Moduli schemes and the Shafarevich conjecture (the arithmetic case) for pseudo-polarized K3 surfaces

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first version, 8/31/99, Univ. of Utah, submitted to J. A.M.S. on 9/24/99

ABSTRACT. We construct moduli schemes of pseudo-polarized (polarized) K3 surfaces as open subschemes of finite type integral canonical models of a particular orthogonal Shimura variety. We draw conclusions pertaining to Shafarevich conjecture, Milne conjecture, connectivity, specializations, and other aspects of pseudo-polarized K3 surfaces in positive characteristic.

Key words: K3 surfaces, moduli stacks, Shimura and abelian varieties, and F-crystals.
Math. Subject Classification 2000: Primary 14J10, 14J28, 14G40, 14F30, 11G18 and 11G35.

Contents

§1. Introduction

1.0. The goal of this paper is to report on some arithmetical aspects of pseudo-polarized (or just polarized) K3 surfaces in mixt and positive characteristic. What we do: we put together some of the ideas and results of [An], [JT1-2] and [Va1-8] to understand more about the arithmetics of (pseudo-) polarized K3 surfaces.

1.1. Our approach to the subject is from the point of view of Shimura varieties of Hodge type, and so (see [Va1, 4.1 and the proof of 5.1]) of abelian varieties. However the general point of view is: one studies (the arithmetics of) abelian varieties as part of the study of algebraic varieties over fields (schemes) of interest. Among such varieties, those which have the closest arithmetics to abelian varieties are its "neighbours", i.e. the smooth projective varieties whose canonical class is zero, and in particular the hyperkählerian varieties (see [Be1] and [An]). The symplest example of hyperkählerian varieties is that of K3 surfaces. In §2 we review the main definitions and properties pertaining to them and needed in the paper (in particular 2.1 contains the definitions and notations to be used below). The arithmetical "closeness" between abelian varieties and K3 surfaces is rooted in Kuga–Satake's construction (see [KS] and [De5] for the case of complex numbers, and see [An, 1.7.1] for the case of a number field; see also 2.7), and in their deformation theory (see [De4], [De6] and 2.8). The arithmetical aspects of pseudo-polarized (or just polarized) K3 surfaces in mixt and positive characteristic to be dealt with here are "modelled" on similar ones for polarized abelian varieties.

All the (pseudo-) polarizations to be mentioned in this introduction are assumed to be primitive.

1.2. For an introduction to Shimura varieties we refer to [De1], [Mi1] and [Va1, §2]. Let $d \in \mathbb{N}$. Let $G := SO(2,19)$ be the special orthogonal group of the quadratic form $x_1^2 + x_2^2 - x_3^2 - ... - x_{20}^2 - x_{21}^2$ in 21 variables. It is an absolutely simple adjoint group over $\mathbb{Q}$ of B_{10} Lie type. Let $G_1 := \mathrm{GSpin}(2,19)$ be the non-trivial central extension of G by $\mathbb{G}_m$. Let

$$\mathcal{H}_{K3} := SO(2,19)(\mathbb{R})/SO(2)(\mathbb{R}) \times SO(19)(\mathbb{R}) = O(2,19)(\mathbb{R})/SO(2)(\mathbb{R}) \times O(19)(\mathbb{R})$$

be two copies of the Hermitian symmetric domain X^0 defined by the connected component $SO_0(2,19)(\mathbb{R})$ of the origin of the Lie group $SO(2,19)(\mathbb{R})$. In [BB] it is shown that $\mathcal{H}_{K3}$ has a natural structure (obtained via G) of a $\mathbb{C}$-scheme: it is obtained by taking a projective limit of smooth quasi-projective $\mathbb{C}$-schemes whose transition morphisms are finite étale.

1.2.1. A global Torelli theorem for marked polarized K3 surfaces of degree d was first given by Piatetskii-Shapiro and Shafarevich (see [PSS]). To explain their work, we recall some facts. Let (Z,L) be a polarized K3 surface of degree d over $\mathbb{C}$ (our convention $L.L = 2d$). The twisted Betti cohomology group $H := H^2(Z,\mathbb{Z})(1)$ is endowed naturally with a perfect symmetric form B (the cup product). The data $(c_1(L) \in H, B)$ is isomorphic to a fixed data $(l_0 \in H_0, B_0)$ (see [Be2, p. 111]). So by a marked structure of (Z,L) we mean an isomorphism

$$m\colon (l_0 \in H_0, B_0) \xrightarrow{\sim} (c_1(L) \in H, B).$$

To such a triple (Z,L,m) it is naturally associated (via Hodge structures) an element (called the period of (Z,L,m))

$$x_{(Z,L,m)} \in \mathcal{H}_{K3}.$$

In [PSS] (see also [LP]) it is shown that this association $x_{(Z,L,m)}$ to the isomorphism class of (Z,L,m) is injective. This implies easily: an open subscheme $\mathcal{H}_{K3}$ is a fine moduli scheme of marked polarized K3 surfaces of degree d.

1.2.2. Of course everything boils down to finite type level, i.e. to the level of finitely marked polarized K3 surfaces. Here by finitely marked we mean: endowed with some finite level structure $\mathcal{L}$, see 2.5.3. The most common such structures are level-n marked and (when possible; see the convention of 2.5.3.5 C)) level-n primitively marked (see 2.5.3.1

and 2.5.3.3), where $n \in \mathbb{N}$, $n \geq 3$. To detail this finite type level, for simplicity, we assume here that the level structures involved are refined enough to "produce" smooth quasi-projective schemes and not just stacks or non-smooth schemes (cf. 2.6 D)). Let K be the compact open subgroup of $G(\mathbb{A}_f)$ defining $\mathcal{L}$ (cf. 2.5.3.1). Using [PSS] one shows (see 2.6) the existence of a moduli scheme $\mathcal{A}_{K3,d,p,\mathcal{L}_{\mathbb{C}}}$ (of isomorphism classes of polarized K3 surfaces of degree d having the level structure $\mathcal{L}$) over $\mathbb{C}$; it is constructed (see 2.6) as an open subscheme $\mathcal{A}_{K3,d,p,\mathcal{L}_{\mathbb{C}}}$ of

$$\mathcal{A}_{K3,d,pp,\mathcal{L}_{\mathbb{C}}} := \Gamma_{d,\mathcal{L}} \backslash \mathcal{H}_{K_3},$$

where $\Gamma_{d,\mathcal{L}} := K \cap G(\mathbb{Q})$ is an arithmetic subgroup of $G(\mathbb{Q})$. We always assume that K is admissible in the sense of 2.5.3.1.

1.2.3. So one naturally comes across (see 2.5 for details):

– an adjoint Shimura pair $\mathrm{Sh}(G, X)$, where $X = \mathcal{H}_{K3}$ is defined as the $G(\mathbb{R})$-conjugacy class of a homomorphism $T \to G_{\mathbb{R}}$, with T a one dimensional compact torus over $\mathbb{R}$;

– a Shimura variety $\mathrm{Sh}(G_1, X_1)$, with $X_1 = X$ identified as a $G_1(\mathbb{R})$-conjugacy class of monomorphisms from the Weil restriction $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ to $G_{1\mathbb{R}}$.

Both these Shimura varieties do not depend on d (cf. 2.5). Moreover the different Kuga–Satake constructions associated to (Z, L) are closely related to injective maps of $\mathrm{Sh}(G_1, X_1)$ into Siegel modular varieties (see 2.7). In 2.7.2 we detail one such injective map: the $\mathbb{Q}$–version of one of Satake over $\mathbb{R}$ (cf. also [Va1, 5.7.5]); we do think it offers some new insights and simplifications. Moreover $\mathcal{H}_{K3}$ is an open subscheme of $\mathrm{Sh}(G, X)_{\mathbb{C}}$ and $\mathcal{A}_{K3,d,pp,\mathcal{L}_{\mathbb{C}}}$ is a union of one or two connected components of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ (see 2.6).

1.3. There is a second approach to the construction of such moduli schemes: via Hilbert schemes and Mumford's geometric invariant theory (see [An, 2.3.3-5] for a quick idea; see also [MFK] and [Vi]). This second approach works over $\mathbb{Q}$, but has some limitations in positive characteristic (cf. [MFK]). A third approach is via Artin's method (see [FC, §4 of ch. I] and [Vi]); its limitations start from the fact that is not explicite. So we felt more inclined in using the first approach (via Shimura varieties) in moving from moduli schemes over $\mathrm{Spec}(\mathbb{C})$ to moduli schemes over (étale covers of) $\mathrm{Spec}(\mathbb{Z})$ punctured in some points. Moreover this first approach relies heavily on the connection between K3 surfaces and abelian varieties (via the Kuga–Satake construction), and so it is more suited for arithmetical purposes.

1.3.1. The work of Kulikov (see [Ku]; see also [PP], [Be3, p. 150] and [KT]) implies that $\mathcal{A}_{K3,d,pp,\mathcal{L}_{\mathbb{C}}}$ itself is a moduli scheme of finitely marked pseudo-polarized K3 surfaces of degree d (so *pp* stands for pseudo-polarized).

1.3.2. The starting point of the paper was: [Va1, 6.4.1, 6.4.2.1, 6.4.4 and 6.4.6 1)]. The promises of [Va1, 3.2.3.2 1), 3.2.7 6) and 6.4.6 1)] (resp. of [Va1, 6.4.7]) are fulfilled in 3.1 (resp. in 3.2). So in 3.1 we present an incipient general theory (based on different extension properties introduced in [Va1, 3.2]) of different integral canonical models of smooth separated schemes of finite type over the field of fractions of some Dedekind integral ring faithfully flat over some localization of Spec($\mathbb{Z}$). It is the natural extension of the theories of [Va1, 3.2-3].

Combining [Va1, 6.4.1, 6.4.2.1, 6.4.4 and 6.4.6 1)] one gets directly the existence of a smooth quasi-projective model $\mathcal{N}_{d,\mathcal{L}}$ of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6dl}])$, where l is the product of the primes where $\mathcal{L}$ does not behave properly. It is uniquely determined (see [Va1, 3.2.3.1 7)]) by three properties: by prescribing its generic fibre, by an extension property (see [Va1, 6.4.6 1)]), and by specifying some normalizations of its pull backs over local rings of $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6dl}])$ (see 2.5.4 C) for details). In the language of 3.1, $\mathcal{N}_{d,\mathcal{L}}$ is an integral canonical model of its generic fibre. For simplicity we assume here that $\mathcal{A}_{K3,d,pp,\mathcal{L}_{\mathbb{C}}}$ is formed by two connected components $\mathcal{C}^0$ and $\mathcal{C}^1$ of $\mathcal{N}_{d,\mathcal{L}_{\mathbb{C}}}$. Let $E(\mathcal{C}^i)$ be the field of definition of $\mathcal{C}^i$, and let O be the ring of integers of the composite field of $E(\mathcal{C}^0)$ and $E(\mathcal{C}^1)$ punctured in $6dl$. Spec(O) is an étale cover of $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6dl}])$. In practice K is normal in the sense of 2.5.3.1: this implies that $\mathcal{C}^0$ is isomorphic to $\mathcal{C}^1$ and we have $E(\mathcal{C}^0) = E(\mathcal{C}^1)$. The basic result (see 3.2) says:

1.3.3. Theorem 1. *a) An open subscheme $\mathcal{A}_{K3,d,p,\mathcal{L}}$ of $\mathcal{N}_{d,\mathcal{L}_O}$ is a moduli scheme of finitely marked polarized K3 surfaces of degree d over Spec(O).*

b) An open subscheme $\mathcal{A}_{K3,d,pp,\mathcal{L}}$ of $\mathcal{N}_{d,\mathcal{L}_O}$ containing $\mathcal{A}_{K3,d,p,\mathcal{L}}$ as a dense subscheme, is a moduli scheme of finitely marked pseudo-polarized K3 surfaces of degree d over Spec(O).

1.3.4. The new recent ideas needed for the proof of Theorem 1 are: the extension property enjoyed by $\mathcal{N}_{d,\mathcal{L}}$ and the Main lemma [An, 1.7.1]. We first detail the part a). For any finitely marked polarized K3 surface (S, L_S) of degree d over an algebraically closed field k of positive characteristic p relatively prime to $6dl$, the deformation theory (see 2.8) shows the existence of a versal deformation of it over $Y := \mathrm{Spec}(W(k)[[x_1, ..., x_{19}]])$. The extension property enjoyed by $\mathcal{N}_{d,\mathcal{L}}$ implies easily the existence of a morphism

$$m_S\colon Y \to \mathcal{N}_{d,\mathcal{L}_{W(k)}}$$

extending the logical one at the level of generic fibres (here $W(k)$ is the ring of Witt vectors of k). Using [An, 1.7.1 and its variant 8.4.3] one can check directly that m_S is a formally étale morphism. This is enough to get 3.1.2 a), via standard arguments. The part b) can be checked following entirely the same pattern, starting from the ideas of part a) and from 1.3.1.

1.3.5. If the finite level structures $\mathcal{L}$ are the level-n marked structures (primitive or not), with $n \geq 3$ such that all its prime divisors divide $6d$, then we can take l to be 1. In [Va7] we will see that in 1.3.3-4 we can replace $6dl$ by $2dl$ (i.e. the things are fine as well for $p = 3$) (see 3.2.1 2)). However, here we will state the things only for primes $p \geq 5$, and refer to the case $p \geq 3$ just as remarks. Different variants and functorial aspects of Theorem 1 are presented in 3.2.3-5.

1.3.6. Let $\mathcal{SQF} := \{m \in \mathbb{N} | m \text{ is square free}\}$. Let $\mathcal{D}$ be the complement of $\mathcal{A}_{K3,d,p,\mathcal{L}}$ in $\mathcal{A}_{K3,d,pp,\mathcal{L}}$, with the induced reduced closed subscheme structure. It has pure codimension one, and often it is called the discriminant locus (cf. [JT1-2]). It is not clear to us for which primes p relatively prime to $6dl$ we do have (with $O_{(p)} := O \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$):

i) $\mathcal{A}_{K3,d,pp,\mathcal{L}_{\mathrm{Spec}(O_{(p)})}}$ is a disjoint union of two connected components of $\mathcal{N}_{d,\mathcal{L}_{\mathrm{Spec}(O_{(p)})}}$;

ii) $\mathcal{D}_{\mathrm{Spec}(O_{(p)})}$ is an ample divisor for $\mathcal{A}_{K3,d,pp,\mathcal{L}_{\mathrm{Spec}(O_{(p)})}}$ (for this part we assume that d is a product of distinct primes, cf. [JT1-2]).

This leads to the introduction (see 3.3) of two functions

$$f\colon \mathbb{N} \to \mathcal{SQF}$$

and

$$g\colon \mathcal{SQF} \to \mathcal{SQF}$$

such that i) (resp. ii)) above is true iff p also does not divide $f(d)$ (resp. $g(d)$). We always assume that $f(d)$ and $g(d)$ are relatively prime to $6d$. Our expectation is:

Expectation. *We have $f = g$ equal to the constant one function.*

For a motivation of this expectation see 5.4.

1.3.7. We view Theorem 1 as an arithmetic global Torelli theorem, and we view the expectation $f(d) = 1$ as the arithmetic Kulikov problem (of surjectivity of the period map).

1.4. In §4 we list some of the immediate applications of Theorem 1. They are grouped in 7 sections (4.1-7). We will not state here all these applications as theorems, as §4 is very much self explanatory; the only exemptions are the results of 4.3 and 4.5. [Va6] shows that $\mathcal{N}_{d,\mathcal{L}}$ has plenty of smooth toroidal projective compactifications. So the schemes $\mathcal{A}_{K3,d,p,\mathcal{L}}$ and $\mathcal{A}_{K3,d,pp,\mathcal{L}}$ have (see 4.1) smooth projective compactifications. This implies (see 4.2) that the moduli stack $\mathcal{A}_{K3,d,pp_{\mathbb{F}_p}}$ (with $\mathbb{F}_r$ as the finite field with r elements) obtained from $\mathcal{A}_{K3,d,pp,\mathcal{L}_{\mathbb{F}_p}}$ by the natural operation of forgetting the level structures, is geometrically connected.

In 4.3 we obtain (by just combining Theorem 1 with [JT1]) variants of the Shafarevich conjecture (for polarized K3 surfaces) over function fields in positive characteristic, similar to one of [Sz, p. 65]. We have:

Theorem 2. *We assume that d is square free. Let p be a prime not dividing $6dg(d)$, and let k be an arbitrary field of characteristic p. Let C be a geometrically connected projective curve over k. Let C_1 be a finite closed k-subscheme of C. We have:*

a) Any polarized K3 surface of degree d over C is isotrivial;

b) There are only a finite number of isomorphism classes of polarized K3 surfaces of degree d over $C_0 := C \setminus C_1$.

