}$ is not contained in $\mathcal{K}_\infty$, the inductive system E_{n,O_ζ} is a Barsotti–Tate group over $\mathrm{Spec}(O_\zeta)$ which, based on Tate's extension theorem, is the pullback of $D_{\mathrm{Spec}(S_\zeta)}$ to $\mathrm{Spec}(O_\zeta)$. This implies that for each $n \in \mathbb{N}^*$, $E_{n,\zeta}$ is the pullback to Z_ζ of $D_{\mathrm{Spec}(S_\zeta)}[p^n]$. Thus the inductive system $E_{n,\zeta}$ is a Barsotti–Tate group over Z_ζ. This implies that Z_ζ and therefore also $\mathbb{P}^1_{k,\zeta}$ does not intersect $\mathcal{K}_\infty$. As two irreducible projective curves in $\mathbb{P}^2_k$ always intersect and as $\mathbb{P}^1_{k,\zeta} \cap \mathcal{K}_n = \emptyset$, we conclude that each $\mathcal{K}_n$ has dimension 0 and therefore (by enlarging $\mathcal{S}_n$) we can assume that for each $n \in \mathbb{N}^*$ we have $\mathcal{S}_n = \mathcal{K}_n$. Thus also $\mathcal{K}_\infty = \mathcal{S}_\infty$ and each E_n is a Barsotti–Tate group of level n over $Z \setminus \mathcal{S}_n$. This ends the proof of the Claim 1. $\square$

6.1.4 Applying Theorem 2

Let G be the fiber of $D_{\mathrm{Spec}(S_\zeta)}$ over the closed point $\mathrm{Spec}(k)$ of $\mathrm{Spec}(S_\zeta)$. For $n \in \mathbb{N}^*$, as $E_{n,\zeta}$ is the pullback of $D_{\mathrm{Spec}(S_\zeta)}[p^n]$ to Z_ζ, the pullback of E_n to $\mathbb{P}^1_{k,\zeta}$ is canonically identified with $G[p^n]_{\mathbb{P}^1_{k,\zeta}}$. From this and Theorem 2 applied with $(l, N, \mathcal{C}, \mathcal{U}_n, \mathcal{D}_n) = (k, 2, \mathbb{P}^1_{k,\zeta}, \mathbb{P}^2_k \setminus \mathcal{K}_n, E_{n,\mathbb{P}^2_k \setminus \mathcal{K}_n})$ we get that the restriction $D_{\mathbb{P}^2_k \setminus \mathcal{S}_\infty}$ of $D_{Z \setminus \mathcal{S}_\infty}$ to $\mathbb{P}^2_k \setminus \mathcal{S}_\infty$ is isomorphic to $G_{\mathbb{P}^2_k \setminus \mathcal{S}_\infty}$ and thus it extends to a constant Barsotti–Tate group $D_{\mathbb{P}^2_k}$ over $\mathbb{P}^2_k$ isomorphic to $G_{\mathbb{P}^2_k}$.

6.1.5 Liftings to infinitesimal neighborhoods of $\mathbb{P}^2_k$ in Z

Let $\mathfrak{m} = (y_1, y_2, y_3)$ be the maximal ideal of R. As $\mathfrak{m}\mathcal{O}_Z$ is the ideal sheaf of $\mathcal{O}_Z$ that defines the closed subscheme $\mathbb{P}^2_k$ of Z, for each $t \in \mathbb{N}$ for the invertible $\mathcal{O}_{\mathbb{P}^2_k}$-module $\mathfrak{m}^t\mathcal{O}_Z/\mathfrak{m}^{t+1}\mathcal{O}_Z$ we have

$$H^1(Z, \mathfrak{m}^t\mathcal{O}_Z/\mathfrak{m}^{t+1}\mathcal{O}_Z) = H^1(\mathbb{P}^2_k, \mathfrak{m}^t\mathcal{O}_Z/\mathfrak{m}^{t+1}\mathcal{O}_Z) = 0.$$

We consider a coherent locally free module $\mathcal{V}_{t+1}$ of rank r over the $t+1$-th infinitesimal neighborhood $\mathbb{P}^2_{k,t+1}$ of $\mathbb{P}^2_k$ in Z (i.e., over the reduction modulo $\mathfrak{m}^{t+2}\mathcal{O}_Z$ of Z) such that the coherent locally free module $\mathcal{V}_t = \mathcal{V}_{t+1}/\mathfrak{m}^{t+1}\mathcal{V}_{t+1}$ over the t-th infinitesimal neighborhood $\mathbb{P}^2_{k,t}$ of $\mathbb{P}^2_k$ in Z is trivial and thus isomorphic to $\mathcal{O}^r_{\mathbb{P}^2_{k,t}}$. As we have $\mathfrak{m}^{t+1}\mathcal{V}_{t+1} = (\mathfrak{m}^{t+1}\mathcal{O}_Z/\mathfrak{m}^{t+2}\mathcal{O}_Z) \otimes_{\mathcal{O}_Z} \mathcal{V}_{t+1} = (\mathfrak{m}^{t+1}\mathcal{O}_Z/\mathfrak{m}^{t+2}\mathcal{O}_Z) \otimes_{\mathcal{O}_Z} \mathcal{V}_t = (\mathfrak{m}^{t+1}\mathcal{O}_Z/\mathfrak{m}^{t+2}\mathcal{O}_Z)^r$, the short exact sequence