In 4.4 we obtain an upper bound estimate of the number of elements of the set $\mathcal{A}_{K3,d,*,\mathcal{L}}(\mathbb{F}_{p^q})$ (here $q \in \mathbb{N}$, while $* \in \{p, pp\}$), as well as a conjectural combinatorial description of the set $\mathcal{A}_{K3,d,pp,\mathcal{L}}(\mathbb{F}_{p^q})$ acted upon by a suitable power of the Frobenius automorphism of $\mathbb{F}_{p^q}$. 4.4 is based (see [Va7]) on [Mi4] and on our work on the Langlands-Rapoport conjecture of [Va1, 1.7].

In 4.5 we show that the mentioned extension property of $\mathcal{N}_{d,\mathcal{L}}$ gets translated in extension properties enjoyed by (pseudo-) polarized K3 surfaces. See [Va1, 3.2.1 1-2)] for the definition of a healthy regular scheme and of an extensible pair; here we just mention that any regular formally smooth scheme over $\mathbb{Z}_{(q)}$, with q an odd prime, is healthy (see [Va1, 3.2.2 1)]. We have :

Theorem 3. *Let $(\mathcal{Y}, \mathcal{U})$ be an extensible pair with $\mathcal{Y}$ a healthy regular scheme, and let $n \in \mathbb{N}$, $n \geq 3$. We assume that $6df(d)$ (resp. $6df(d)n$) is invertible in $\mathcal{Y}$. Then any polarized (resp. level-n marked pseudo-polarized) K3 surface of degree d over $\mathcal{U}$ extends to a polarized (resp. level-n marked pseudo-polarized) K3 surface over $\mathcal{Y}$.*

In 4.6 we show that the Milne's conjecture proved in [Va4] for Shimura p-divisible groups has an analogue version for K3 surfaces.

In 4.7 we prove different specialization and local deformation results for K3 surfaces in positive characteristic, similar to the ones of [Va2] (for abelian varieties).

1.5. In §5 five extra (besides the one of 1.3.6) open problems are formulated. Here we mention partially just two of them. First (see 5.1) is the construction of an arithmetic à la Tate (i.e. similar to one for elliptic curves; see [Si, p. 46]) for K3 surfaces in the projective space $\mathbb{P}^3$. Second, Theorem 1 allows the introduction of Kuga–Satake polarized abelian varieties attached to a (pseudo-) polarized K3 surface in positive characteristic relatively prime to 6 times the degree of the polarization; it is desirable to have a more direct description of how to construct directly such polarized abelian varieties (see 5.5).

The open problems reflect our taste and interest, and are mainly inspired from the context of polarized abelian variaeties.

1.6. In 4.1-2 (resp. 4.4) we rely on the work in progress [Va6] (resp. [Va7]); so these three sections are starred. Most of the results presented here were mentioned in a letter to Prof. A. Todorov, dated 6/3/96; so we are oblidged to [JT1-2]. We are also oblidged to [An]: though this work was obtained independently of [An], loc. cit. produced some simplifications and improvements (see 2.7.2.1). We benefited from discussions with Prof. A. Ogus and A. Todorov, for which we thank them.

The main motivation for this work: we desired to make more accessible some of the ideas and results of [Mi1-6] and [Va1-8] to specialist working with other classes of polarized projective varieties. The topics touched in this paper are forming what we call "standard aritmetics of polarized projective varieties minus rational points" (see [Va9]).

Here are some (recent) examples of classes of polarized projective varieties we have in mind and for which the (different) period maps are having an open image: cubic fourfolds (see [Vo] and [An]), (the known to exist) polarized hyperkählerian varieties with second Betti number greater than 3 (see [An] and 3.4 1)), cubic surfaces (see [ACT]), and non-hyperelliptic curves of genus three or four (see [Ko]). It is not dificult to see (see [Va9] for details) that in the first and the last two of these examples (see 3.4 1) for the case of polarized hyperkählerian varieties), following the pattern of the present paper, one can descend from moduli schemes over $\mathbb{C}$ to ones over $\mathrm{Spec}(\mathbb{Z})$ punctured in some points, and moreover §4-5 (under proper formulation) apply as well (in particular, as in 4.7 B) and C), we obtain Shimura-canonical lifts of Shimura-ordinary cubic fourfolds and surfaces over perfect fields, which are worth further study and attention). The general philosophy is:

GP. *Whenever we have a global (resp. local) Torelli theorem (at least in the context of Hermitian symmetric domains) for some class of polarized projective varieties, an arithmetic global (resp. local) Torelli theorem should exist (in the same sense as of 1.3.7).*

Warning: if we are in the context of a local Torelli theorem whose period map does not map into an open subscheme of (an arithmetic quotient of) a Hermitian symmetric domain, then a great part of §4-5 (like 4.1-2, 4.4, 4.6, etc.) does not apply.

As here we deal only with Shimura varieties of abelian type (see [Va1, defs. 3 of 2.5]), all the references to [Va1] are not starred (cf. the discussion of the last paragraph of [Va1, 1.4]). We would like to thank University of Utah for providing us with very good conditions for the writing of this paper. This research was partially supported by the NSF grant DMF 97-05376.

§2. Preliminaries

We introduce the notations and mostly review some well known facts which are needed in the subsequent chapters.

2.0. Notations and conventions. Reductive groups over fields are always assumed to be connected. Reductive group schemes are understood to have connected fibres. For a reductive group G over a scheme we denote by G^{der}, $Z(G)$, G^{ab} and G^{ad}, respectively, the derived group of G, the center of G, the maximal abelian quotient of G and the adjoint group of G. We say that a reductive group G over $\mathbb{Q}$ is unramified over $\mathbb{Q}_p$ (p being a rational prime) if $G_{\mathbb{Q}_p}$ is unramified over $\mathbb{Q}_p$. Different classical groups like $SO(2)$, $SO(2,19)$, $\mathrm{GSpin}(2,19)$, etc., are viewed as being over $\mathbb{Q}$.

The expression (G,X) always denotes a pair defining a Shimura variety $\mathrm{Sh}(G,X)$, while $E(G,X)$ denotes its attached reflex field (see [Va1]). We use freely the notations and conventions of [Va1, 2.2-4]. So we denote by $\mathbb{S}$ the Weil restriction from $\mathbb{C}$ to $\mathbb{R}$ of the one dimensional torus. For the terminology and notations involving a standard Hodge situation (to be abbreviated as SHS) we refer to [Va2, 2.3]. For the terminology involving Shimura crystals we refer to [Va2, 2.2].

Let p be a rational prime. By $\mathbb{Z}_{(p)}$ we denote the localization of $\mathbb{Z}$ with respect to p. The ring of finite adèles $\widehat{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q}$ (resp. of adèles) is denoted by $\mathbb{A}_f$ (resp. by $\mathbb{A}:=\mathbb{R}\times\mathbb{A}_f$) and the ring of finite adèles with the p-component omitted is denoted by $\mathbb{A}_f^p$. We use freely different Tate-twists: $\mathbb{Q}(1)$, $\mathbb{Q}_p(1)$, $\mathbb{Z}_p(1)$, $\mathbb{A}_f(1)$ etc. For G a linear group over $\mathbb{Q}$, $G(\mathbb{A})$ is endowed with the coarser topology which makes all the maps $G(\mathbb{A})\to\mathbb{A}^1_{\mathbb{Q}}(\mathbb{A})=\mathbb{A}$, induced by morphisms $G\to\mathbb{A}^1_{\mathbb{Q}}$, continuous ($\mathbb{A}^1_{\mathbb{Q}}$ being the affine line over $\mathbb{Q}$). Similarly for $G(\mathbb{A}_f)$. If G is a linear group over the field K of fractions of a discrete valuation ring (abbreviated DVR), then $G(K)$ is endowed in the same manner with a topology. We denote by $\mathbb{F}_{p^r}$ the field with p^r ($r\in\mathbb{N}$) elements and by $\mathbb{F}$ its algebraic closure.

The quasi-projective (projective) morphisms of schemes are in the sense of [Hart]. If Y is a scheme and Y_1 is a Y-scheme, then for any Y-scheme Z_Y, we denote by Z_{Y_1} its pull back to Y_1. For every free module M of finite rank over a commutative ring R we denote by M^* its dual. For any non-negative integer n, we denote by $M^{\otimes n}$ the tensor product of n-copies of M. By the tensor algebra of M we mean $\oplus_{n\in\mathbb{N}\cup\{0\}}M^{\otimes n}$. By the essential tensor algebra of $M\oplus M^*$ we mean

$$\mathcal{T}(M):=\oplus_{n,m\in\mathbb{N}\cup\{0\}}M^{\otimes n}\otimes M^{*\otimes m}.$$

If $v_\alpha \in M^{\otimes n} \otimes M^{*\otimes m}$, with n and m non-negative integers, we denote by $\deg(v_\alpha) := n+m$ its degree. A family of tensors of the tensor algebra of M is usually denoted in the form $(v_\alpha)_{\alpha\in\mathcal{J}}$, with $\mathcal{J}$ a set. A bilinear form on M is called perfect if it induces an isomorphism from M into its dual M^*. A pair (M,ψ) with M as above and with ψ a perfect alternating form on it, is called a symplectic space over R. We use the same notation for two nondegenerate forms if they are obtained one from another by extension of scalars.

The Witt ring of a perfect field k is denoted by $W(k)$. For the standard language pertaining to different Fontaine's categories $\mathcal{MF}_{[-1,1]}(*)$ and $\mathcal{MF}_{[0,1]}(*)$ we refer to [Va2, 2.1] and [Fa1]. The algebraic closure of a field E is denoted by $\bar{E}$.

Let $d \in \frac{1}{2}\mathbb{N}$. Let Y be an arbitrary scheme. Let k be an arbitrary algebraically closed field.

2.1. Definitions. a) A hyperkählerian scheme V over Y is a smooth projective scheme Z over Y of (constant) even relative dimension $2r$, whose geometic fibre over any algebraically closed field k is connected, simply connected, and $\Omega^2_{Z_k/k}$ has a global section w such that w^r vanishes nowhere.

b) A polarization of a hyperkählerian scheme V over Y of degree $d \in \mathbb{N}$ is (defined by) an Y-ample line bundle L on Z of (warning) degree $2d$.

c) An algebraic K3 surface over Y is a hyperkählerian scheme V over Y of relative dimension 2. We always drop the word algebraic.

d) A pseudo-polarized K3 surface (alternative terminology polarized generalized K3 surface) over Y is a pair (Z,L), where Z is a projective flat Y-scheme of relative dimension 2, whose geometric fibre Z_k over any algebraically closed field k is obtained by blowing down (-2)-curves on a K3 surface $\tilde{Z}_k$, and where L is an ample line bundle on Z. As in b) we speak about its degree (provided it is constant on its geometic fibres) as being half the self intersection number of L_k. We also refer to the pair $(\tilde{Z}_k,\tilde{L}_k)$, where $\tilde{L}_k$ is the pull back of L_k to $\tilde{Z}_k$, as a K3 surface with a pseudo-polarization. Similarly we define a pseudo-polarization of a K3 surface Z over Y and we speak about a K3 surface with a pseudo-polarization.

e) A pseudo-polarized K3 surface (Z,L) over Y is said to be primitive if for any geometric fibre Z_k, L_k is not the n-th power ($n\in\mathbb{N}$, $n\geq 2$) of another line bundle.

2.1.0. Remark. In 2.1 d), the singularities of Z_k are double rational (see [Ba, ch. 3-4] and especially [Ba, 3.31] for different characterizations of such singularities).

Through the rest of this chapter we consider a K3 surface Z over Y. From now on $d\in\mathbb{N}$.

2.2. Some numerical properties. We assume here that $Y=\mathrm{Spec}(k)$. We have (see [Ba, p. 136]): the canonical class of Z is zero, $b_1(Z)=0$, $b_2(Z)=22$, $\chi(O_Z)=0$, $\mathrm{Pic}(Z)=\mathbb{Z}^r$, with $r\in\mathbb{N}$, $r\leq 22$. Moreover there is a unique Hilbert polynomial $P(x):=dx^2+2$ associated to pseudo-polarized K3 surfaces of degree d (easy check starting from the above numerical properties and [Ba, 1.3 b)]).

2.3. Examples. For examples of hyperkählerian varieties of arbitrary even dimension see [Be1, ch. 6 and 7]. The first examples of K3 surfaces are obtained by considering smooth complete intersections in a projective space over Y. There are three possibilities:

a) a smooth quartic in $\mathbb{P}^3_Y$;
b) the smooth intersection of a quadric with a cubic in $\mathbb{P}^4_Y$;
c) the smooth intersection of three quadrics in $\mathbb{P}^5_Y$.

In each of these three cases, we obtain (see [GH, p. 591-3]) versal families of polarized K3 surfaces of dimension 19 (the degrees being respectively 2, 3 and 4).

d) The second standard example of K3 surfaces is that of Kummer surfaces. They are constructed as follows (see [Ba, 10.5 and 10.7 b)] for details). We assume that Y has no points of characteristic 2. Let A be an abelian surface over Y. Let $\tau\colon A \xrightarrow{\sim} A$ be the automorphism (it is an involution) taking x to $-x$. Its fixed scheme defines the finite flat group subscheme $A[2]$ of 2-torsion points of A. Let $\tilde{A}$ be obtained from A by blowing up along $A[2]$. τ lifts to an automorphism $\tilde{\tau}$ of $\tilde{A}$. The quotient of $\tilde{A}$ by it is a K3 surface Z. [FC, 1.10 a) of p. 7] guarantees that Z is indeed projective over Y. Another way to obtain Z: we first take the quotient A_1 of A through τ, and then we blow up A_1 along its singular locus. In this way we obtain families of K3 surfaces of dimension 3 whose Kodaira-Spencer map is injective. If L_1 is a polarization of A_1 then, (A_1, L_1) is a pseudo-polarized K3 surface.

e) If d is square free than any pseudo-polarization of a K3 surface is primitive.

2.4. The "H^2" and its primitive versions. We assume here that $Y = \mathrm{Spec}(k)$. Below all the perfect symmetric bilinear forms (pairings) are defined by the usual cup product (and not by its additive inverse). We distinguish three cases.

Case 1: $k = \mathbb{C}$. We have (cf. 2.2; see [Me, 1.2-3]): $H^0_B(Z,\mathbb{Z}) = \mathbb{Z}$, $H^1_B(Z,\mathbb{Z}) = 0$, and $H^2_B(Z,\mathbb{Z}) = \mathbb{Z}^{22}$; here the lower index B refers to the Betti cohomology. Moreover we have a perfect symmetric form $H^2_B(Z,\mathbb{Z})(1) \otimes H^2_B(Z,\mathbb{Z})(1) \to \mathbb{Z}$.

Case 2: characteristic of k is zero. Let p be a prime. We have (cf. 2.2 and Case 1) $H^i_{\text{ét}}(Z,\mathbb{Z}_p)$ is $\mathbb{Z}_p$, 0, or $\mathbb{Z}_p^{22}$, depending on i being 0, 1, or 2. Moreover we have a perfect symmetric form $H^2_{\text{ét}}(Z,\mathbb{Z}_p)(1) \otimes H^2_{\text{ét}}(Z,\mathbb{Z}_p)(1) \to \mathbb{Z}_p$.

Case 3: characteristic of k is a prime p. Let q be a prime different from p. We have (cf. 2.2) $H^i_{\text{ét}}(Z,\mathbb{Z}_q)$ is $\mathbb{Z}_q$, 0, or $\mathbb{Z}_q^{22}$, depending on i being 0, 1, or 2. Moreover we have a perfect symmetric form $H^2_{\text{ét}}(Z,\mathbb{Z}_q)(1) \otimes H^2_{\text{ét}}(Z,\mathbb{Z}_q)(1) \to \mathbb{Z}_q$.

We also have (see [De4, p. 59]) $H^i_{\text{crys}}(Z/W(k))$ is $W(k)$, 0, or $W(k)^{22}$, depending on i being 0, 1, or 2. Moreover we have a perfect (even for $p = 2$, cf. [De4, 2.2 c)]) symmetric form $H^2_{\text{crys}}(Z/W(k))(-1) \otimes H^2_{\text{crys}}(Z/W(k))(-1) \to W(k)$.

2.4.1. We assume now that Z has a pseudo-polarization L of degree $d \in \mathbb{N}$. With the conventions of 2.1 b) we have $L.L = 2d$. We denote by $< L >$ its first Chern class in any of the $H^2(*)$ introduced above, twisted or not by the usual Tate twist (1) or (-1) (like $\mathbb{Z}(1)$, $\mathbb{Z}_p(1)$, $W(k)(-1)$). We detail here the first case when $k = \mathbb{C}$ (see 2.8 for the third case, and see 2.5.3 and 4.6 for the second one). So till the end of 2.5.0 we assume that $Y = \mathrm{Spec}(\mathbb{C})$.

Let

$$H := H^2(Z,\mathbb{Z})(1).$$

We have a perfect symmetric bilinear form $B : H \otimes H \to \mathbb{Z}$. Its signiture is $(3, 19)$ and its main property is: $B(x,x) \in 2\mathbb{Z}$, $\forall x \in H$ (see [Me, 1.3]). The pair (H, B) does not depend on Z (see [Me, 1.3.2] and [Se, th. 5 of p. 54]). In other words it is isomorphic to $(\tilde{L}_0, \tilde{B}_0)$, where $\tilde{L}_0 = \mathbb{Z}^{22}$, and where B_0 is a fixed nondegenerate symmetric form on L_0, which w.r.t. the standard ordered basis $\{l_1, ..., l_{22}\}$ of $\tilde{L}_0$ is described by the quadratic form $2l_1l_2+2l_3l_4+2l_5l_6-E_8(l_7,...,l_{14})-E_8(l_{15},...,l_{22})$, with $E_8(x_1,...,x_8)$ as the quadratic form in 8 variables defined by the Cartan matrix of the E_8 Lie type.

2.4.2. Convention. From now on without otherwise stated, all the pseudo-polarizations to be considered are primitive.

2.4.3. The above convention implies that the $\mathbb{Z}$-submodule of H generated by $< L >$ is a direct summand (this is an easy consequence of [Me, 2.3]). From [Be2, p. 111] we deduce that an isomorphism $(H, B) \tilde{\to} (\tilde{L}_0, \tilde{B}_0)$ can be chosen such that $< L >$ is mapped into $l_1 + dl_2$.

Let $H_{\mathrm{pr}}(Z, L, \mathbb{Q})$ be the perpendicular subspace of $H \otimes \mathbb{Q}$ on $< L >$ w.r.t. B. Let

$$H_{\mathrm{pr}} = H_{\mathrm{pr}}(Z, L, \mathbb{Z}) := H_{\mathrm{pr}}(Z, L, \mathbb{Q}) \cap H.$$

We still denote by B its restriction to H_{pr}. Let L_0 be the free $\mathbb{Z}$-submodule of $\tilde{L}_0$ generated by $l_1 - dl_2$, l_3,..., l_{22}. Its rank is 21. So $L_0 \oplus < L > \mathbb{Z}$ is a subgroup of $\tilde{L}_0$ of index $2d$, and the resulting quotient group is cyclic. Let d' be the square free natural number dividing $2d$ and such that $\frac{2d}{d'}$ is a square. Let B_0 be the restriction of $\tilde{B}_0$ to L_0. The quadratic form associated to B_0 and the subbases $\{l_1 - dl_2, l_3, l_4, l_5, l_6\}$ of L_0 is $-2dx_1^2 + 2x_2x_3 + 2x_4x_5$; so the discriminant of B_0 is $-2d$ (when viewed over $\mathbb{Q}$ it is $-d'$).

We get a polarized Hodge $\mathbb{Z}[\frac{1}{2d}]$-structure $(H_{\mathrm{pr}}[\frac{1}{2d}], B)$ of weight 0 (that was the role of using Tate twists), of type $(-1, 1)$, $(0, 0)$, $(1, -1)$, and of signature $(2, 19)$. Moreover the Hodge numbers are $h^{-1,1} = h^{1,-1} = 1$ and $h^{0,0} = 19$. Let

$$h_Z : \mathbb{S} \to SO(H_{pr} \otimes_{\mathbb{Z}} \mathbb{R}, B)$$

be the homomorphism defining this polarized Hodge $\mathbb{Z}[\frac{1}{2d}]$-structure. Let G_Z be the smallest subgroup of $SO(H_{\mathrm{pr}} \otimes \mathbb{Q}, B)$ through which h_Z factors. It is known (for instance see [De2-3]) that G_Z is a reductive group over $\mathbb{Q}$, called the Mumford-Tate group of Z. Warning: in general it is not a semisimple group, while G_Z^{der} is not an adjoint group.

2.5. The standard Shimura varieties attached to K3 surfaces. Let

$$G := SO(2, 19; d')$$

be the special orthogonal group of $Q_{d'}$, where Q_m (for $m \in \mathbb{N}$) is the quadratic form $x_1^2 + x_2^2 - x_3^2 - ... - x_{20}^2 - mx_{21}^2$. We view it as an absolutely simple adjoint group over $\mathbb{Q}$. It is of B_{10} Lie type. If $d' > 1$ then $Q_{d'}$ and Q_1 are not equivalent over $\mathbb{Q}$, as their discriminants do not differ by a square of $\mathbb{Q}$. But $d'Q_{d'}$ and Q_1 are equivalent over $\mathbb{Q}$. This can be checked easily starting from the fact that any prime is a sum of four squares and that $d'Q_{d'}$ is equivalent (over $\mathbb{Z}[\frac{1}{2d}]$) to the quadratic form $x_1x_2 + x_3x_4 - d'(x_5^2 + ... + x_{20}^2) - x_{21}^2$. We deduce:

Fact. *G is isomorphic to the special orthogonal group $SO(2, 19)$ of Q_1.*

We refer to 2.4.1. As the pair (H_{pr}, B) does not depend on (Z, L) but only on d (cf. 2.4.1), we can identify (cf. 2.4.3) G with $SO(H_{\mathrm{pr}} \otimes \mathbb{Q}, B) = SO(L_0, B_0)$. So we can view h_Z as a homorphism

$$h_Z \colon \mathbb{S} \to G_{\mathbb{R}},$$

uniquely determined up to conjugation by an element of

$$\Gamma_d := SO(L_0, B_0)(L_0) \cap \Gamma_d^O,$$

where

$$\Gamma_d^O := \{g \in O(\tilde{L}_0, \tilde{B}_0)(\tilde{L}_0) | g \text{ fixes } l_1 + dl_2\} \subset O(L_0, B_0)(L_0).$$

Similarly we define

$$\Gamma_d^{SO} := \{g \in SO(\tilde{L}_0, \tilde{B}_0)(\tilde{L}_0) | g \text{ fixes } l_1 + dl_2\}.$$

Let $\tilde{L}_1$ (resp. L_1)) be a $\mathbb{Z}$-lattice of $L_0 \otimes \mathbb{Q}$ such that (cf. 2.4.1 and [Se, p. 51]):

– there is a $\mathbb{Z}$-bases $\{\tilde{e}_1, ..., \tilde{e}_{21}\}$ (resp. $\{e_1, ..., e_{21}\}$) of it with respect to which the quadratic form of B_0 is $x_1^2 + x_2^2 - x_3^3 - ... - x_{20}^2 - 2dx_{21}^2$ (resp. is $x_1^2 + x_2^2 - x_3^3 - ... - x_{20}^2 - d' x_{21}^2$);

– $\tilde{L}_1 \otimes \mathbb{Z}\left[\frac{1}{2}\right] = L_0 \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ (resp. $L_1 \otimes \mathbb{Z}\left[\frac{1}{2d}\right] = L_0 \otimes \mathbb{Z}\left[\frac{1}{2d}\right]$).

Let L_{st} be a $\mathbb{Z}$-lattice of $L_0 \otimes \mathbb{Q}$ such that w.r.t. to a $\mathbb{Z}$-bases of it the quadratic form of $d' B_0$ has the standard form Q_1 and $L_{\mathrm{st}}\left[\frac{1}{2d}\right] = L_0\left[\frac{1}{2d}\right]$ (cf. to the proof of the above Fact).

Let X (resp. X_Z) be the $G(\mathbb{R})$-conjugacy class (resp. the $G_Z(\mathbb{R})$-conjugacy class) of h_Z. It is easy to see that the axioms SV1-3 of [Va1, 2.3] are satisfied for (G, X) and (G_Z, X_Z). We obtain an adjoint Shimura variety $\mathrm{Sh}(G, X)$ and an injective map $f_Z \colon (G_Z, X_Z) \hookrightarrow (G, X)$.

X can be described as follows (cf. [De2, 1.2.8]): it is the $G(\mathbb{R})$-conjugacy class of the homorphism $h_{\mathrm{standard}} \colon \mathbb{S} \to G_{\mathbb{R}}$ factoring through $SO(2)_{\mathbb{R}}$, and where $SO(2)_{\mathbb{R}}$ acts trivially on e_3,...,e_{21} and through the natural inclusion on the real subspace generated by e_1 and e_2. Obviously h_{standard} is definable over $\mathbb{Q}$. Moreover loc. cit. implies that (G, X) is uniquely determined by G.

Let $G_1 := \mathrm{GSpin}(2, 19)$ be the non-trivial central extension of G by $\mathbb{G}_m$. Let $\mathrm{Sh}(G_1, X_1)$ be the Shimura variety constructed in [Va1, 5.7.5] for $l = 10$. It is a classical Spin variety of dimension 19 and rank 2 (cf. the terminology of loc. cit.). It comes equiped with an injective map into a Siegel modular variety

$$f_1 \colon \mathrm{Sh}(G_1, X_1) \hookrightarrow \mathrm{Sh}(GSp(W, \psi), S),$$

where (W, ψ) is a symplectic space over $\mathbb{Q}$. We have (cf. loc. cit.) $\dim(W) = 2^{10}$ (as 10, the rank of G, is congruent to 2 mod 4). We recall that $G_1^{\mathrm{ad}} = G$, that $G_1^{\mathrm{ab}} = \mathbb{G}_m$, and that $G_1^{\mathrm{der}} = \mathrm{Spin}(2, 19)$ is simply connected. Strictly speaking in loc. cit. we worked with the signature $(19, 2)$, but the same remains true (under the standard isomorphism $SO(2, 19) \xrightarrow{\sim} SO(19, 2)$) for the case of the signature $(2, 19)$ (we have $SO(L_0, B_0) = SO(L_0, -B_0)$).

From [De1, 3.8] we get $E(G,X)=E(G_1,X_1)=\mathbb{Q}$. Here X_1 is the $G_1(\mathbb{R})$-conjugacy class of the unique monomorphism

$$h_{1Z}\colon \mathbb{S} \hookrightarrow G_{1\mathbb{R}}$$

lifting h_Z and which restricted to the subgroup $\mathbb{G}_m$ of $\mathbb{S}$ acts as scalar multiplications on $W\otimes\mathbb{R}$.

We denote by $q:(G_1,X_1)\to(G_1^{\mathrm{ad}},X_1^{\mathrm{ad}})=(G,X)$ the natural adjoint map. Following the conventions of [Va1, 2.4] we still denote by q the different morphisms between quotients of Shimura varieties it induces naturally. Let $G_{1Z}=q^{-1}(G_Z)$; it is a reductive group over $\mathbb{Q}$. Obviusly h_{1Z} factors through $G_{1Z\mathbb{R}}$, giving birth to a Shimura variety (G_{1Z},X_{1Z}). Similarly we denote by $q_Z\colon(G_{1Z},X_{1Z})\to(G_{1Z}^{\mathrm{ad}},X_{1Z}^{\mathrm{ad}})=(G_Z,X_Z)$ the natural adjoint map.

The faithful representation $f_{1\mathbb{C}}\colon G_{1\mathbb{C}}^{\mathrm{der}}\hookrightarrow GL(W\otimes\mathbb{C})$ defined by f_1 is irreducible. From [De2, 1.3.9] we deduce that $\dim(W)$ is the smallest possible dimension for embedding $\mathrm{Sh}(G_1,X_1)$ in a Siegel modular variety.

We refer to $\mathrm{Sh}(G_1,X_1)$ and its adjoint $\mathrm{Sh}(G,X)$ as the companion Shimura varieties attached to pseudo-polarized K3 surfaces (of degree d). They do not depend on d. We refer to $\mathrm{Sh}(G_Z,X_Z)$ as the Shimura variety attached to Z (or to (Z,L)) itself.

2.5.0. The d-invariant. We denote by $d_Z\in\mathbb{N}$ the smallest number such that G_Z can be described as the subgroup of $GL(H_{\mathrm{pr}}\otimes\mathbb{Q})=GL(L_0\otimes\mathbb{Q})$ fixing a family of homogeneous tensors of the essential tensor algebra of $(L_0\oplus L_0^*)\otimes\mathbb{Q}$ of degrees at most d_Z (cf. [De3, 3.1 c)]). We refer to it as the d-invariant of Z. It is quite common to have $d_Z=2$.

We also fix an arbitrary family $(v_\alpha)_{\alpha\in\mathcal{J}_Z}$ of such tensors (with no restriction on the degrees involved) such that it contains B_0 (i.e. there is $\alpha_0\in\mathcal{J}_Z$ with $v_{\alpha_0}=B_0$) and G_Z is the subgroup of $GL(L_0\otimes\mathbb{Q})$ fixing its members.

2.5.1. Shimura subvarieties of $\mathrm{Sh}(G,X)$. Let $(G_0,X_0)\hookrightarrow(G,X)$ be an injective map. As the $\mathbb{R}$-rank of G is 2, from [He, p. 518] and [De2, 1.3.9-10] we deduce that:

a) either G_0^{ad} is a product of at most two $\mathbb{Q}$–simple adjoint groups of some A_n Lie type, or

b) G_0^{ad} is a simple $\mathbb{Q}$–group of B_2, B_3,..., B_{10}, D_4, D_5,...., D_9 or D_{10} Lie type, with just one non-compact factor over $\mathbb{R}$.

From [Va3, (1) of 3.0] we deduce easily that all the possibilities of b) show up. In case G_0^{ad} is of D_n Lie type with $n\in\{6,...,10\}$, (G_0,X_0) is of $D_n^{\mathbb{R}}$ type (see [De2] for the meaning of this). For a discussion of the possibilities of a) we refer to [Za]. Moreover from the work of Kulikov recalled in 2.6.1 (see also [Za]) it results that any such Shimura subvariety $\mathrm{Sh}(G_0,X_0)$ of $\mathrm{Sh}(G,X)$ with the property that there is no subgroup of G_0 through which all the homorphisms $\mathbb{S}\to G_{0\mathbb{R}}$ defined by elements of X_0 factor, is the Shimura variety attached to a pseudo-polarized K3 surface of degree d.

If $G_Z=G$ then the group $\mathrm{Pic}(Z)$ is freely cyclic, having (cf. 2.4.2) L as its generator.

2.5.2. Connected components. Let K (resp. K_1) be a compact subgroup of $G(\mathbb{A}_f)$ (resp. of $G_1(\mathbb{A}_f)$). Let K_1^{ab} be the image of K_1 in $G_1^{\mathrm{ab}}(\mathbb{A}_f)=\mathbb{G}_m(\mathbb{A}_f)$. From [De1, 2.4-5] we deduce that the set of connected components of $\mathrm{Sh}_{K_1}(G_1,X_1)$ is a principal homogeneous space of the group $\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q})K_1^{\mathrm{ab}}$. So the situation is entirely similar to the one of Siegel modular varieties. Moreover the natural ("adjoint") morphism

$$q\colon \mathrm{Sh}(G_1, X_1) \to \mathrm{Sh}(G, X)$$

is a cover in the sense of [Va1, 2.4]. From this and [Mi4, 4.10-13], as $\mathbb{G}_m(\mathbb{A}_f) = \mathbb{G}_m(\mathbb{Q})\mathbb{G}_m(\widehat{\mathbb{Z}})$, we get:

Fact. *If K_1 is the maximal compact subgroup of $G_1(\mathbb{A}_f)$ with $q(K_1) = K$ (we recall that q induces an epimorphism $q\colon G_1(\mathbb{A}) \to G(\mathbb{A})$), then the natural morphism $\mathrm{Sh}_{K_1}(G_1, X_1) \to \mathrm{Sh}_K(G, X)$ is an isomorphism.*

This is one of the reasons (see also 2.7) why it is more convenient to work part of what follows with (G_1, X_1) instead of (G, X).

$X_1 = X$ is a disjoint union of two connected components, each one being (with the standard notations of [He]) $BDI(p = 2, q = 19)$. In other words $G(\mathbb{R})$ and $G_1(\mathbb{R})$ have two connected components and a connected component of X is (isomorphic to)

$$X^0 := G(\mathbb{R})^0/SO(2)(\mathbb{R}) \times SO(19)(\mathbb{R})$$

(the upper index 0 for $G(\mathbb{R})$ refers to the connected component of the origin).

2.5.3. Level structures. In what follows we include a detail treatment of the level structures as we hope it will be useful in other contexts as well.

We still assume that $Y = \mathrm{Spec}(\mathbb{C})$. We say that (Z, L) is marked if an isomorphism

$$(\tilde{L}_0, \tilde{B}_0, l_1 + dl_2) \tilde{\to} (H, B, < L >)$$

is fixed. So there are Γ_d^O such marked structures of (Z, L).

Let p be a prime. We assume now that Y is an arbitrary scheme where p is invertible. Let $\pi\colon Z \to Y$ be a K3 surface, and let L be a pseudo-polarization of it of degree d. We say that (Z, L) is p-marked if an isomorphism

$$k_0\colon (\tilde{L}_0 \otimes \mathbb{Z}_p, \tilde{B}_0, l_1 + dl_2)_Y \tilde{\to} (R^2\pi_*(\mathbb{Z}_p)(1), B, < L >)$$

is given (here we still denote by B the symmetric bilinear form on $R^2\pi_*(\mathbb{Z}_p)(1)$ defined by the cup product, while the lower index Y refers to the fact that we view $(L_0 \otimes \mathbb{Z}_p, B_0)$ as a constant polarized étale sheaf on Y). We refer to the triple (Z, L, k_0) as a p-marked polarized K3 surface if L is a polarization, and as a p-marked K3 surface with a pseudo-polarization otherwise. If p does not divide $2d$, then there are precisely $O(L_0, B_0)(L_0 \otimes \mathbb{Z}_p)$ such p-marked structures of (Z, L). If instead of $\mathbb{Z}_p$ we work with $\mathbb{Z}/p^m\mathbb{Z}$ ($m \in \mathbb{N}$), we speak about level-p^m marked polarized K3 surfaces and about a level-p^m marked K3 surface with a pseudo-polarization (notation again by a triple (Z, L, k_{p^m})).