$$0 \to \mathfrak{m}^{t+1}\mathcal{V}_{t+1} \to \mathcal{V}_{t+1} \to \mathcal{V}_t \to 0$$

of coherent $\mathcal{O}_Z$-modules induces an exact complex

$$H^0(Z, \mathcal{V}_{t+1}) \to H^0(Z, \mathcal{V}_t) \to H^1(Z, \mathfrak{m}^{t+1}\mathcal{O}_Z/\mathfrak{m}^{t+2}\mathcal{O}_Z)^r = 0.$$

Therefore the reduction homomorphism $H^0(Z, \mathcal{V}_{t+1}) \to H^0(Z, \mathcal{V}_t)$ is surjective and this implies that $\mathcal{V}_{t+1}$ is as well a free $\mathcal{O}_{\mathbb{P}^2_{k,t+1}}$-module of rank r.

By induction on $t \in \mathbb{N}$ we check that the constant Barsotti–Tate group $D_{\mathbb{P}^2_k}$ over $\mathbb{P}^2_k$ lifts uniquely (up to unique isomorphism) to a Barsotti–Tate group $D_{\mathbb{P}^2_{k,t}}$ over $\mathbb{P}^2_{k,t}$ in such a way that its restriction to the reduction modulo $\mathfrak{m}^{t+1}$ of $Z \setminus \mathcal{S}_\infty$ is induced naturally by $D_{Z\setminus\mathcal{S}_\infty}$ and moreover the coherent $\mathcal{O}_{\mathbb{P}^2_{k,t}}$-module associated naturally to the structure sheaf of the truncation $D_{\mathbb{P}^2_{k,t}}[p^n]$ of level n is free for all $n \in \mathbb{N}^*$ (in particular, each $D_{\mathbb{P}^2_{k,t}}[p^n]$ extends the reduction of E_n modulo $\mathfrak{m}^{t+1}$).

The case $t = 0$ was accomplished in Subsubsection 6.1.4. The passage from t to $t+1$ goes as follows. From the Grothendieck–Messing deformation theory we get that the lifts of the Barsotti–Tate group $D_{\mathbb{P}^2_{k,t}}$ to a Barsotti–Tate group over $\mathbb{P}^2_{k,t+1}$ are parametrized by the global sections of a torsor under the group of global sections of a coherent locally free $\mathcal{O}_{\mathbb{P}^2_k}$-module $\mathcal{F}_t$. Similarly, for each integer $\tilde{t} \geq 1$ the lifts of the truncated Barsotti–Tate group of level $t+\tilde{t}$ which is the reduction modulo $\mathfrak{m}^{t+1}$ of $E_{t+\tilde{t}}$ to truncated Barsotti–Tate groups of level $t+\tilde{t}$ over the reduction modulo $\mathfrak{m}^{t+2}$ of $Z \setminus \mathcal{S}_{t+\tilde{t}}$, are parametrized by a torsor under the group $H^0(\mathbb{P}^2_k \setminus \mathcal{S}_{t+\tilde{t}}, \mathcal{F}_t)$ in a way compatible in $\tilde{t}$ (cf. [14], Thm. 4.4 c) and Cor. 4.7). As $H^0(\mathbb{P}^2_k \setminus \mathcal{S}_{t+\tilde{t}}, \mathcal{F}_t) = H^0(\mathbb{P}^2_k \setminus \mathcal{S}_\infty, \mathcal{F}_t)$ does not depend on $\tilde{t}$, we get that the lifts of the Barsotti–Tate group which is the restriction of $D_{Z\setminus\mathcal{S}_\infty}$ to the reduction modulo $\mathfrak{m}^{t+1}$ of $Z\setminus\mathcal{S}_\infty$ to Barsotti–Tate groups over the reduction modulo $\mathfrak{m}^{t+2}$ of $Z\setminus\mathcal{S}_\infty$, are parametrized by a torsor under the group $H^0(\mathbb{P}^2_k \setminus \mathcal{S}_\infty, \mathcal{F}_t)$. As we have $H^0(\mathbb{P}^2_k, \mathcal{F}_t) = H^0(\mathbb{P}^2_k \setminus \mathcal{S}_\infty, \mathcal{F}_t)$, we conclude that there exists a unique (up to unique isomorphism) Barsotti–Tate group over $\mathbb{P}^2_{k,t+1}$ which lifts $D_{\mathbb{P}^2_{k,t}}$ in such a way that its restriction to the reduction modulo $\mathfrak{m}^{t+2}$ of $Z \setminus \mathcal{S}_\infty$ is induced naturally by $D_{Z\setminus\mathcal{S}_\infty}$. Based on the previous paragraph, we get that the coherent $\mathcal{O}_{\mathbb{P}^2_{k,t+1}}$-module associated naturally to the structure sheaf of the truncation $D_{\mathbb{P}^2_{k,t+1}}[p^n]$ of level n is free for all $n \in \mathbb{N}^*$. This ends the induction on $t \in \mathbb{N}$.