2.5.3.1. Definitions. A subgroup (resp. normal subgroup) of $SO(L_0, B_0)(L_0 \otimes \widehat{\mathbb{Z}})$ is called weakly admissible (resp. normal). A weakly admissible subgroup is called admissible (resp. strongly admissible) if it is contained in $\mathcal{SOA} := \{g \in SO(\tilde{L}_0, \tilde{B}_0)(\tilde{L}_0 \otimes \widehat{\mathbb{Z}}) | g$ fixes l_1+

$dl_2\} \subset SO(L_0,B_0)(L_0\otimes\widehat{\mathbb{Z}})$ (resp. is contained in the closure of Γ_d^{SO} in $SO(L_0,B_0)(L_0\otimes\widehat{\mathbb{Z}})$). Similarly we define (weakly or strongly) admissible subgroups of $O(L_0,B_0)(L_0\otimes\widehat{\mathbb{Z}})$.

Let

$$K_O(n) := \{g \in O(L_0,B_0)(L_0\otimes\widehat{\mathbb{Z}}) | g \text{ mod n is the identity}\}$$

and let

$$K(n) := K_O(n) \cap SO(L_0,B_0)(L_0\otimes\widehat{\mathbb{Z}}).$$

It is easy to see that any subgroup of $K_O(2d)$ (and so of $K(2d)$) is admissible. The notions weakly admissible and admissible coincide iff $d=1$.

As above we define the notion of n-marked (or of level-n marked) polarized K3 surface (resp. K3 surface with a pseudo-polarization) over Y, where $n\in\mathbb{N}$ (of course we assume that n is invertible in Y). We apply the convention: an n-marked structure is the same as r-marked structure for any prime r dividing n. More generally, if K is an admissible compact subroup of $O(L_0,B_0)(L_0\otimes\widehat{\mathbb{Z}})$ or of $SO(L_0,B_0)(L_0\otimes\widehat{\mathbb{Z}})$, then we speak similarly about level-K marked polarized K3 surface (resp. K3 surface with a pseudo-polarization); so if $(n,2d)=1$ then the level-n marked structure is the same as level-$\tilde{K}_O(n)$ marked structure, where $\forall m\in\mathbb{N}$

$$\tilde{K}_O(m) := \{g\in O(\tilde{L}_0,\tilde{B}_0)(\tilde{L}_0\otimes\widehat{\mathbb{Z}}) | g \text{ fixes } l_1+dl_2 \text{ and is congruent to 1 mod m}\}.$$

If $n\geq 3$, then $K_O(n)=K(n)$. It is desirable for the sake of flexibility not to assume K to be an open subgroup.

2.5.3.2. Discussion. One might inquire: what is the right group to work with $SO(L_0,B_0)$ or $O(L_0,B_0)$? The answer is: as 19 and 21 are odd, multiplication by -1 is an isomorphism of (L_0,B_0) which is not an element of $SO(L_0,B_0)$; so the end result is the same (see (1) below). For instance, the set $\mathcal{PSS}$ of isomorphism classes of marked polarized K3 surfaces of degree d is identifiable with a subset of

$$\mathrm{Sh}(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A})/(SO(2)(\mathbb{R})\times SO(19)(\mathbb{R}))';$$

(for the last equality cf. [De2, 2.1]). Here $(SO(2)(\mathbb{R})\times SO(19)(\mathbb{R}))'$ is the maximal compact subgroup of $G(\mathbb{R})$ containing $SO(2)(\mathbb{R})\times SO(19)(\mathbb{R})$ as a normal subgroup of index 2.

The fact that we have a set map from $\mathcal{PSS}$ to $\mathrm{Sh}(G,X)(\mathbb{C})$ is obtained in the same manner as in [Va1, 4.1] (where the case of Shimura varieties of Hodge type is treated; with our notations, we just have to repeat the construction of the map $g_{(G_1,X_1,W,\psi)}$ of loc. cit.), starting from the natural identification:

$$(1) \qquad G(\mathbb{R})/SO(2)(\mathbb{R})\times SO(19)(\mathbb{R}) = O(L_0,B_0)(\mathbb{R})/SO(2)(\mathbb{R})\times O(19)(\mathbb{R}).$$

In [PSS] it is shown that this set map is injective. There are two other useful natural identifications:

$$G(\mathbb{R})^0/SO(2)(\mathbb{R})\times SO(19)(\mathbb{R}) = G(\mathbb{R})/(SO(2)(\mathbb{R})\times SO(19)(\mathbb{R}))'$$

and

$$G(\mathbb{R})/(SO(2)(\mathbb{R}) \times SO(19)(\mathbb{R}))' = O(L_0, B_0)(\mathbb{R})/O(2)(\mathbb{R}) \times O(19)(\mathbb{R});$$

they allow the replecement of $O(*)$ by $SO(*)$. Moreover the connected components of $\mathrm{Sh}(G, X)(\mathbb{C})$ are all identifiable (as complex manifolds) with X^0. (This part should be compared with [An, 3.1-2], where are also treated situations with even numbers, like 20 and 22 instead of 19 and 21.) However (see 2.7) it is more convenient to work with special orthogonal groups than to orthogonal groups; that is why we most commonly speak about level-K marked structures, for K an admissible compact subgroup of $SO(L_0, B_0)(L_0 \otimes \widehat{\mathbb{Z}})$.

2.5.3.3. The primitive counterpart. One can define level structures by just using the primitive parts of the different H^2's involved (this is the approach used in [An], see 2.7.2.1 below). For instance we say that (Z, L) is primitively marked if an isomorphism

$$(L_0, B_0) \tilde{\to} (H_{\mathrm{pr}}, B)$$

is given. So there are $O(L_0, B_0)(L_0)$ such primitively marked structures of (Z, L). Similarly we speak about a level-n primitively marked structure, or about a primitively n-marked structure, or about a level-K primitively marked structure, for K a compact (not necessarily open or admissible) subgroup of $SO(L_0, B_0)(L_0 \otimes \widehat{\mathbb{Z}})$ or of $O(L_0, B_0)(L_0 \otimes \widehat{\mathbb{Z}})$. The advantage of primitively marked structures compared with the marked structure is that the different subgroups involved are "simpler" ($K_O(n)$ is "simpler" than $\tilde{K}_O(n)$). The disadvantage is: the subgroups of $G(\mathbb{A}_f)$ involved are often not admissible. So a level-n primitively marked structure makes sense for any $n \in \mathbb{N}$; if $(2d, n) = 1$ then it is the same as a level-$K_O(n)$ primitively marked structure. Obviously a level-n marked structure defines naturally a level-n primitively marked structure; if n is a multiple of $4d$, then a level-n primitively marked structure defines naturally a level-$\frac{n}{2}$ marked structure.

2.5.3.4. Remark. Let $n \in \mathbb{N}$ and let (Z, L, k_n) be a level-n marked polarized K3 surface over a field k_0 of characteristic zero. Then k_0 can not be too small, in the sense that it must contain some number field (which of course depends on n). For instance, if $n = 3$ does not divide d, then from [An, 8.4.3] we deduce that k_0 must contain a root of unity of order 3. Starting from 2.8.1 it can be checked easily that the same remains true if the characteristic of k_0 is positive and relatively prime to 3.

However this phenomenon is subtler than the one for abelian varieties (see 3.2.4 C) for details).

2.5.3.5. Comments. Let (Z, L) be an arbitrary K3 surface over Y with a pseudo-polarization.

A) We assume here that Y is a $\mathbb{Q}$–scheme. By a marked structure of (Z, L) we mean a level-n marked structure k_n of it for any $n \in \mathbb{N}$, such that there is a $\mathbb{Q}$–scheme Y_1 having the properties:

– Y factors through it;

– the data $(Z, L, (k_n)_{n \in \mathbb{N}})$ descends over Y_1;

– the $\mathbb{C}$-valued points of Y_1 are Zariski dense in Y_1;

– for any $\mathbb{C}$-valued point y of Y_1, these level structures match together into a marked structure of the K3 surface with a pseudo-polarization we get over $\mathbb{C}$ via y.

B) Warning: if $Y = \mathbb{C}$, a marked structure of (Z, L) is not the same thing as a system of compatible level-n marked structures, $n \in \mathbb{N}$.

C) We come back to an arbitrary scheme Y. It is not clear to us when (in general) a level-n marked (resp. primitively marked) structure lifts (after replacing Y by a pro-étale cover of it) to an n-marked (resp. primitively n-marked) structure. Of course this is so if $(n, 2d) = 1$ (cf. 2.5.4 A) below). Also it is not clear to us when a level-n marked or primitively marked structure lifts (after replacing Y by a pro-étale cover of it) to a marked structure. This leads to the following convention:

Convention. All the level-K marked structures to be considered from now on, except otherwise stated, are assumed to be defined by admissible subgroups of $SO(L_0, B_0)(L_0 \otimes \widehat{\mathbb{Z}})$ and are assumed to be liftable (to characteristic zero) to marked structures.

This convention implies that from now on a level-n marked structure is the same thing as a level-$\tilde{K}_{SO}(n)$ marked structure, where $\tilde{K}_{SO}(n) := \tilde{K}_O(n) \cap SO(L_0, B_0)(L_0 \otimes \widehat{\mathbb{Z}})$; if $n \geq 3$ then $\tilde{K}_O(n) = \tilde{K}_{SO}(n)$. Warning: this convention does not apply to the primitively marked structures; so when the things are true in a larger context then the one of the convention, we use primitively marked structures. Based on this convention we could have avoided in what follows the use of the adelic language; however, for greater generality and flexibility, we felt that it is still more appropriate to use it.

D) We consider a level-n marked (resp. primitively marked) K3 surface with a pseudo-polarization (Z, L, k_n) over a field k_0 of characteristic zero. We assume that $n \geq 3$. Then the natural Galois representation $\mathrm{Gal}(\overline{k_0}/k_0) \to GL(H^2_{\mathrm{pr}}(Z_0, \mathbb{A}_f(1)))$, under a natural identification as in 2.5 (obtained via a complex model of (Z, L, k_n)), factors through $\tilde{K}_{SO}(n)$ (resp. through $K(n)$).

2.5.4. Integral aspects. A) The closure $\mathcal{G}$ of G in $GL(L_{\mathrm{st}}[\frac{1}{2}])$ is a reductive group over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. So G_1 and G are unramified over $\mathbb{Q}_p$, for any odd prime p. Moreover, as the bilinear form over $\mathbb{Z}$ associated to the quadratic form $E_8(x_1, ..., x_8)$ is unimodular, we deduce that G and G_1 are unramified over $\mathbb{Q}_2$ as well. But the closure of G in $GL(L_0)$ becomes semisimple only after inverting $2d$ (cf. 2.4.1). We can assume the existence of a a $\mathbb{Z}$-lattice L_W of W such that we get a perfect alternating form $\psi\colon L_W \otimes L_W \to \mathbb{Z}$, and the closure $\mathcal{G}_1$ of G_1 in $GL(L_W[\frac{1}{2}])$ is a reductive group having $\mathcal{G}$ as its adjoint. This is an easy consequence of the fact that for any odd prime p, $\mathcal{G}_{1\mathbb{Z}_p}$ is split, and the spin representation $G_{1\mathbb{Q}_p} \hookrightarrow GL(W \otimes \mathbb{Q}_p)$ is symplectic and irreducible; so multiplying eventually ψ by a non-zero rational number, we can assume that such a $\mathbb{Z}$-lattice L_W does exist. The same argument shows that we can also assume that the closure of G_1 in $GL(L_W \otimes \mathbb{Z}_{(2)})$ is a reductive group over $\mathbb{Z}_{(2)}$. Let $\mathcal{G}_Z$ be the closure of G_Z in $\mathcal{G}$ (or in $GL(L_1[\frac{1}{2}])$). Let $\mathcal{G}_{1Z}$ be the closure of G_{1Z} in $\mathcal{G}_1$ (or in $GL(L_W[\frac{1}{2}])$).

Exercise 1. Let $p \geq 5$ be a prime. Show that the triple $(f_1, L_W \otimes \mathbb{Z}_{(p)}, p)$ is a SHS. Hint: If $p \neq 19$, this is handled in [Va1, 5.7.5], cf. [Va2, 2.3.6]; if $p = 19$ use [Va2, 4.2.1, P1 of 4.6 and 4.6.12] and [BLR, Th. 1 of p. 109].

B) Let $\mathcal{S}$ be a finite set of primes containing the divisors of $6d$. Let l be the product of primes of the subset $\mathcal{S}_0$ of $\mathcal{S}$ of primes relatively prime to $6d$; if $\mathcal{S}_0$ is empty we take $l = 1$. In most applications $l = 1$, but for the sake of generality we do not assume this. Let K and K_1 be as in 2.5.2. We assume that:

i) for any prime p relatively prime to $6dl$ we have $K^p := \mathcal{G}(L_0 \otimes \mathbb{Z}_p) \subset K$ and $K_1^p := \mathcal{G}_1(L_W \otimes \mathbb{Z}_p) \subset K_1$;

ii) K_1 maps (via q) into K;

iii) $\mathrm{Sh}_{K_1}(G_1, X_1)$ and $\mathrm{Sh}_K(G, X)$ are pro-étale covers of smooth quasi-projective $\mathbb{Q}$–schemes (i.e., with the terminology of [Va1, 2.11], K_1 and K are smooth for (G_1, X_1) and respectively for (G, X)).

For instance we can take K_1 to be the subgroup of $G_1(L_W \otimes \widehat{\mathbb{Z}})$ acting trivially modulo n, for some $n \in \mathbb{N}$, $n \geq 3$, and we can take $K = q(K_1)$. This can be easily deduced from Serre's lemma (see [Mu, p. 207]) and the existence of the embedding f_1. From i) we deduce that:

$$K_1^p = K_{1\mathcal{S}} \times K_1^{\mathcal{S}} := K_{1\mathcal{S}} \times \prod_{\text{j a prime}, j \notin \mathcal{S}} K_1^j,$$

where $K_{1\mathcal{S}}$ is a compact subgroup of $G_1(\prod_{j \in \mathcal{S}} \mathbb{Q}_j)$; similarly

$$K = K_{\mathcal{S}} \times K^{\mathcal{S}} := K_{\mathcal{S}} \times \prod_{\text{j a prime}, j \notin \mathcal{S}} K^j,$$

with $K_{\mathcal{S}}$ a compact subgroup of $G(\prod_{j \in \mathcal{S}} \mathbb{Q}_j)$.

C) From B) and [Va1, 2.11, 6.4.4 and 6.4.6 1)] we deduce the existence of a pro-étale cover $\mathcal{N}_1$ (resp. $\mathcal{N}$) of a smooth quasi-projective scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6dl}])$, uniquely determined by the following three properties:

a) its generic fibre is $\mathrm{Sh}_{K_1}(G_1, X_1)$ (resp. $\mathrm{Sh}_K(G, X)$);

b) it has the following extension property: for any extensible pair $(\mathcal{Y}, \mathcal{U})$ with $\mathcal{Y}$ a healthy regular scheme (see [Va, 3.2.1 1-2)] for def.), any morphism $\mathcal{U} \to \mathcal{N}_1$ (resp. $\mathcal{U} \to \mathcal{N}$) extends uniquely to a morphism $\mathcal{Y} \to \mathcal{N}_1$ (resp. $\mathcal{Y} \to \mathcal{N}$).

c) for any prime p relatively prime to $6dl$, the normalization of $\mathcal{N}_{1\mathbb{Z}_{(p)}}$ (resp. of $\mathcal{N}_{\mathbb{Z}_{(p)}}$) in $\mathrm{Sh}_{K_1^p}(G_1, X_1)$ (resp. in $\mathrm{Sh}_{K^p}(G, X)$) has naturally a continuous right action by $G_1(\mathbb{A}_f^p)$ (resp. by $G(\mathbb{A}_f^p)$) with respect to which it is the integral canonical model (see [Va1, 3.2.3 6) and 3.2.6] for def.) of $\mathrm{Sh}(G_1, X_1)$ (resp. of $\mathrm{Sh}(G, X)$) with respect to p, and is a pro-étale cover of $\mathcal{N}_{1\mathbb{Z}_{(p)}}$ (resp. of $\mathcal{N}_{\mathbb{Z}_{(p)}}$).

The unicity part is implied by [Va1, 3.2.4] and c). Related to b) what we need to know in what follows: any regular formally smooth scheme over a DVR of mixt characteristic which is faithfully flat over a localization of $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6dl}])$ and has index of ramification one, is a healthy regular scheme, cf. [Va1, 3.2.2]. With the notations of [Va1, 6.4.4 and 6.4.6 2)] we have $\mathcal{N}_1 = \mathcal{M}_1(K_{1\mathcal{S}})$ and $\mathcal{N} = \mathcal{M}(K_{\mathcal{S}})$.

D) We recall how these schemes $\mathcal{N}_1$ an $\mathcal{N}$ are constructed. We define $K_W(n) := \{g \in GSp(L_W \otimes \widehat{\mathbb{Z}}) | \text{g mod n is the identity}\}$, and $K_1(n) := K_W(n) \cap G_1(\mathbb{A}_f)$. First we assume that:

iv) $K_1 = K_1(n)$ for some $n \in \mathbb{N}$, $n \geq 3$ being a product of primes dividing $6d$, and that $l = 1$.

Under this assumption we have a closed embedding of $\mathbb{Q}$–schemes

$$\mathrm{Sh}_{K_1}(G_1, X_1) \hookrightarrow \mathrm{Sh}_{K_W(n)}(GSp(W, \psi), S).$$

Let $g_W := 2^9$, and let $\mathcal{A}_{g_W,1,n}$ be the moduli scheme over $\operatorname{Spec}(\mathbb{Z}[\frac{1}{6d}])$ of isomorphism classes of principally polarized abelian schemes of dimension g_W having level-n symplectic similitude structure (cf. [Va1, 3.2.9]). From loc. cit. we deduce that $\mathcal{A}_{g_W,1,n_{\mathbb{Q}}}$ can be identified with $\mathrm{Sh}_{K_W(n)}(GSp(W,\psi),S)$. From [Va1, 3.2.12 and the proof of 3.4.1] we get directly:

Theorem. *$\mathcal{N}_1$ is the normalization $\mathcal{N}_1(n)$ of the closure of $\mathrm{Sh}_{K_1}(G_1,X_1)$ in $\mathcal{A}_{g_W,1,n}$.*

If we "put aside" the prime 19, the same thing is implied by [Va1, 5.7.5] and 5.6.1], cf. the proof of [Va1, 3.4.1]; moreover the prime 19 is also resolved by [Va2, 2.3.11] and Exercise 1. See [Va1, 5.6.4] why morally we do not need to take this normalization.

We treat now the arbitrary case (i.e. we are not any more under the assumption iv)). We can assume that K_1 is open. We choose $n \in \mathbb{N}$, $n \geq 3$, n a product of primes (not necessarily distinct) dividing $6dl$ and such that $K_1(n)$ is a normal subgroup of a normal subgroup K_1' of K_1. Then $\mathcal{N}_1$ is obtained by taking the quotient by the finite group K_1/K_1' of the quotient of $\mathcal{N}_1(n)_{\mathbb{Z}[\frac{1}{6dl}]}$ by the finite group $K_1'/K_1(n)$ (cf. [Va1, def. 3.4.8, 6.4.2.1 and 6.4.4 a)]).

From [Va1, 6.2.2 a)] we deduce that $\mathcal{N}$ is obtained by the same method of taking quotients from a scheme $\mathcal{N}_1$ obtained as above but for a K_1 chosen such that K_1 is maximal (still compact subgroup of $G_1(\mathbb{A}_f)$) with the property that $q(K_1)$ is a normal subgroup of K.

E) Let K_1' (resp. K') be a compact subgroup of K_1 (resp. of K). Let $\mathcal{S}'$ be a set of primes containg $\mathcal{S}$. Similarly to l of 2.5.4 B), we define $l' \in \mathbb{N}$. We assume that $q(K_1') \subset K'$ and that i) above is satisfied for K_1' and K', but for l being replaced by l'. So i-iii) bove are satisfied for our "primed" situation. As in B) we get a pro-étale cover $\mathcal{N}_1'$ (resp. $\mathcal{N}'$) of a smooth quasi-projective $\operatorname{Spec}(\mathbb{Z}[\frac{1}{6dl'}])$-scheme, by working with K_1' (resp. K') instead of K_1 (resp. of K). We have the following functorial property described by a commutative diagram of pro-étale covers

$$\begin{array}{ccc} \mathcal{N}_1' & \xrightarrow{q_Z} & \mathcal{N}' \\ \mathrm{nat}\downarrow & & \downarrow\mathrm{nat} \\ \mathcal{N}_{1\mathbb{Z}[\frac{1}{2dl'}]} & \xrightarrow{q} & \mathcal{N}_{\mathbb{Z}[\frac{1}{2dl'}]}. \end{array}$$

Here nat are the natural morphisms (cf. c) and def. 6) of [Va1, 3.2.3]), while the q morphisms are defined naturally starting from [Va1, 3.2.7 4)]. The fact that morphisms involved are pro-finite étale is implied by [Va1, 6.4.5.1].

F) We present now quickly the relative situation. Let $\mathcal{S}_Z$ be a finite set of primes such that it contains $\mathcal{S}$ and the primes j such that $\mathcal{G}_{Z\mathbb{Z}_{(j)}}$ is not reductive. Let l_Z be the product of the primes of $\mathcal{S}_Z$ not dividing $6d$ (we take l_Z to be one if such primes do not exist). We similarly can work with K_{1Z} (resp. K_Z) a compact subgroup of $G_{1Z}(\mathbb{A}_f^p)$ (resp. of $G_Z(\mathbb{A}_f^p)$) such that:

i_Z) for any prime p relatively prime $6dl_Z$ we have $K_{1Z}^p := \mathcal{G}_{1Z}(L_W \otimes \mathbb{Z}_p) \subset K_{1Z}$ and $K_Z^p := \mathcal{G}_Z(L_0 \otimes \mathbb{Z}_p) \subset K_Z$;

ii_Z) we have $q(K_{1Z}) \subset K_Z$;

iii_Z) the schemes $\mathrm{Sh}_{K_{1Z}}(G_{1Z}, X_{1Z})$ and $\mathrm{Sh}_{K_Z}(G_Z, X_Z)$ are pro-étale covers of quasi-projective smooth $E(G_Z, X_Z)$-schemes.

Let $E_Z := E(G_Z, X_Z) = E(G_{1Z}, X_{1Z})$. Warning: it is quite common that $E_Z \neq \mathbb{Q}$ (plenty of such examples can be constructed with G_Z a torus). Let $O_Z := O_{E_Z}\left[\frac{1}{6dl_Z}\right]$, where O_{E_Z} is the ring of integers of E_Z. As in C) above we deduce the existence of pro-étale covers $\mathcal{N}_{1Z}$ and $\mathcal{N}_Z$ of quasi-projective smooth O_Z-schemes, having entirely the same properties. In particular we have a pro-étale cover $q_Z\colon \mathcal{N}_{1Z} \to \mathcal{N}_Z$. The functorial aspects of E) still hold in this relative situation.

G) If in C) to F) above the compact groups are also open, then the pro-étale covers become étale covers.

2.6. Moduli schemes over $\mathbb{C}$. A) From [BB] we deduce that $\mathrm{Sh}(G, X)_{\mathbb{C}}$ is a projective limit of smooth quasi-projective $\mathbb{C}$-schemes, with finite étale transition morphisms (see also [Va1, 2.3]). In particular it has a natural structure as a $\mathbb{C}$-scheme. In [PSS] (see also [LP] and [Be2]) it is shown the existence of a coarse moduli scheme $\mathcal{K}3_{d,p,\mathrm{marked}}$ over $\mathbb{C}$ of marked polarized K3 surfaces of degree d. We have $\mathcal{K}3_{d,p,\mathrm{marked}}(\mathbb{C}) = \mathcal{PSS}$. In [PSS] it is shown that $\mathcal{PSS}$ is the set of complex points of an open dense subscheme of a union $\mathcal{K}3_{d,pp,\mathrm{marked}}$ (see 2.6.1 for an explanation of the notation; pp stands for pseudo-polarized) of two distinct connected components of $\mathrm{Sh}(G, X)_{\mathbb{C}}$, which can be identified to $\mathcal{K}3_{d,p,\mathrm{marked}}$.

The two connected components are interchanged naturally by any element of $\Gamma_d^{SO} \setminus G(\mathbb{R})^0$.

B) It is easy to see that

$$\Gamma_d^0 := \Gamma_d \cap G(\mathbb{R})^0$$

is a normal subgroup of Γ_d of index 2. Obviously it is an arithmetic subgroup of $G(\mathbb{Q})$ which acts naturally (it is a left action) on X^0. The loc. cit. (see also [Be2]) shows as well the existence of a coarse quasi-projective moduli scheme $\mathcal{K}3_{d,p}$ over $\mathbb{C}$ of unmarked polarized K3 surfaces of degree d. We have a canonical identification:

$$\mathcal{K}3_{d,p} = \Gamma_d \backslash \mathcal{K}3_{d,p,\mathrm{marked}}.$$

$\mathcal{K}3_{d,p}$ has just one connected component and is isomorphic to an open subscheme of one connected component $\mathcal{K}3_{d,pp}$ of $\mathrm{Sh}_{K(d,SO)}(G, X)_{\mathbb{C}}$, where

$$K(d, SO) := \{g \in SO(\tilde{L}_0, \tilde{B}_0)(\tilde{L}_0 \otimes \widehat{\mathbb{Z}}) | g \text{ fixes } l_1 + dl_2\}.$$

So $\mathcal{K}3_{d,p}$ can be identified with $\Gamma_d^0 \backslash X^0$.

C) Let $n \in \mathbb{N}$. Let $\Gamma_{d,n} := \Gamma_d \cap \tilde{K}_{SO}(n)$ and let $\Gamma_{d,n}^0 := \Gamma_d^0 \cap \tilde{K}_{SO}(n)$. Then

$$\Gamma_{d,n}^0 \backslash X^0$$

is a connected component of $\mathrm{Sh}_{\tilde{K}_{SO}(n)}(G, X)_{\mathbb{C}}$, and there is a coarse quasi-projective moduli scheme $\mathcal{K}3_{d,p,n}$ over $\mathbb{C}$ of level-n marked polarized K3 surfaces of degree d, which has one (resp. two) connected component (resp. components) if $\Gamma_{d,n} \neq \Gamma_{d,n}^0$ (resp. if $\Gamma_{d,n} =$

$\Gamma^0_{d,n}$) and is an open subscheme of a union $\mathcal{K}3_{d,pp,n}$ of one (resp. two distinct) connected component (resp. components) of $\mathrm{Sh}_{\tilde{K}_{SO(n)}}(G,X)_{\mathbb{C}}$.

If $n \geq 3$ then it is easy to see that $\Gamma_{d,n} = \Gamma^0_{d,n}$. If n divides $n_1 \in \mathbb{N}$, then we have a natural étale cover

$$\mathcal{K}3_{d,p,n_1} \to \mathcal{K}3_{d,p,n}.$$

D) If moreover $n \geq 3$, from 2.5.4 B) we deduce that $\mathcal{K}3_{d,p,n}$ is smooth over $\mathbb{C}$. It is a moduli scheme over $\mathbb{C}$ of level-n marked polarized K3 surfaces of degree d. This can be checked in the same manner as for the case of principally polarized abelian schemes (see [FC, Ch. I, 4.11]), starting from the following well-known result (Serre's lemma for K3 surfaces):

Theorem. *Let $m \in \mathbb{N}$, $m \geq 3$. We assume that $2md$ is invertible in Y. Then any automorphism ϕ of a level-m primitively marked pseudo-polarized K3 surface (Z, L) of degree d over Y is trivial.*

Proof: Using Rudakov–Shafarevich theorem (see [RS1] and [De4, 1.1 b)]) we deduce the absence of infinitesimal automorphisms of a pseudo-polarized K3 surface. So we can assume that Y is the spectrum of a field. We first recall briefly the argument in characteristic zero. We can assume that $Y = \mathrm{Spec}(\mathbb{C})$. A theorem of Beauville (see [Be3, Prop. 6] for a proof) shows that we have a natural monomorphism

$$\mathrm{Aut}(Z) \hookrightarrow \mathrm{Aut}(H^2(Z,\mathbb{Z})(1))^{\mathrm{opp}}.$$

So Serre' s lemma (see [Mu, p. 207]) applies directly (cf. 2.5.4 B)).

We recall now the argument in the case Y is of positive characteristic p. Standard arguments (of specializing to finite fields) show that ϕ has finite order. From here there are two ways to proceed further. The first one: from Serre's lemma we deduce that ϕ acts trivially on $H^2(Z, \mathbb{Z}_r)$ for a suitable prime r dividing n; using this, it can be deduced easily that Φ is trivial by considering the quotient surface $\tilde{Z}$ of Z by ϕ and analysis the situation in the $\mathbb{Z}_r$-étale context.

The second way is to use the characteristic zero result and the fact expressed in the following exercise:

Exercise 2. Show that any automorphism of prime order of (Z, L) lifts to characteristic zero. Hint: Use [Va1, 3.4.5 and 6.2.2 G) and H)] and the deformation theory recalled in 2.8.

E) Even better, if $\Gamma^0_{d,K} := \Gamma^0_d \cap K$, with K an admissible compact subgroup of $SO(L_0, B_0)(L_0 \otimes \widehat{\mathbb{Z}})$, then

$$\Gamma^0_{d,K} \backslash X^0$$

is a connected component of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$, and there is a coarse moduli scheme $\mathcal{K}3_{d,p,K}$ of level-K marked polarized K3 surfaces of degree d, which has one or two connected components and is an open dense subscheme of a union $\mathcal{K}3_{d,pp,K}$ of one or two connected components of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$; it is a moduli scheme if K is smooth for (G,X) (for instance if $K \subset K(n)$ with $n \geq 3$).

2.6.1. The pseudo-polarized context. From [Ku] (see also [PP], [Be3, p. 150] and [KT]) we deduce that $\mathcal{K}3_{d,pp,marked}$ (resp. $\mathcal{K}3_{d,pp}$) is the moduli scheme (resp. the coarse moduli scheme) parametrizing isomorphism classes of marked (resp. unmarked) pseudo-polarized K3 surfaces of degree d. To show that the similar statements remain true for $\mathcal{K}3_{d,pp,n}$ and $\mathcal{K}3_{d,pp,K}$ (with K as above), we just need to define what is a level-K marked structure for a pseudo-polarized K3 surface (Z, L) over an arbitrary scheme Y (over which such a structure can be defined). There are a couple of possibilities in defining it. One way is to try to follow the pattern (see [FC, §6 of ch. IV]) of semi-abelian varieties in comparison to abelian varieties. The difficulty in loc. cit. is: a semi-abelian scheme of some relative dimension $g \in \mathbb{N}$, can have fibres whose toric part is of dimension some (varying) number of the set $\{0, ..., g\}$. In our case the singularities of the geometric fibres of Z are varying in number: their number can be any number of the set $\{0, ..., 19\}$. Of course by specialization the number of singular points is not decreasing. Here we adopt a simpler way (somehow non-standard).

2.6.1.1. Definition. We say that (Z, L) is level-K marked if:

– for any point y of Y with values in a field k_0, Z_y can be lifted to a pseudo-polarized K3 surface (Z_V, L_V) over a DVR V having k_0 as its residue field in such a way that its generic fibre $(Z_{K(V)}, L_{K(V)})$ over the field of fractions $K(V)$ of V is a polarized K3 surface having a level-K marked structure $\mathcal{L}_{K(V)}$;

– the level-K structures are compatible in the usual sense, that if two such data $(Z_{K(V)}, L_{K(V)}, \mathcal{L}_{K(V)})$ and $(Z_{K(U)}, L_{K(U)}, \mathcal{L}_{K(U)})$ are given which specializes to the same one obtained through the point y, then they are obtained by specializing a particular level-K marked polarized K3 surface.

2.6.1.2. Remark. We assume that Y is the spectrum of a field k_0 of characteristic relatively prime to $2d$. It seems to us (cf. also 2.7.2.2 4)) that if (Z, L) has level-n marked structure for $n \geq 3$, then the singularities of Z are defined over k_0.

2.6.2. Remark. If $n \geq 3$, using the Theorem of D) above, we can define as well a moduli scheme $\mathcal{K}3_{d,p,n,pr}$ (resp.$\mathcal{K}3_{d,p,n,pr}$) of isomorphism classes of level-n primitively marked polarized (resp. pseudo-polarized) K3 surfaces over $\mathbb{C}$-schemes. Warning: as the convention of 2.5.3.5 C) does not apply for primitively marked structure, they can have more than two connected components.

2.7. The Kuga–Satake construction. Let $f_2: (G_1, X_1) \hookrightarrow (GSp(W_2, \psi_2), S_2)$ be an arbitrary embedding into a pair defining a Siegel modular variety. Let

$$f_3: (G_1, X_1) \hookrightarrow (GSp(W_3, \psi_3), S_3)$$

be the injective map defined in [KS]. To recall in detail how f_3 is defined we follow the language of [De5] and [An, §4], in the context of present notations. We consider the even Clifford algebra $C^+(L_0)$ over $\mathbb{Z}$. We also consider the even Clifford group G_2 over $\mathbb{Q}$. Its $\mathbb{Q}$–rational points are the invertible elements γ of $C^+(L_0)(\mathbb{Q})$ such that

$$\gamma L_0 \otimes \mathbb{Q} \gamma^{-1} = L_0 \otimes \mathbb{Q}.$$

It is naturally identified to G_1. Let L_3 be a free (left) $C^+(L_0)$-module of rank one. We have naturally (but not canonically; see [De5] or [An, 4.3]) a nondegenerate (perfect if we invert $2d$) sympletic form $\psi_3\colon L_3 \otimes L_3 \to \mathbb{Z}$, fixed by G_2. Let $W_3 := L_3 \otimes \mathbb{Q}$. Then the natural inclusion $G_1 = G_2 \hookrightarrow GSp(W_3, \psi_3)$ defines f_3.