6.1.6 End of the proof in the case $d = 3$

For $z \in \mathcal{S}_n$, let $\tau \in \mathcal{O}_{Z,z}$ be such that we have $\mathrm{Spec}(\mathcal{O}_{Z,z}/(\tau)) = \mathrm{Spec}(\mathcal{O}_{Z,z}) \times_Z \mathbb{P}^2_k$. As $\mathcal{O}_{Z,z}$ is a regular local ring of dimension $d = 3$, for $t \in \mathbb{N}$ we have $\mathrm{depth}_{\mathcal{O}_{Z,z}}(\mathcal{O}_{Z,z}/\tau^{t+1}\mathcal{O}_{Z,z}) = d - 1 \geq 2$ (cf. Auslander–Buchsbaum formula) and thus for each coherent locally free $\mathcal{O}_{\mathbb{P}^2_{k,t}}$-module $\mathfrak{F}_t$ we have a canonical identification $H^0(\mathbb{P}^2_{k,t} \setminus \mathcal{S}_n, \mathfrak{F}_t) = H^0(\mathbb{P}^2_{k,t}, \mathfrak{F}_t)$. Thus the direct image via the open embedding $\mathbb{P}^2_{k,t} \setminus \mathcal{S}_n \to \mathbb{P}^2_{k,t}$ of the $\mathcal{O}_{\mathbb{P}^2_{k,t} \setminus \mathcal{S}_n}$-module associated to the reduction of E_n modulo $\mathfrak{m}^{t+1}$ is the $\mathcal{O}_{\mathbb{P}^2_{k,t}}$-module associated to $D_{\mathbb{P}^2_{k,t}}[p^n]$. Based on this, from Lemma 4 applied to the local rings $\mathcal{O}_{Z,z}$ we get that for each $n \in \mathbb{N}^*$ the locally free $\mathcal{O}_{Z \setminus \mathcal{S}_n}$-module associated to E_n extends (uniquely up to unique isomorphism) to a locally free $\mathcal{O}_Z$-module whose reduction modulo each $\mathfrak{m}^{t+1}$ is the $\mathcal{O}_{\mathbb{P}^2_{k,t}}$-module associated to $D_{\mathbb{P}^2_{k,t}}[p^n]$. This implies that each E_n extends to a Barsotti–Tate group E_n^+ of level n over Z which lifts each $D_{\mathbb{P}^2_{k,t}}[p^n]$. For all $n, m \in \mathbb{N}^*$, the closed embedding homomorphisms $E_{n, Z \setminus \mathcal{S}_{n+m}} \to E_{n+m}$ extend to homomorphisms $E_n^+ \to E_{n+m}^+$ over Z which are closed embeddings identifying E_n^+ with $E_{n+m}^+[p^n]$, as their restrictions to $\mathbb{P}^2_{k,0} = \mathbb{P}^2_k$ are so. Thus the inductive system E_n^+ is a Barsotti–Tate group E^+ over Z.

The $D_{\mathbb{P}^2_{k,t}}$'s define a Barsotti–Tate group $D_{\widehat{Z}}$ over the formal scheme $\widehat{Z}$ of the completion of Z along $\mathbb{P}^2_k$ and moreover the coherent $\mathcal{O}_{\widehat{Z}}$-module $\mathcal{O}_{D_{\widehat{Z}}[p^n]}$ is free for all $n \in \mathbb{N}^*$. From [6], Thm. 5.1.4 we get that $D_{\widehat{Z}}$ is the formal completion along $\mathbb{P}^2_k$ of a uniquely determined (up to unique isomorphism) Barsotti–Tate group D_Z over Z and that for all $n \in \mathbb{N}^*$ the coherent $\mathcal{O}_Z$-module $\mathcal{O}_{D_Z[p^n]}$ is free. Due to the uniqueness part we have $E^+ = D_Z$.

As the morphism $Z \to X$ is birational and projective and as R is normal, the ring of global functions on Z is R. From this and the fact that for all $n \in \mathbb{N}^*$ the coherent $\mathcal{O}_Z$-module $\mathcal{O}_{E_n^+} = \mathcal{O}_{D_Z[p^n]}$ is free, we get that $E^+ = D_Z$ is the pullback of a uniquely determined (up to unique isomorphism) Barsotti–Tate group D over X which extends D_U and whose truncations $D[p^n]$ are defined by identities $H^0(X, D[p^n]) = H^0(Z, E^+[p^n])$ (to be compared with the Hopf algebra argument involving ω_n's at the end of Section 5).