Let $n \in \mathbb{N}$. We now review Kuga–Satake construction at the level of level-n (resp. level-n marked) polarized abelian varieties (resp. K3 surfaces with a pseudo-polarization). Let L_3' be an arbitrary $\mathbb{Z}$-lattice of W_3 such that $\psi_3\colon L_3' \otimes L_3' \to \mathbb{Z}$ is perfect and $L_3\big[\frac{1}{2d}\big] = L_3\big[\frac{1}{2d}\big]$. Let (Z, L, k_n) be a level-n marked K3 surface with a pseudo-polarization of degree d over $\mathbb{C}$. Let (cf. Riemann's theorem; see [De1, 4.7]) A_Z (resp. $(A_Z', p_{A_Z'})$) be the complex abelian variety (resp. principally polarized abelian variety) defined by (L_3, h_{1Z}) (resp. defined by (L_3', h_{1Z}, ψ_3). The association: A_Z to (Z, L), is called the Kuga–Satake construction. The association: $(Z_Z', p_{A_Z'})$ to (Z, L) will be called here as a modified p.p. Kuga–Satake construction (here p.p. stands for principally polarized). Obviously A_Z and A_Z' are isogeneous. Moreover, if n is realtively prime to $2d$, then to k_n it is associated naturally a level-n symplectic similitude structure of $(A_Z', p_{A_Z'})$ (still denoted by k_n).

Similarly, for any $\mathbb{Z}$-lattice L_2 of W_2 such that $\psi_2\colon L_2 \otimes L_2 \to \mathbb{Z}$ is perfect we speak about a modified p.p. Kuga–Satake construction. In what follows, for the case $f_2 = f_1$ and $L_2 = L_W$ we will speak about the Satake construction, as the embedding f_1 is just the $\mathbb{Q}$–version of the $\mathbb{R}$-one of [Sa, p. 458] and as [Sa] is an earlier work than [KS].

2.7.1. Exercise 3. Show that through all these modified p.p. Kuga–Satake constructions (starting from a fixed pair (Z, L)), the different abelian varieties obtained are isogeneous to powers of the abelian variety associated (to (Z, L)) via f_1 and L_W. Hint: use [De2, 1.3.9] and the fact that $G_1^{\mathrm{ab}} = \mathbb{G}_m$ to show that the representation $G_1^{\mathrm{der}} \to GL(W_3)$ is a direct sum of a number of copies of the representation $G_1^{\mathrm{der}} \to GL(W_1)$.

This exercise motivates why in general we prefer to work with f_1 instead of f_3 (cf. also the smallest dimension possible property enjoyed by $\dim(W)$; see 2.5).

2.7.2. A digression. A natural question arises:

Q. *How unique is the Satake construction, and what advantages it offers in comparison to the Kuga–Satake construction?*

To give an answer to this question we start remarking that the symplectic form ψ is uniquely determined by f_1 up to multiplication by a non-zero rational scalar (as $f_{1\mathbb{C}}$ is irreducible; see 2.5); the scalar has to be positive in order to ensure that $\varepsilon\psi$ and $r\varepsilon\psi$ are both defining a polarization of the Hodge $\mathbb{Q}$–structure of W defined by h_{1Z} (here $\varepsilon \in \{-1, 1\}$). Let $r \in \mathbb{Q}_+ := \mathbb{Q} \cap (0, \infty)$, and let L_W^1 be another $\mathbb{Z}$-lattice of $W \otimes \mathbb{Q}$ such that:

i) $r\psi\colon L_W^1 \otimes L_W^1 \to \mathbb{Z}$ is perfect;
ii) $\mathcal{G}_1$ is a reductive subgroup of $GSp(L_W^1\big[\frac{1}{2}\big], \psi)$.

Let p be a prime not dividing $2d$. The spin representations of the split group $\mathcal{G}_{1\mathbb{F}_p}$ on $L_W \otimes \mathbb{F}_p$ and on $L_W^1 \otimes \mathbb{F}_p$ are irreducible (see [Bo]). This implies that $L_W \otimes \mathbb{Z}_p$ and $L_W^1 \otimes \mathbb{Z}_p$ are proportional. We deduce the existence of an element $\tilde{m} \in \mathbb{Q}$ such that $\tilde{m}L_W \otimes \mathbb{Z}\big[\frac{1}{2d}\big] = L_W^1 \otimes \mathbb{Z}\big[\frac{1}{2d}\big]$. So we can assume that:

iii) $L_W \otimes \mathbb{Z}\big[\frac{1}{2d}\big] = L_W^1 \otimes \mathbb{Z}\big[\frac{1}{2d}\big]$, and that
iv) r is an invertible element of $\mathbb{Z}\big[\frac{1}{2d}\big]$.

So r is a product of primes dividing $2d$ and of their inverses. We do not know what is the best way to handle these primes. Of course a model of $\mathcal{G}_1$ over $\mathrm{Spec}(\mathbb{Z})$ has to be chosen; this implies a first choice (as natural as possible) of a model $\mathcal{G}_{\mathbb{Z}}$ of $\mathcal{G}$ over $\mathrm{Spec}(\mathbb{Z})$. A first choice (the most natural one) for $\mathcal{G}_{\mathbb{Z}}$ would be the closure of G in $GL(L_0)$; a second choice (also quite natural) is to take the closure $\tilde{\mathcal{G}}$ of G in $GL(L_0')$, where L_0' is generated by $\frac{1}{d_1}(l_1 - dl_2)$, l_3,..., l_{21}, with $d_1 \in \mathbb{N}$ defined by $d = d_1^2 d'$. We can assume that (cf. the proof of the Fact of 2.5) that $L_{\mathrm{st}}\left[\frac{1}{d'}\right] = L_0'\left[\frac{1}{d'}\right]$. Obviously $\tilde{\mathcal{G}}_{\mathbb{Z}[\frac{1}{d'}]}$ is a reductive group. If $\tilde{\mathcal{G}}_1$ is the non-trivial extension of $\tilde{\mathcal{G}}$ by $\mathbb{G}_m$, then we get (cf. 2.4.5 A)):

v) $\tilde{\mathcal{G}}_{1\mathbb{Z}[\frac{1}{d'}]}$ is a reductive subgroup of $GL(L_W\left[\frac{1}{d'}\right])$.

If we impose that $\tilde{\mathcal{G}}_{1\mathbb{Z}[\frac{1}{d'}]}$ is a reductive subgroup of $GL(L_W^1\left[\frac{1}{d'}\right])$ as well, then as bove we can assume that:

vi) r is invertible in $\mathbb{Z}\left[\frac{1}{d'}\right]$, and we have $L_W\left[\frac{1}{d'}\right] = L_W^1\left[\frac{1}{d'}\right]$.

Moreover we can assume that:

vii) r is square free.

2.7.2.1. A comparison. Here we put together what we can get using vi) and [An, 1.7 and 8.4.3]. Let $\tilde{\mathcal{G}}_1^{KS}$ be the closure of G_1 in $GL(L_3)$. From the very constructions we have:

$$\tilde{\mathcal{G}}^{KS}_{1\mathbb{Z}[\frac{1}{2d}]} = \mathcal{G}_{1\mathbb{Z}[\frac{1}{2d}]}. \tag{2}$$

The representation of G_1 on W_3 is isomorphic to 2^{10} copies of the representation of G_1 on W_1 (see also [An, 4.2]). Moreover from (2) above we deduce (the argument is the same as the one used to get iii) and iv) above), that we can choose a G_1-isomorphism

$$i_Z : W^{2^{10}} \tilde{\to} W_3$$

such that:

viii) $L_W^{2^{10}}\left[\frac{1}{2d}\right]$ is mapped onto $L_3\left[\frac{1}{2d}\right]$.

Let (A, p_A) (resp. (A_1, p_{A_1}) be the principally polarized abelian scheme defined by the triple (L_W, ψ, h_{1Z}) (resp. by $(L_W^1, r\psi, h_{1Z})$). Let A_Z be as above in 2.7. The isomorphism i_Z achieves a $\mathbb{Z}\left[\frac{1}{2d}\right]$-isogeny between $A^{2^{10}}$ and A_Z. Moreover A and A_1 are $\mathbb{Z}\left[\frac{1}{d'}\right]$-isogeneous. Let $m \in \mathbb{Z}\left[\frac{1}{2d}\right] \cap \mathbb{Q}_+$ be an invertible element such that (via i_Z) we have $mL_W^{2^{10}} \subset L_3$. Let $m_1 \in \mathbb{N}$ be the smallest number such that $m_1 L_3 \subset mL_W^{2^{10}}$. We usually choose m and m_1 so that m_1 takes the smallest possible value $m_{d'} \in \mathbb{N}$. We have:

A. Proposition. *Let k_0 be an arbitrary subfield of $\mathbb{C}$. We assume that (Z, L) is obtained by extension of scalars from a K3 surface with a pseudo-polarization (Z_0, L_0) over k_0. Then there is a finite field extension k_1 of k_0 such that there is a principally polarized abelian variety (A_0, p_{A_0}) over k_0 having the properties:*

a) its extension to $\mathbb{C}$ is (A, p_A) (so its Mumford-Tate group can be identified naturally with the subgroup G_{1Z} of G_1);

b) it does not depend on the embedding of k_0 in $\mathbb{C}$;

c) if l is a prime relatively prime to $2d$ (resp. dividing $2d$), then the natural Galois representation $Gal(\bar{k}_0/k_0) \to \mathcal{G}_{\mathbb{Z}_l}(\mathbb{Z}_l)$ (resp. $Gal(\bar{k}_0/k_0) \to G(\mathbb{Q}_l)$) defined by A_0 (via the adjoint map of its factorization through $\mathcal{G}_{1\mathbb{Z}_l}$) is isomorphic to the one defined by (Z_0, L_0).

Proof: We first remark that the standard arguments (see [Hart, 7.10 of p. 161]) allow us to pass to a polarized context (see also [An, 4.1.5]). So [An, 1.7.1 and 8.5] applies directly.

Following [An, 4.4] we denote by $\mathbb{K}_n^{\mathrm{ad}}$ the image in $G(L_0 \otimes \widehat{\mathbb{Z}})$ of the subgroup of $G_1(L_3 \otimes \widehat{\mathbb{Z}})$ formed by elements congruent to 1 mod n. If $(n, 2d) = 1$, then $\mathbb{K}_n^{\mathrm{ad}} = K(n)$. Putting the above proposition together with [An, 8.4.3] we obtain :

B. Proposition. *We assume also that (Z_0, L_0) is level-$m_{d'}$ primitively marked and level-$\mathbb{K}_n^{\mathrm{ad}}$ primitively marked, for some $n \in \{3, 4\}$. Then in the above proposition we can take $k_1 = k_0$.*

Proof: From loc. cit. we deduce that it is enough to show that p_{A_0} is defined over k_0. But this is an immediate consequence of the following fact (with l an arbitrary odd prime):

Fact. *$G(\mathbb{Q}_l)$ is its own normalizer in $GL(W \otimes \mathbb{Q}_l)$.*

The Fact itself is implied by the fact that $f_{1\mathbb{C}}$ is irreducible, and that $G_{\mathbb{Q}_l}$ is split (being unramified) and has no outer automorphism (being of B_{10} Lie type).

2.7.2.2. Remarks. 1) Properties vi) and vii) above can be used as a substitute of the rigid conditions used in [An, 5.4]; so they can be used as well to perform the necessary descent of 2.7.2.1 A. So one can reobtain [An, 1.7.1 and 8.4.3] for K3 surfaces, working (in the context of the Satake construction) with them.

2) The number $m_{d'}$ depends only on d'. It is a product of primes dividing d'. It is easy to obtain upper bounds for it, but this is entirely irrelevant for what follows.

3) If $d' = 1$ (i.e. if d is twice a square) and we are in the context of 2.7.2.1 B, then (A_0, p_{A_0}) will be referred (following the pattern of [An, 8.5.2-3]) as the Satake canonical principally polarized abelian variety attached to (Z_0, L_0).

4) The conditions of 2.7.2 1 B are automatically satisfied if (Z_0, L_0) is $2d'$-marked.

5) In [An, 8.4.3] we can not replace $n \in \{3, 4\}$ by an arbitrary $n \geq 3$ (to be compared with 3.2.4 C)). So in 2.7.2.1 B we can not perform this general replacement as well (of course depending on $m_{d'}$, some finite values of n might still work).

6) We refer to 2.6.1.2. If (Z, L) is level-$\mathbb{K}_n^{\mathrm{ad}}$ primitively marked, for some $n \in \{3, 4\}$, then its singularities are defined over k_0. In characteristic zero, this is an easy consequence of [An, 8.4.3], while Claim 2 of the proof of 3.2 allows us to lift the positive characteristic situation to a characteristic zero situation.

2.8. Deformation theory. The main references for this section are [De4] and [De6]. We will state the results for a perfect field (not necessarily algebraically closed). Let now k be a perfect field of characteristic $p > 0$. Let Z be an arbitrary K3 surface. In [De4, 1.2] it is shown that there is a versal deformation of Z over $\mathcal{Z} := \mathrm{Spf}(W(k)[[t_1, ..., t_{20}]])$, where t_i's are independent variables. Let now L be a non-trivial line bundle on Z. In [De4, 1.6] it is shown the existence of $f_L \in W(k)[[t_1, ..., t_{20}]]$ not divisible by p and such that $\mathcal{ZL}_f := \mathrm{Spf}(W(k)[[t_1, ..., t_{20}]]/(f_L))$ is the formal scheme of deformations of (Z, L). If moreover L is ample, then Grothendieck's algebrization theorem (see [EGA, 5.4.5]) shows that we have a versal deformation of (Z,L) over

$$\mathcal{ZL} := \mathrm{Spec}(W(k)[[t_1,...,t_{20}]]/(f_L)).$$

2.8.1. We consider now the case when $L.L = 2d$, with $(p,2d) = 1$. In this case the perfect form $H^2_{\mathrm{crys}}(Z/W(k)) \otimes H^2_{\mathrm{crys}}(Z/W(k)) \to W(k)(2)$ is such that we have a direct sum decomposition

$$H^2_{\mathrm{crys}}(Z/W(k)) =< L > W(k) \oplus H^2_{\mathrm{pr}},$$

where H^2_{pr} is the perpendicular to $< L >$ with respect to this form. This implies (cf. [De6, 2.1.11]; see also below) that (Z,L) has a formal deformation over the formal spectrum of the ring of formal power series in 19 variables, whose Kodaira-Spencer map is injective. So $\mathcal{ZL}_f$ is formally smooth. In particular (Z,L) has a lift (Z_1,L_1) over $\mathrm{Spf}(W(k))$. Associated to it, $H^2 := H^2_{\mathrm{crys}}(Z/W(k))$, gets a natural (Hodge) filtration $H^2 = F^0(H^2) \supset F^1(H^2) \supset F^2(H^2) \supset F^3(H^2) = 0$ (see [De4, p. 66]); $F^1(H^2)$ and $F^2(H^2)$ are direct summands of H^2 of ranks 21 and respectively 1. With respect to the cup product, we have ([De4, p. 67]) $F^2(H^2) = F^1(H^2)^\perp$. Moreover (cf. [De6, 2.1.1]), the lifts of Z to $\mathrm{Spf}(W(k))$ are in one-to-one correspondace with such a filtration of H^2.

As one usually twists $H^2_{\text{ét}}$ with $\mathbb{Z}_r$ coefficienrs by $\mathbb{Z}_r(1)$ (here r is a prime), in the same manner H^2 is twisted by $W(k)(-1)$. So $H_{\mathrm{pr}} := H^2_{\mathrm{pr}}(-1)$ has a natural filtration by direct summands

$$H_{\mathrm{pr}} = F^{-1}(H_{\mathrm{pr}}) \supset F^0(H_{\mathrm{pr}}) \supset F^1(H_{\mathrm{pr}}) \supset F^2(H_{\mathrm{pr}}) = 0,$$

such that the rank of $F^0(H_{\mathrm{pr}})$ (resp. of $F^1(H_{\mathrm{pr}})$) is 20 (resp. 1) and we have

$$F^0(H_{\mathrm{pr}}) = F^1(H_{\mathrm{pr}})^\perp$$

with respect to the induced perfect form B_{crys} on H_{pr}. From [De6, 2.1.1] we deduce that the lifts of (Z,L) to $\mathrm{Spf}(W(k))$ are in one-to-one correspondance to such filtrations of H_{pr}. This implies that $\mathcal{ZL}_f$ is canonically identifiable with the formal spectrum of the completion of $SO(H_{\mathrm{pr}}, B_{\mathrm{crys}})/P^1_{SO}$ in the origin, where P^1_{SO} is the parabolic subgroup of $SO(H_{\mathrm{pr}}, B_{\mathrm{crys}})$ leaving invariant the filtration of H_{pr} defined by the (arbitrarily fixed) lift (Z_1,L_1).

We denote by (H_{pr},ϕ_Z) the F-crystal defined by (or associated to) (Z,L). So

$$\mathbb{H} := (H_{\mathrm{pr}}, F^0(H_{\mathrm{pr}}), F^1(H_{\mathrm{pr}}), \phi_Z)$$

is a p-divisible object of $\mathcal{MF}_{[-1,1]}(W(k))$ (cf. the terminology of [Va2, 2.1]). When we want to emphasize B_{crys}, we speak about the polarized F-crystal defined by (Z,L), or by a polarized p-divisible object of $\mathcal{MF}_{[-1,1]}(W(k))$ (defined by (Z_1,L_1)).

Let $\mathfrak{g} := \mathrm{Lie}(SO(H_{\mathrm{pr}}, B_{\mathrm{crys}}))$. It has a natural filtration $F^i(\mathfrak{g})_{i=\overline{-1,1}}$ defined by the above filtration of H_{pr}. We obtain (cf. the terminology of [Va2, 2.2]) a Shimura filtered Lie σ-crystal

$$(\mathfrak{g}, F^0(\mathfrak{g}), F^1(\mathfrak{g}), \phi_Z)$$

associated to (Z_1,L_1).

2.8.2. We do not assume anymore that $(p,2d) = 1$. As above we define H_{pr} and ϕ_Z. By the Newton polygon of (Z,L) we mean the Newton polygon of the isocrystal defined by $(H_{\mathrm{pr}}[\frac{1}{p}], \phi_Z)$.

§3. The basic result: the construction of moduli stacks (and schemes)

Let $d \in \mathbb{N}$. In this chapter we show (see 3.2.4 A)) that the moduli $\mathbb{Z}$-stack $\mathcal{A}_{K3,d,p}$ (resp. $\mathcal{A}_{K3,d,pp}$) of polarized (resp. pseudo-polarized) K3 surfaces of degree d, over an étale connected cover $\mathrm{Spec}(O_d)$ of $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6d}])$ is an open substack of a suitable $\mathrm{Spec}(O_d)$-stack of the Shimura variety $\mathrm{Sh}(G,X)$. We start by extending part of [Va1, 3.2-3] to the finite type context in order to have a convenient context and language for 3.2. 3.2 starts with the context of smooth moduli schemes and ends with its variants and direct consequences of its proof. 3.3-4 refer to some possibilities of extending the results of 3.2.

3.1. Integral canonical models. Let D be an integral Dedekind ring and let K_D be its field of fractions. We assume that K_D is of characteristic zero and that all the maximal ideals of D are of positive characteristic. To our knowledge, Néron was the first one (see [BLR] for a comprehensive exposition) to define good integral models over D of some smooth quasi-projective schemes over K_D, using some extension property. In [Mi4, 2.1-9] (see also [Mi3, footnote p. 513]) this idea (of using extension properties) is used (for the first time to our knowledge) in the context of moduli schemes. The ideas of loc. cit. were exploited successfully in [Va1] for the construction of integral canonical models of Shimura varieties of preabelian type.

For a general incipient theory of (and terminology involving) healhy schemes and of different extension properties (like EP, EEP, WEP, REP, SEP, LEP, QEP, QEEP, etc.) they define we refer to [Va1, 3.2-3]. Loc. cit. is well adapted when we are dealing with projective limits (sort of "infinite level structures") of schemes. Here we adapt part of loc. cit. to the context of smooth separated K_D-schemes of finite type. The main goal is: to create a convenient language and context for 3.2 below, and to provide a framework for other contexts where one can define moduli schemes by (or by just using) some extension property.

Let Y_{K_D} be a smooth separeted integral K_D-scheme of finite type. The general problem is to define "good" integral models of Y_{K_D} over D. For simplifying the presentation we often assume that $D = V$ is a DVR, faithfully flat over $\mathbb{Z}_{(p)}$, with p a prime.

3.1.1. Definitions. a) A smooth separated faithfully flat D-scheme Y of finite type is said to be an integral canonical model (resp. an integral E-canonical model) of Y_{K_D} if:

i) its generic fibre is Y_{K_D};

ii) for any extensible pair $(\mathcal{Y},\mathcal{U})$ over D, with Y a healthy regular scheme (resp. with Y an almost healthy normal scheme), every morphism $\mathcal{U} \to Y$ extends (uniquely) to a morphism $\mathcal{Y} \to Y$.

b) If in ii) above we work with healthy regular formally smooth schemes $\mathcal{Y}$ over Dedekind D-algebras whose local rings have the same indices of ramification as the local rings of D they dominate, then we speak about integral S-canonical models or about integral smooth-canonical models.

c) If in ii) above we work with abstract (resp. quasi-compact) healthy regular D-schemes $\mathcal{Y}$, then we speak about integral W-canonical (resp. Q-canonical) models or about integral weak canonical models.

d) Following the same pattern we speak about integral L-canonical models (resp. about integral R-canonical models), if in ii) above we use locally healthy regular schemes (resp. R-healthy regular schemes) $\mathcal{Y}$.

In what follows $* \in \{-, E, S, W, L, R, Q\}$. If $* = -$ then by integral $*$-canonical model we mean integral canonical model.

3.1.2. Examples. a) An abelian scheme A over D is an integral S-canonical model of its generic fibre A_{K_D} (cf. [BLR, p. 12]). The same remains true for the Néron model A over D of an abelian variety A_{K_D} over D, provided all the residue fields of D of positive characteristic are perfect (cf. [BLR, p. 176]).

b) The schemes $\mathcal{M}(H_{\mathcal{S}})$ introduced in [Va1, 6.4.4], starting from an arbitrary Shimura pair (G, X) of preabelian type, are integral canonical models of their generic fibres, and many of them are in fact integral E-canonical models of their generic fibres (cf. [Va1, 6.4.9]). We will refer to (the extension to Dedekind rings faithfully flat over some localization of $\mathbb{Z}$ of) such schemes, as a finite type integral canonical model of $\mathrm{Sh}(G, X)$, for not creating confusion with the terminology of [Va1, 3.2-3].

c) A smooth faithfully flat D-scheme finite over a scheme Y which is an integral canonical $*$-model, is itself an integral $*$-canonical model of its generic fibre (cf. [Va1, 3.2.3.1 5)]).

d) If a faithfully flat D-scheme Y_0 is an open subscheme of an integral $*$-canonical model Y such that $Y \setminus Y_0$ is of pure codimension 1 in Y, then Y_0 is an integral $*$-canonical model.

e) The notion of integral $*$-canonical model is well behaved with respect to finite products.

f) Let Y be a smooth separated faithfully flat D-scheme of finite type, such that it has a pro-étale cover Y_∞ having the EP. Then Y is an integral canonical model (cf. [Va1, 3.2.2 4)] and the classical purity theorem). The same remains true for integral $*$-canonical models, provided $* \in \{S, L, W, R, Q\}$.

g) Abelian schemes over integral $*$-canonical models, are integral $*$-canonical models (cf. [BLR, th. 1 of p. 109]).

h) Regardless of how D is, the projective spaces $\mathbb{P}^n_D$ are not integral S-canonical models.

3.1.3. Remarks. a) Different implications between different extension properties (as examples see [Va1, 3.2.3.1 4)]), imply implications between different integral canonical models. We will not state here all these implications. Just one example: in 3.1.2 b) we get integral canonical models which are also integral S-canonical models (cf. [Va1, 3.2.2 1)]). It is worth mentioning that part of implications in the case $D = V$ depend on the fact that the index of ramification of V is smaller than $p - 1$ or not (for instance see loc. cit.).

b) For different philosophies underlying different types of healthy normal schemes, we refer to [Va1, 3.2.7 6) and 3.2.2 5)].

c) It is hard to predict what extra (geometric) conditions one needs to impose on Y_{K_D} in order to have an integral $*$-canonical model. For instance in 3.1.2 a) we have examples where the canonical class is zero, while 3.1.2 b) provides us with plenty of examples of varieties of general type.

d) If Y is an integral S-canonical model, then there is no smooth separated scheme Y_1 containing Y_0 as an open subscheme, and such that $Y_{0K_D} = Y_{1K_D}$ and $Y_1 \setminus Y_0$ is non-empty.

e) Any affine smooth faithfully flat D-scheme is an integral $*$-canonical model. This means: the above definitions are useful if Y_{K_D} (or Y) is somehow "far from being affine".

3.1.4. The unicity part. A natural question arises.

Q. *To what extend an integral $*$-canonical model is unique?*

Of course the most interesting case is when $* = S$ (or more generally, when an integral $*$-canonical model is automatically an integral S-canonical model). If Y is an integral S-canonical model of Y_{K_D}, and if one of its fibres in positive characteristic has more than one connected component, than by removing from Y one such connected component, we obtain an open subscheme Y_0 of Y which is still an integral S-canonical model. So to answer this question precautions are in order. There are a couple of ways in assuring the unicity part.

i) One way, in case Y_{K_D} is proper, is to impose (if possible) Y to be proper as well. Obviously we have the unicity part of a proper integral S-canonical model.

ii) Another way is to specify the local rings of Y which are DVR's of mixt characteristic (i.e. to specify what we want to get in codimension one).

iii) A third way is to search for a minimal (see below) such integral S-canonical model Y of Y_{K_D}.

In case $D = V$, a fourth potential way is to impose (if possible) Y to have the maximality property (as defined in [Va1, 3.2.3.2.1 1)). But in this case we do not know if indeed we get the unicity part.

3.1.4.1. Definition. An integral $*$-canonical model Y of Y_{K_D} with the property that for any other integral $*$-canonical model Y_1 of Y_{K_D}, there is a unique morphism $Y_1 \to Y$ which achieves the logical identification at the level of generic fibres, is called the minimal integral $*$-canonical model of Y_{K_D}.

Obviously a minimal integral $*$-canonical model is unique. Moreover a proper integral $*$-integral model is minimal. It seems to us possible to prove (starting from [Ar, §5]) that, provided Y_{K_D} has integral S-canonical models and is "far from being affine" (we have in mind Y_{K_D} proper), then there is a uniquely determined minimal integral S-canonical model of Y.

Let now $\mathcal{S}$ and l be as in 2.5.4 B). In what follows we consider primes p such that $(p, 6dl) = 1$. Let K_1 (resp. K) be a compact subgroup of $G_1(\mathbb{A}_f)$ (resp. of $G(\mathbb{A}_f)$) such that the requirements of 2.5.4 B) hold. We assume that K is admissible and that (cf. 2.6 E)) $\mathcal{K}3_{d,p,K}$ is a moduli scheme over $\mathbb{C}$ of polarized K3 surfaces of degree d having two connected components. Let $\mathcal{N}$ and $\mathcal{N}_1$ have the same significance as in 2.5.4 C). Let $\mathcal{C}^0$ and $\mathcal{C}^1$ be the connected components of $\mathcal{K}3_{d,pp,K}$. Let $E(\mathcal{C}^i)$ be the field of definition of $\mathcal{C}^i$ (or more precisely of the connected component of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ containing $\mathcal{C}^i$). Let E be the composite field of $E(\mathcal{C}^0)$ and $E(\mathcal{C}^1)$. Let O_E be the ring of integers of E, and let $O := O_E\left[\frac{1}{6dl}\right]$. From [Va2, 6.4.4] and [Mi3, 4.7] we deduce that $\mathrm{Spec}(O)$ is an étale cover of $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2dl}])$. Obviously E, O_E and O depend on K; so sometimes we write $E(K)$ and $O(K)$ instead of E and O. In practice K is normal as well; this assumption implies $\mathcal{C}^0 = \mathcal{C}^1$ and $E(\mathcal{C}^0) = E(\mathcal{C}^1)$. Typical example: $K = \tilde{K}_{SO}(n)$, with $n \geq 3$. In what follows we do not assume K normal. We have the following basic result:

3.2. Theorem. *a) An open subscheme $\mathcal{A}_{K3,d,p,K}$ of $\mathcal{N}_O$ is a moduli scheme of (isomorphism classes of) level-K marked polarized K3 surfaces of degree d over Spec(O)-schemes.*

b) An open subscheme $\mathcal{A}_{K3,d,pp,K}$ of $\mathcal{N}_O$ containing $\mathcal{A}_{K3,d,p,K}$ as a dense subscheme, is a moduli scheme of (isomorphism classes of) level-K marked pseudo-polarized K3 surfaces of degree d over Spec(O)-schemes.

Proof: We first proof part a). Standard arguments of descent show that we can pass from things over $\mathrm{Spec}(O)$ to things over a pro-étale cover of $\mathrm{Spec}(O)$. So we can assume that O contains the roots of unity of order any power of $6dl$. Moreover we can assume that K is as small as desired, subject to the requirements of 2.5.4 B). We assume from now on that $K_{\mathcal{S}}$ and $K_{1\mathcal{S}}$ are trivial; so $q(K_1) = K$. So we are dealing with $6dl$-marked polarized K3 surfaces of degree d.

We can identify a connected component of $\mathcal{N}_O$ with a quotient of a connected component of $\mathcal{N}_{1O}$ by the group $\mathbb{Z}/2\mathbb{Z}^{i_{6dl}-1}$, where i_{6dl} is the number of distinct primes dividing $6dl$; this is an immediate consequence of 2.5.2 and of [Mi4, 4.10-13], starting from the fact that the natural homomorphism $\mathbb{G}_m = Z(G_1) \to G_1^{\mathrm{ab}} = \mathbb{G}_m$ is the square homomorphism. In particular there is a universal principally polarized abelian scheme $(\mathcal{A}_1, \mathcal{P}_{\mathcal{A}_1})$ over $\mathcal{N}_1$ having level-m symplectic similitude structure, for any m dividing some power of $6dl$; it is obtained as in [Va2, 2.3] starting from the data of (f_1, L_W) of 2.5. Moreover $\mathcal{N}_O$ is the moduli space of principally polarized abelian schemes over O-schemes having up to scalar multiplication by -1 level-r symplectic structure, for any prime r dividing $6dl$. In particular $(\mathcal{A}_1, \mathcal{P}_{\mathcal{A}_1})_O$ is obtained from a principaly polarized abelian scheme $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ over $\mathcal{N}_O$ by natural pull back.

Using Hilbert schemes and geometric invariant theory (see also [An, 2.3.4]) we do get that $\mathcal{K}3_{d,p,K}$ is obtained from an E-scheme $\mathcal{K}3_{d,p,K_E}$, which is the moduli scheme of level-K marked polarized K3 surfaces of degree d over E-schemes, by extension of scalars. 2.7.2.2 4) implies that 2.7.2.1 B applies. From 2.7.2.1 B we deduce the existence of a rational map $r_E\colon \mathcal{K}3_{d,p,K_E} \to \mathcal{N}_E$ which over $\mathbb{C}$ is an open embedding. So r_E is a morphism which is an open embedding. So $\mathcal{K}3_{d,p,K_E}$ is (canonically identifiable with) an open subscheme of $\mathcal{N}_E$. In other words we got the characteristic zero part of a). In particular we have a universal $6dl$-marked polarized K3 surface $(\mathcal{Z}_E, \mathcal{L}_E, k_{6dl})$ of degree d over $\mathcal{K}3_{d,p,K_E}$.

Let $\mathcal{A}_{K3,d,p,K}$ be the maximal open subscheme of $\mathcal{N}_O$ having $\mathcal{K}3_{d,p,K_E}$ as its generic fibre, and to which $(\mathcal{Z}_E, \mathcal{L}_E)$ extends to a polarized K3 surface $(\mathcal{Z}, \mathcal{L})$ over it. $(\mathcal{Z}, \mathcal{L})$ is naturally $6dl$-marked, and we still denote by k_{6dl} its $6dl$-marked structure (as well as other $6dl$-marked structures to be introduced below). We need to show: any $6dl$-marked polarized K3 surface (Z, L, k_{6dl}) of degree d over an O-scheme Y, is obtained from $(\mathcal{Z}, \mathcal{L}, k_{6dl})$ by pulling back through a uniquely determined morphism $Y \to \mathcal{A}_{K3,d,p,K}$. We can assume that Y is local, of perfect residue field having characteristic p relatively prime to $6dl$. Standard arguments on descent (based on 2.6 D)) show that we can assume that Y is complete. Moreover from 2.8 we deduce that we can assume that $Y = \mathrm{Spec}(W(k)[[t_1, ...t_{19}]])$, and that (Z, L) is a versal deformation. From the above characteristic zero part we already have a natural morphism $m_0: Y_{\mathbb{Q}} \to \mathcal{A}_{K3,d,p,K}$. Standard arguments based on 2.7.2.1 A) (see [An, 9.3.1] for details) show that the abelian scheme $A_{Y_{\mathbb{Q}}}$ over $Y_{\mathbb{Q}}$ we get (by pulling back $\mathcal{A}_E$ via m_0) extends to one A_U over U, where (Y,U) is an extensible pair; moreover the natural principal polarization of $A_{Y_{\mathbb{Q}}}$ extends as well (cf. [FC, 2.7]) to one of A_U. So we get a morphism $m_U: U \to \mathcal{N}_O$ extending m_0. From b) of 2.5.4 C) we deduce that m_U extends uniquely to a morphism $m_Y: Y \to \mathcal{N}_O$.