6.2 The inductive step

In this subsection we will assume that $d \geq 4$ and that Theorem 1 (b) holds for regular rings of dimension at most $d - 1$. As in Subsection 6.1, for each

d-tuple $\zeta = (\zeta_1, \ldots, \zeta_d) \in W(k)^d$ such that its reduction modulo p is not zero, we consider the regular ring $S_\zeta = R/(\sum_{i=1}^d \zeta_i y_i)$. As k is infinite, from Proposition 3 we get that we can choose ζ such that S_ζ is isomorphic to $W(k)[[x_1, x_2, \ldots, x_{d-1}]]/(h_\zeta)$, where $h_\zeta \in W(k)[[x_1, x_2, \ldots, x_{d-1}]]$ is such that its reduction modulo p does not belong to the ideal $(x_1^p, x_2^p, \ldots, x_{d-1}^p) + (x_1, x_2, \ldots, x_{d-1})^{2p-2}$ of $k[[x_1, x_2, \ldots, x_{d-1}]]$. We know that S_ζ is p-quasi-healthy, by the inductive assumption that Theorem 1 (b) holds for regular rings of dimension at most $d-1$. Thus D_U modulo $(\sum_{i=1}^d \zeta_i y_i)$ extends to a Barsotti–Tate group over $\mathrm{Spec}(S_\zeta)$. From this and Lemma 5 applied with $(R, y, R/(y), D_U) = (R, \sum_{i=1}^d \zeta_i y_i, S_\zeta, D_U)$ we get that D_U extends to a Barsotti–Tate group over X. Thus R is p-quasi-healthy. This ends the induction and the proof of Theorem 1 (b). $\square$

7 Proof of Theorem 1 (a) and Corollary 1

In the situation of Theorem 1, from Theorem 1 (b) and Proposition 5 we get that each local ring of X of mixed characteristic $(0, p)$ and of dimension at least 2 is p-quasi-healthy. Thus Theorem 1 (a) is a particular case of the following general lemma whose proof relies on the purity of the branch locus.

Lemma 8 *Let Y be an integral scheme flat over $\mathbb{Z}$ such that $Y[\frac{1}{p}]$ is regular and each local ring of Y is normal noetherian. We assume that the following two conditions hold:*

(i) *every local ring of Y of mixed characteristic $(0, p)$ and dimension at least 2 is p-quasi-healthy;*

(ii) *there exists an affine open cover $(W_\lambda)_{\lambda \in \Lambda}$ of Y such that for each $\lambda \in \Lambda$ there exists $N_\lambda \in \mathbb{N}^*$ with the property that for every maximal point $\eta_0 \in Y_{\mathbb{F}_p} \cap W_\lambda$, the absolute ramification index of $\mathcal{O}_{Y,\eta_0}$ is at most N_λ.*

If D_η is a Barsotti–Tate group over the generic point η of Y which extends to every one dimensional local ring of Y, then D_η extends to Y (uniquely up to unique isomorphism).

Proof: As the local rings of Y are normal noetherian, the same arguments as in the proof of [20], Thm. 4 give that the functor

$$(\text{BT groups over } Y) \longrightarrow (\text{BT groups over } \eta)$$

is fully faithful. Thus it is enough to prove the assertion locally and we can assume that $Y = \mathrm{Spec}(A) = W_\lambda$.

In this paragraph we check that the Barsotti–Tate group D_η extends to a Barsotti–Tate group $D_{Y[\frac{1}{p}]}$ over $Y[\frac{1}{p}] = \mathrm{Spec}(A[\frac{1}{p}])$. For $n \in \mathbb{N}^*$, we consider the normalization $D_{Y[\frac{1}{p}],n}$ of $Y[\frac{1}{p}]$ in $D_\eta[p^n]$. As the finite étale group scheme $D_\eta[p^n]$ extends to a finite étale group scheme over the spectrum of each local ring of $Y[\frac{1}{p}]$ of dimension 1 (i.e., of each local ring of $Y[\frac{1}{p}]$ which is a discrete valuation ring), the morphism $\vartheta_n : D_{Y[\frac{1}{p}],n} \to Y[\frac{1}{p}]$ is finite and étale over each local ring of $Y[\frac{1}{p}]$ of dimension 1. From this and the purity of the branch locus (see [7], Exp. X, Thm. 3.4 (i)]) we get that the morphism ϑ_n is finite and étale over each local ring of $Y[\frac{1}{p}]$. It is easy to see that this implies that ϑ_n defines a finite étale scheme over $Y[\frac{1}{p}]$ which extends $D_\eta[p^n]$ and thus has a unique group scheme structure which extends the group scheme structure on $D_\eta[p^n]$, and this defines a Barsotti–Tate group $D_{Y[\frac{1}{p}],n}$ of level n over $Y[\frac{1}{p}]$. For $n, m \in \mathbb{N}^*$, the inclusions $D_\eta[p^n] \to D_\eta[p^{n+m}]$ extend to closed embedding homomorphisms $D_{Y[\frac{1}{p}],n} \to D_{Y[\frac{1}{p}],n+m}$ and the inductive system $D_{Y[\frac{1}{p}],n}$ is the unique (up to unique isomorphism) Barsotti–Tate group $D_{Y[\frac{1}{p}]}$ over $Y[\frac{1}{p}]$ which extends D_η.

To check that $D_{Y[\frac{1}{p}]}$ extends to Y we consider two cases as follows.

Case 1: A/pA is noetherian. Let $O_1, \ldots, O_t$ be all local rings of R which are discrete valuation rings of mixed characteristic $(0, p)$; they correspond to the maximal points $\eta_1, \ldots, \eta_t$ (respectively) of $Y_{\mathbb{F}_p}$. For $i \in \{1, \ldots, t\}$ let D_{O_i} be the unique (up to unique isomorphism) Barsotti–Tate group over $\mathrm{Spec}(O_i)$ which extends D_η. For each $n \in \mathbb{N}^*$ we consider the largest open subscheme U_n of Y which contains $Y[\frac{1}{p}] \cup \{\eta_1, \ldots, \eta_t\}$ and over which there exists a finite flat (locally free) commutative group scheme $D_{U_n,n}$ which extends compatibly $D_{Y[\frac{1}{p}]}[p^n]$ and $D_{O_i}[p^n]$ for all $i \in \{1, \ldots, t\}$.[5] Let W_n be the largest open subscheme of U_n with the property that the restriction of $D_{U_n,n}$ to W_n is a truncated Barsotti–Tate group of level n over W_n. We have a chain of inclusions $W_1 \supset W_2 \supset W_3 \supset \cdots \supset W_m \supset \cdots$. Thus for $Y_i = Y \setminus W_i$, we have a chain of inclusions $Y_1 \subset Y_2 \subset Y_3 \subset \cdots \subset Y_m \subset \cdots$ between reduced closed subschemes of $Y_{\mathbb{F}_p}$. The codimension of each Y_m is at least 2. Note that

[5] In the general non-noetherian case "finite flat" does not imply locally free, but in our case this follows from [10], Prop. 2.4.19 applied to classical extensions of the type $A \subset A[\frac{1}{p}]$.

the formation of the open subschemes U_n and W_n commutes with passage to spectra of local rings of Y.

We will show that the assumption that there exists $q \in \mathbb{N}^*$ such that Y_q is non-empty leads to a contradiction. We can choose $q \in \mathbb{N}^*$ such that for all $m \in \mathbb{N}^*$ we have $\mathrm{codim}(Y_q) = \mathrm{codim}(Y_{q+m}) = c \geq 2$.

Let z be a generic point of an irreducible component of Y_q of codimension c and let $\tilde{R} = \mathcal{O}_{Y,z}$. We have $\dim(\tilde{R}) = c \geq 2$ and therefore $\tilde{R}$ is p-quasi-healthy. Let $\tilde{X} = \mathrm{Spec}(\tilde{R})$ and let $\tilde{U}$ be the punctured spectrum of $\tilde{R}$. From the very definitions and the choice of z we get that $\tilde{U} \cap Y_n = \emptyset$ for all $n \in \mathbb{N}^*$. Thus there exists a unique (up to unique isomorphism) Barsotti–Tate group $D_{\tilde{U}}$ over $\tilde{U}$ which extends the restriction of $D_{Y[\frac{1}{p}]}$ to $\tilde{X}[\frac{1}{p}] = \mathrm{Spec}(\tilde{R}[\frac{1}{p}])$. Therefore $D_{\tilde{U}}$ extends uniquely (up to unique isomorphism) to a Barsotti–Tate group over $\tilde{X}$ and thus for each $n \in \mathbb{N}^*$ we have $z \notin Y_n$. This contradicts the fact that $z \in Y_q$. Thus our assumption leads to a contradiction. Therefore for all $n \in \mathbb{N}^*$ we have $U_n = W_n = Y$. This implies that D_η extends to a Barsotti–Tate group over Y.

Case 2: general case. Let $n \in \mathbb{N}^*$. Let $s \in \mathbb{N}$ be such that it depends only on N_λ and [1], Thm. E or [24], Cor. 3 applies to all discrete valuation rings which are local rings of Y at maximal points of $Y_{\mathbb{F}_p}$ (see [24], Exs. 2 and 4). Let z be a point of $Y_{\mathbb{F}_p}$. From Case 1 applied to $\mathcal{O}_{Y,z}$ we get that the restriction of $D_{Y[\frac{1}{p}]}$ to $\mathrm{Spec}(\mathcal{O}_{Y,z}[\frac{1}{p}])$ extends to a Barsotti–Tate group D_z over $\mathrm{Spec}(\mathcal{O}_{Y,z})$. Thus $D_z[p^{n+s}]$ extends to a Barsotti–Tate group $E_{n+s,z}$ of level $n+s$ over an affine open subscheme $W_z = \mathrm{Spec}(A_z)$ of Y with $z \in W_z$.

From [1], Thm. E or [24], Cor. 3 and the property (ii) we get that for each maximal point $\eta_0 \in Y_{\mathbb{F}_p}$ that belongs to W_z, the restriction of $E_{n+s,z}[p^n]$ to $\mathcal{O}_{Y,\eta_0}$ is (canonically identified with) the truncation of level n of the Barsotti–Tate group D_{η_0} over $\mathcal{O}_{Y,\eta_0}$ which extends D_η. Based on Fact 1 (c) applied to the local rings of Y we get that $D_{Y[\frac{1}{p}]}[p^n]$ and the $E_{n+s,z}[p^n]$'s with $z \in Y_{\mathbb{F}_p}$ glue together to define a Barsotti–Tate group D_n^+ of level n over Y which extends $D_\eta[p^n]$ and whose restriction to each $\mathcal{O}_{Y,\eta_0}$ is $D_{\eta_0}[p^n]$.

If $m \in \mathbb{N}^*$, from Fact 1 (c) applied to local rings of Y we get that the closed embedding homomorphism $D_{Y[\frac{1}{p}]}[p^n] \to D_{Y[\frac{1}{p}]}[p^{n+m}]$ extends to a homomorphism $D_n^+ \to D_{n+m}^+$. The fact that the inductive system D_n^+ is a Barsotti–Tate group over Y which extends $D_{Y[\frac{1}{p}]}$ follows from the fact that its restriction to each $\mathrm{Spec}(\mathcal{O}_{Y,z})$ is canonically identified with the inductive system $D_z[p^n]$. This ends the proofs of the lemma and of Theorem 1. $\square$

7.1 Proof of Corollary 1

In the situation of Corollary 1 (a), the condition (i) of Lemma 8 holds. The condition (ii) of Lemma 8 also holds: for each affine open cover $(W_\lambda)_{\lambda\in\Lambda}$ of Y we can take all N_λ's to be $p-1$, cf. property (ii) of Section 3. Thus Lemma 8 implies that Corollary 1 (a) holds. In view of the uniqueness (up to unique isomorphism) of the extension from the generic point of Y, Corollary 1 (b) is a particular case of Corollary 1 (a). $\square$

8 Complements to Corollary 1 and Lemma 8

For a topological space $\mathcal{Y}$, let $\mathcal{Y}^{\min}$ be the subspace of maximal points of $\mathcal{Y}$. We recall that if $\mathcal{Y}$ is a locally spectral space (e.g., the underlying topological space of a scheme), then $\mathcal{Y}^{\min}$ is retrocompact in $\mathcal{Y}$ if and only if it is pro-constructible, and for $\mathcal{Y}$ spectral if and only if it is quasi-compact, see [19], Cors. 2.6 (i) and 2.7.

For the sake of completeness, for extending Barsotti–Tate groups from $Y[\frac{1}{p}]$ to Y we have the following variant of Lemma 8 and Corollary 1.

Lemma 9 *Let Y be scheme on which p is a non-zero-divisor, which is integrally closed in $Y[\frac{1}{p}]$, and whose local rings at points of residue characteristic p are noetherian.*[6] *We assume that the condition (i) of Lemma 8 holds. Then the following two properties hold:*

(a) *If the condition (ii) of Lemma 8 holds, then each Barsotti–Tate group $D_{Y[\frac{1}{p}]}$ over $Y[\frac{1}{p}]$ which extends at each maximal point of $Y_{\mathbb{F}_p}$, extends to a Barsotti–Tate group over Y (uniquely up to unique isomorphism).*

(b) *If $Y_{\mathbb{F}_p}^{min}$ is retrocompact (i.e., the morphism $Y_{\mathbb{F}_p}^{min} \to Y$ is quasi-compact), then each Barsotti–Tate group D_W over an open subscheme W of Y that contains $Y[\frac{1}{p}]$ and $Y_{\mathbb{F}_p}^{min}$ extends to a Barsotti–Tate group over Y (uniquely up to unique isomorphism); thus if Y is a faithfully flat regular Spec$(\mathbb{Z}_{(p)})$-scheme, then it is p-healthy regular.*

Proof: Due to the assumptions of the first sentence of the lemma, the functor

$$(\text{BT groups over } Y) \longrightarrow (\text{BT groups over } Y[\frac{1}{p}])$$

[6] Thus in view of [12], Cor. 7, for each $z \in Y_{\mathbb{F}_p}$, the local ring $\mathcal{O}_{Y,z}$ is either a discrete valuation ring or a noetherian ring of both dimension and depth at least 2. We will only use this characterization and not the integrally closed condition itself.

is fully faithful.

Thus to prove (a) we can assume that $Y = W_\lambda$ is affine. The remaining part of the proof of (a) is similar to Cases 1 and 2 of the proof of Lemma 8, with just one difference. Referring to the Case 2 of the proof of Lemma 8, as we are not assuming $Y[\frac{1}{p}]$ regular, once we obtain $E_{n+s,z}$ over an affine open subscheme $W_z = \mathrm{Spec}(A_z)$, we have to add that by replacing W_z by an affine open subscheme of it we can assume that $E_{n+s,z}[\frac{1}{p}]$ is isomorphic to the restriction of $D_{Y[\frac{1}{p}]}[p^{n+s}]$ to $\mathrm{Spec}(A_z[\frac{1}{p}])$ under an isomorphism which extends the known isomorphism over $\mathrm{Spec}(\mathcal{O}_{Y,z}[\frac{1}{p}])$.

To prove (b) we can assume that Y is quasi-compact and quasi-separated. By hypotheses, $Y^{\mathrm{min}}_{\mathbb{F}_p}$ is quasi-compact. Thus W can be exhausted by quasi-compact open subschemes of it which contain $Y[\frac{1}{p}] \cup Y^{\mathrm{min}}_{\mathbb{F}_p}$ and it is enough to prove the extension assertion for each such open subscheme of W. Therefore we can assume W is quasi-compact. Let $n \in \mathbb{N}^*$. Let $z \in Y_{\mathbb{F}_p}$. From (a) we get that the restriction of D_W to $W \cap \mathrm{Spec}(\mathcal{O}_{Y,z})$ extends to a p-divisible group D_z over $\mathrm{Spec}(\mathcal{O}_{Y,z})$. A standard limit argument shows that $D_z[p^n]$ spreads out to an extension of $D_W[p^n]_{W_z \cap W}$ to a Barsotti–Tate group over W_z, where W_z is an affine open neighborhood of z in Y. These extensions are unique (up to unique isomorphism) and glue as we have $\iota_*(\mathcal{O}_W) = \mathcal{O}_Y$, where $\iota : W \to Y$ is the open embedding. Thus each $D_W[p^n]$ extends to a Barsotti–Tate group of level n over Y, and by the uniqueness part, these extensions constitute a Barsotti–Tate group over Y. $\square$

Example 1 *Let $\mathbb{Q} \subset K_1 \subset K_2 \subset \cdots$ be a tower of finite field extensions unramified above p such that the union $K_\infty = \cup_{n\in\mathbb{N}^*} K_n$ has infinitely many places above p. Let O_n be the integral closure of $\mathbb{Z}_{(p)}$ in K_n; so $O_\infty = \cup_{n\in\mathbb{N}^*} O_n$ is the integral closure of $\mathbb{Z}_{(p)}$ in K_∞. Let $\mathfrak{m} \in Max(O_\infty)$ be a non-isolated point. For instance, if $K_n = \mathbb{Q}(\sqrt{l_1}, \ldots, \sqrt{l_n})$, where the l_i's are distinct primes which are squares in $\mathbb{Q}_p$, then $Max(O_\infty)$ is a Cantor set with no isolated point. Let $\pi_n \in O_n$ be a generator of the product of all maximal ideals of O_n different from $\mathfrak{m} \cap O_n$. Let x be an indeterminate and let $A = \varinjlim O_n[\frac{x}{\pi_n}]$ be the filtered union of subrings of $O_\infty[\frac{x}{p}]$. Then the ring A is regular of dimension 2 and $Spec(A/pA)^{min}$ is not retrocompact, being a topological space homeomorphic to the disjoint union of $\{\mathfrak{m}\}$ and $Max(O_\infty) \backslash \{\mathfrak{m}\}$. This example can be modified in a way which allows ramification above p to get examples of faithfully flat $\mathbb{Z}_{(p)}$-schemes Y which are regular of dimension 2 and for which Y^{min} is not retrocompact and the condition (ii) of Lemma 8 does not hold.*

Let R be a regular local ring of mixed characteristic $(0,p)$ and dimension $d \geq 1$. For $e \in \mathbb{N}^*$ we consider the following condition on R:

($\diamondsuit_e$) *The reduced ring $(R/pR)_{red}$ is regular and we have $div(p) = e\, div(p)_{red}$ (as divisors on $X = Spec(R)$).*

If condition ($\diamondsuit_e$) holds for R, then it also holds for $\widehat{R}$ and for each local ring of X of mixed characteristic $(0,p)$. If R is formally smooth (or only flat with regular special fiber) over a discrete valuation ring O of mixed characteristic $(0,p)$ and absolute ramification index e, then condition ($\diamondsuit_e$) holds for R. If $e > 1$ there are examples in which condition ($\diamondsuit_e$) holds for $\widehat{R}$ but not for R and moreover R/pR is an integral domain. Moreover, we have the following converse which is related to the applicability of Corollary 1:

Lemma 10 *If condition ($\diamondsuit_e$) holds for R and p does not divide e, then $\widehat{R}$ is a formal power series ring over a complete discrete valuation ring O as above.*

Proof: Let $\mathfrak{m}$ be the maximal ideal of R; so $k = R/\mathfrak{m}$. We consider a Cohen ring $C(k) \subset \widehat{R}$. As R is a unique factorization domain, from the identity $\mathrm{div}(p) = e\mathrm{div}(p)_{\mathrm{red}}$ of divisors of X we get that there exist $\pi \in \widehat{R}$ and a unit u_π of $\widehat{R}$ such that we have $p = \pi^e u_\pi$. We write $u_\pi = u^{-1}u_1$, where $u \in C(k)$ and $u_1 \in 1 + \mathfrak{m}\widehat{R}$. By Hensel's Lemma, u_1 has an e-th root $u_1^{\frac{1}{e}}$ in $\widehat{R}$. Thus, by replacing π with $\pi u_1^{\frac{1}{e}}$ we can assume that $u_1 = 1$. As $pu = \pi^e$, the desired discrete valuation subring of $\widehat{R}$ is $O = C(k)[\pi] = C(k)[x_d]/(x_d^e - pu)$. If $x_1, \ldots, x_{d-1} \in \widehat{R}$ lift a regular system of parameters of $(\widehat{R}/p\widehat{R})_{\mathrm{red}}$, it is easy to see that the natural homomorphism $C(k)[[x_1, \ldots, x_d]]/(x_d^e - pu) = O[[x_1, \ldots, x_{d-1}]] \to \widehat{R}$ is an isomorphism. $\square$

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References

[1] M.V. Bondarko, *The generic fiber of finite group schemes; a "finite wild" criterion for good reduction of abelian varieties,* Izv. Ross. Akad. Nauk Ser. Mat. **70** (4) (2006) 21–52 (in Russian); transl. in: Izv. Math. **70** (4) (2006) 661–691.

[2] C.-L. Chai, B. Conrad, and F. Oort, *Complex multiplication and lifting problems,* Mathematical Surveys and Monographs, **195**, Amer. Math. Society, Providence, RI, 2014

[3] I. S. Cohen, *On the structure and ideal theory of complete local rings,* Trans. Amer. Math. Soc. **59** (1946), 54–106

[4] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties. With an appendix by David Mumford,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **22**, Springer-Verlag, Berlin, 1990

[5] O. Gabber and A. Vasiu, *The classification of p-quasi-healthy henselian regular rings of dimension* 2, manuscript 2018

[6] A. Grothendieck, *Éléments de géométrie algébrique. III, Étude cohomologique des faisceaux cohérents. I,* Inst. Hautes Études Sci. Publ. Math. No. **11**, 1961

[7] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2),* Advanced Studies in Pure Mathematics, Vol. **2**, North-Holland Publishing Co., Amsterdam; Masson & Cie, Éditeur, Paris, 1968

[8] A. Grothendieck, *Éléments de géométrie algébrique. IV, Étude locale des schémas et des morphismes des schémas. I,* Inst. Hautes Études Sci. Publ. Math. No. **20**, 1964

[9] A. Grothendieck, *Éléments de géométrie algébrique. IV, Étude locale des schémas et des morphismes des schémas. IV,* Inst. Hautes Études Sci. Publ. Math. No. **32**, 1967

[10] O. Gabber and L. Ramero, *Almost ring theory,* Lecture Notes in Mathematics, **1800**, Springer-Verlag, Berlin, 2003

[11] R. Hartshorne, *Stable reflexive sheaves,* Math. Ann. **254** (1980), no. 2, 121–176

[12] J. Kollár, *Variants of normality for Noetherian schemes,* Pure Appl. Math. Q. **12** (2016), no. 1, 1–31

[13] H. Hironaka and H. Matsumura, *Formal functions and formal embeddings,* J. Math. Soc. Japan **20** (1968), 52–82

[14] L. Illusie, *Déformations de groupes de Barsotti–Tate (d'après A. Grothendieck),* Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell (Paris, 1983/84), 151–198, Astérisque **127**, Soc. Math. de France, Paris, 1985

[15] H. Matsumura, *Commutative ring theory. Second edition,* Cambridge Studies in Advanced Mathematics, **8**, Cambridge University Press, Cambridge, 1989

[16] W. Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes,* Lecture Notes in Mathematics, Vol. **264**, Springer-Verlag, Berlin-New York, 1972

[17] B. Moonen, *Models of Shimura varieties in mixed characteristics,* Galois representations in arithmetic algebraic geometry (Durham, 1996), 267–350, London Math. Soc. Lecture Note Ser., **254**, Cambridge Univ. Press, Cambridge, 1998

[18] M. Raynaud, *Schémas en groupes de type* $(p, \cdots, p)$*,* Bull. Soc. Math. France **102** (1974), 241–280

[19] N. Schwartz and M. Tressl, *Elementary properties of minimal and maximal points in Zariski spectra,* Journal of Algebra **323** (2010), 698–728

[20] J. Tate, p*-divisible groups,* Proc. Conf. Local Fields (Driebergen 1966), 158–183, Springer-Verlag, 1967

[21] A. Vasiu, *Integral canonical models of Shimura varieties of preabelian type,* Asian J. Math. **3** (1999), no. 2, 401–518

[22] A. Vasiu, *A purity theorem for abelian schemes,* Michigan Math. J. **52** (2004), no. 1, 71–81

[23] A. Vasiu and T. Zink, *Purity results for* p*-divisible groups and abelian schemes over regular bases of mixed characteristic,* Doc. Math. **15** (2010), 571–599

[24] A. Vasiu and T. Zink, *Boundedness results for finite flat group schemes over discrete valuation rings of mixed characteristic*, J. Number Theory **132** (2012), no. 9, 2003–2019

[25] O. Zariksi and P. Samuel, *Commutative algebra. Vol. I,* The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York 1958

[26] O. Zariksi and P. Samuel, *Commutative algebra. Vol. II,* The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York 1960

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