Claim 1. *The morphism $m'_Y: Y \to \mathcal{N}_{W(k)}$ defined by m_Y is formally étale.*

Proof: We can assume that k is algebraically closed and that $W(k)$ can be embeddable in $\mathbb{C}$. So m'_Y lifts to a morphism $m'_{1Y}: Y \to \mathcal{N}_{1W(k)}$. Let y (resp. z) be the resulting k-valued (resp. $W(k)$-valued) point of $\mathcal{N}_{1W(k)}$ defined by m'_Y (resp. by m'_Y and by taking $t_i = 0$, $i = \overline{1,19}$). Let $(M_y, F^1_z, \phi_y, \mathcal{G}_{1W(k)})$ be the Shimura filtered σ-crystal defined by z, cf. [Va2, 2.3.10]. Let $P_{1W(k)}$ be the parabolic subgroup of $\mathcal{G}_{1W(k)}$ leaving invariant F^1_z. Let $P_{W(k)}$ be its image in $\mathcal{G}_{W(k)}$. The completion of the local ring of y can be identified (cf. [Va1, 5.5.1]) with the completion of $\mathcal{H} := \mathcal{G}_{1W(k)}/P_{1W(k)} = \mathcal{G}_{W(k)}/P_{W(k)}$ in the origin. The same identification can be achieved for Y (cf. 2.8.1). Moreover under these identifications, the morphism m'_{1Y} is nothing else but the identity. As this statement can be checked at the level of $W(k)$-valued points, it is a consequence of 2.7.2.1 A and B. In other words, the Shimura filtered Lie σ-crystal associated to a polarized K3 surface over $W(k)$ obtained from (Z,L) via a $W(k)$-valued point z_1 of Y is the same as the Shimura filtered Lie σ-crystal associated to $z_1 \circ m'_{1Y}$. This is a consequence of Fontaine's comparison theory and of 2.7.2.1 A and B (as $p \geq 5$ [Fa1, 2.6] or [Fa2, §4] applies directly). This ends the proof of the Claim 1.

Obviously the Claim 1 implies that m_Y factors through $\mathcal{A}_{K3,d,p,K}$. This implies (for instance cf. [An, 9.1]) that $(Z, L, k_{6dl}) = m_Y^*((\mathcal{Z}, \mathcal{L}, k_{6dl}))$; this equality ends the proof of part a) of the theorem.

To prove part b) we have to proceed similarly. We define similarly the maximal open subscheme $\mathcal{A}_{K3,d,pp,K}$ of $\mathcal{N}_O$ whose generic points are the generic points of $\mathcal{A}_{K3,d,pp,K}$, and to which $(\mathcal{Z}, \mathcal{L})$ extends to a pseudo-polarized K3 surface $(\tilde{\mathcal{Z}}, \tilde{\mathcal{L}})$. It can be obtained in the standard way using Hilbert-schemes and the fact that $\mathcal{L}^m$ is a very ample line bundle locally in the Zariski topology, for $m \in \mathbb{N}$ big enough (in fact $m = 3$ suffices). Obviously $\mathcal{A}_{K3,d,pp,K_{\mathbb{C}}} = \mathcal{K}3_{d,pp,K}$; this takes care of the characteristic zero part (cf. 2.6.1). Let now p and Y be as above, and let (Z,L) be a level-K marked pseudo-polarized K3 surface of degree d over Y. As above we can assume that Y is local, complete, and has a residue field a perfect field k of characteristic p.

Let (Z_k, L_k) be the fibre of (Z,L) over k. Let $(\tilde{Z}_k, \tilde{L}_k)$ be the K3 surface with a pseudo-polarization defined by (Z_k, L_k). So we have a k-morphism $f_k: \tilde{Z}_k \to Z_k$ obtained by blowing up the singularities of Z_k, and $\tilde{L}_k = f_k^*(L_k)$. We have $f_{k*}(T_{\tilde{Z}_k/k}) = T_{Z_k/k}$, where T_* denotes the relative tangent sheaf. From the Lerray spectral sequence and the theorem of formal functions (see [Hart, p. 277]) we get immediately that $H^i(Z_k, T_{Z_k/k}) = H^i(\tilde{Z}_k, T_{\tilde{Z}_k/k})$, for $i \in \{0,1\}$. So to deform (Z_k, L_k) is the same thing as to deform $(\tilde{Z}_k, \tilde{L}_k)$. As above (cf. 2.8) we can assume now that $Y = \mathrm{Spec}(W(k)[[x_1, ..., x_{19}]])$, and that (Z,L) is a versal deformation.

Claim 2. *There is a lift of f_k to $W(k)$.*

Proof: The number of singularities of $Z_{\bar{k}}$ is at most 19. Let E_1,..., E_m, with $m \in \mathbb{N} \cup \{0\}$, be the fibres of the extension of f_k to $\bar{k}$ above the singular points. Let $\tilde{E}_i$ be the line bundle defined by E_i. We can assume $m \geq 1$. Their self intersection numbers are -2. We deduce that we have

$$H^2_{\text{crys}}(\tilde{Z}_{\bar{k}}/W(\bar{k})) = L_0 \oplus L_0^\perp,$$

where

$$L_0 := < L > W(\bar{k}) \oplus \oplus_{i=1}^m < C_{1\text{crys}}(E_i) > W(\bar{k}).$$

Here $C_{1\text{crys}}$ refers to the crystalline first Chern class, while perpendicularity is with respect to the cup-product. The Frobenius of $H^2_{\text{crys}}(\tilde{Z}_k/W(k))[\frac{1}{p}]$ takes $pL_0^\perp$ into $L_0^\perp$. We deduce easily that the deformation space of $(\tilde{Z}_{\bar{k}}, L_{\bar{k}}, \tilde{E}_1, ..., \tilde{E}_m)$ is $\text{Spec}(W(\bar{k})[[x_1, ..., x_{19-m}]])$ ([Hart, 7.10 of p. 161]) shows that the formal K3 surface we get over it has an ample line bundle, and so is algebraizable). But, as E_i is just $\mathbb{P}^1$ and has -2 as self intersection number, the deformation space of $(\tilde{Z}_{\bar{k}}, L_{\bar{k}}, E_1, ..., E_m)$ is the same as the one of $(\tilde{Z}_{\bar{k}}, L_{\bar{k}}, \tilde{E}_1, ..., \tilde{E}_m)$. So the Claim 2 follows, once we remark that $L_0^\perp$ is obtained from a direct summand of $H^2_{\text{crys}}(Z_k/W(k))$ by extension of scalars (to $W(\bar{k})$).

Claim 2 implies: the fibre of (Z, L) over the generic point of $\text{Spec}(k([[x_1, ..., x_{19}]])$ is a polarized K3 surface. As in the proof of a) above we get a morphism $m'_Y : Y \to \mathcal{N}_{W(k)}$. The same proof applies to show that is formally étale and it factors through $\mathcal{A}_{K3,d,pp,K_{W(k)}}$. This ends the proof of the theorem.

3.2.1. Remarks. 1) We view the above theorem as an arithmetic global Torelli theorem.

2) In [Va7] we will see that we can replace everywhere in this paper $6d$ by $2d$ (i.e. the things are fined for $p = 3$ as well).

3) A level-K marked K3 surface with a pseudo-polarization of degree d over a (perfect) field of characteristic relatively prime to $2d$ defines a level-K marked pseudo-polarized K3 of degrre d over the field. In characteristic zero, with K small enough, this results from 2.7.2.1 B) and 3.2; general descent handles fully the characteristic zero case. Claim 2 of the proof of 3.2, reduces the positive characteristic situation to a characteristic zero situation. Moreover we can replace a perfect field by any complete local integral scheme whose residue field is perfect (cf. the proof of 3.2 b)).

4) Of course 3.2 has a variant where just one connected component shows up (i.e. when $\mathcal{K}3_{d,pp,K}$ is connected), or when we concentrate on just one connected component $\mathcal{C}^i$ (so we can work with $E(\mathcal{C}^i)$ instead of E). We do not detail these variants here.

5) All the fibres of $\mathcal{A}_{K3,d,p,K}$ over $\text{Spec}(O)$ are non-empty, cf. 5.4 a).

3.2.2. The discriminant locus. The complement of $\mathcal{A}_{K3,d,p,K}$ in $\mathcal{A}_{K3,d,pp,K}$ with its induced reduced structure is often called (see [JT1-2]) the discriminant locus. It is known that $\mathcal{D}_{\mathbb{Q}}$ is of pure codimension one (see [Be2, p. 149]). From the proof of 3.2 we get $\mathcal{D}_{\mathbb{Q}}$ is dense in $\mathcal{D}$, and so $\mathcal{D}$ is of pure codimension one in $\mathcal{A}_{K3,d,pp,K}$.

3.2.3. Some functorial aspects. A) In what folows $* \in \{p, pp\}$. Let K' be a open subgroup of K. We have the following functorial property: there is a natural étale cover

$$\mathcal{A}_{K3,d,*,K'} \to \mathcal{A}_{K3,d,*,K_{O(K_1)}}.$$

B) When $K = K(n)$ for some $n \geq 3$, then we denote $\mathcal{A}_{K3,d,*,K}$ by $\mathcal{A}_{K3,d,*,n}$. As a particular case of A) above we have (with $n_1 \in \mathbb{N}$) a natural étale cover

$$\mathcal{A}_{K3,d,*,nn_1} \to \mathcal{A}_{K3,d,*,n\mathbb{Z}[\frac{1}{6dnn_1}]}.$$

3.2.4. An anlysis. It seems slightly unpleasant to pass to $E(\mathcal{C}^0)$, $E(\mathcal{C}^1)$ and to O. There are a couple of ways in avoiding the "unpleasant" part.

A) One way is to use stacks: the stack $\mathcal{A}_{K3,d,p}$ (resp. $\mathcal{A}_{K3,d,pp}$) of unmarked polarized (resp. pseudo-polarized) is defined over $\mathbb{Z}$. Let O_d be the normalization of $\mathbb{Z}[\frac{1}{6d}]$ in

$$E_d := \cap_{n \geq 3} E(\tilde{K}_{SO}(n)).$$

$\mathrm{Spec}(O_d)$ is an étale cover of $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6d}])$. One example: $O_1 = \mathbb{Z}[\frac{1}{6}]$, cf. C) below. The stack $\mathcal{A}_{K3,d,p_{O(K)}}$ (resp. $\mathcal{A}_{K3,d,pp_{O(K)}}$) is obtained in the standard way by taking a quotient of $\mathcal{A}_{K3,d,p,K}$ (resp. of $\mathcal{A}_{K3,d,pp,K}$) by the equivalence relation that "eliminates" the level-K marked structures. We deduce:

Corollary. *$\mathcal{A}_{K3,d,p_{O_d}}$ and $\mathcal{A}_{K3,d,pp_{O_d}}$ are open substacks of the (Shimura) stack over Spec(O_d) defined by the quotient scheme $\mathcal{N}_{O_d}/C_K$ (its generic fibre is $\mathrm{Sh}_{G(L_0\otimes\widehat{\mathbb{Z}})}(G,X)_{E_d}$), where for K assumed to be also normal, we have $C_K := \mathcal{SOA}/K$.*

We do not know when $O_d = \mathbb{Z}[\frac{1}{6d}]$, or when in the above corollary we can replace O_d by $\mathbb{Z}[\frac{1}{6d}]$.

B) A second way is to add extra connected components to $\mathcal{K}3_{d,p,K}$, and to modify the moduli problem so that we end up in a $\mathbb{Q}$–context. We will not detail this approach. We just mention briefly: following the pattern of [Mi3, §3] and of [Va1, 4.1], we can work with abelian motives $\mathbb{Q}$–isogeneous to the ones of polarized K3 surfaces of degree d, so that the resulting moduli scheme over $\mathbb{C}$ is a dense open subscheme of $\mathrm{Sh}_K(G,X)$. Its arithmetic counterpart will be defined over $\mathbb{Q}$, and so we can "move" from things over O to things over $\mathbb{Z}[\frac{1}{6dl}]$.

C) A third way is to work with a K for which $E(\mathcal{C}^i)$ has an easy description. Of course the most interesting cases are when $K = \tilde{K}_{SO}(n)$, $K = K(n)$ or $K = \mathbb{K}_n^{\mathrm{ad}}$ (here $n \geq 3$). Here we treat the case when K is open and we have $K = q(K_1)$ (it is easy to see that always such a K_1 exists, if $K \in \{K(n), \mathbb{K}_n^{\mathrm{ad}}\}$). We can assume that K_1 is the maximal compact subgroup of $G_1(\mathbb{A}_f)$ with this property. From the Fact of 2.5.2 we deduce that $\mathcal{N}_1 = \mathcal{N}$, and that the set of connected components of $\mathcal{N}_{\mathbb{C}}$ is a principal homogeneous space of the group $\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q})K_1^{\mathrm{ab}}$ (the notations are as in loc. cit.). Let $m \in \mathbb{N}$ be the smallest number such that $K(m) \subset K$. Then denoting by $K(m)_1$ and $K(m)_1^{\mathrm{ab}}$ the analogues of K_1 and K_1^{ab}, we have a natural epimorphism

$$\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q})K(m)_1^{\mathrm{ab}} \twoheadrightarrow \mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q})K_1^{\mathrm{ab}}.$$

If m is relatively prime to $2d$, then $\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q})K(m)_1^{\mathrm{ab}}$ is the group of invertible elements of $\mathbb{Z}/m\mathbb{Z}$ modulo squares. Warning: in the above examples, meant just to give an idea about how complicated is the problem of finding fields of definitions (they make sense regardless of the fact that K is admissible or not), the groups K are not admissible if $d > 1$.

Example 1. We assume that $(d, 3) = 1$ and that $l = 3$. Then for $K = K(3)$ we have $E(\mathcal{C}^1) = E(\mathcal{C}^0) = \mathbb{Q}(\zeta_3)$.

Example 2. We assume that $(m, 2d) = 1$ and that l is the product of primes dividing m. Then for $K = K(m)$ we have $E(\mathcal{C}^1) = E(\mathcal{C}^0)$ equal to the maximal subfield E of $\mathbb{Q}(\zeta_m)$ with the property that $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/E)$ is the subgroup of squares of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. So E is the composite field E_m of some quadratic extensions of $\mathbb{Q}$, equal in number to the number of primes dividing m. If we just have $K(m) \subset K$, then E is a subfield of E_m, but we might have $E(\mathcal{C}^0) \neq E(\mathcal{C}^1)$.

The situation when $(m, 2d) > 1$ is more complicated and it will not be treated here.

3.2.5. Variant (the relative setting). As before $* \in \{p, pp\}$. We refer to 2.5.4 F). We assume that K is as above. Let O^Z be the composite of O_Z and $O[\frac{1}{l_Z}]$. $\mathrm{Spec}(O^Z)$ is a pro-étale cover of $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6dl_Z}])$. We have a natural pro-finite morphism

$$m_{\mathrm{nat}}: \mathcal{N}_{ZO^Z} \to \mathcal{N}_{O^Z},$$

(cf. [Va1, 3.2.7 4)] for the existence part and [Va1, 3.2.12] for the pro-finiteness part). Let $\mathcal{A}_{K3,d,*,K_Z,h_Z}$ be the pull back of $\mathcal{A}_{K3,d,*,K_{O_Z}}$ via m_{nat}. It is an open subscheme of $\mathcal{N}_{ZO^Z}$. If $* = p$, then it is the moduli scheme of level-K_Z marked polarized K3 surfaces of degree d which can be lifted in characteristic zero in such a way (cf. [Va1, 4.1]) that:

– there is a family $(t_\alpha)_{\alpha \in \mathcal{J}_Z}$ of Hodge cycles indexed by the set $\mathcal{J}_Z$ (of 2.5.0), with t_{α_0} corresponding to the bilinear form on the primitive part of H^2 defined by the cup product;

– there is a complex model $(Z_1, L_1, k_{K_Z}, (t_{1\alpha})_{\alpha \in \mathcal{J}_Z})$ of this lift, such that there is an isomorphism

$$H_{\mathrm{pr}}(Z_1, L_1, \mathbb{Z}) \tilde{\to} L_0$$

(defined by a marked structure of (Z_1, L_1)) taking:

i) $t_{1\alpha}$ into v_α $\forall \alpha \in \mathcal{J}_Z$ (so it respects the cup products);
ii) the homorphism h_{Z_1} into a $G_Z(\mathbb{R})$-conjugate of h_Z;
iii) k_{K_Z} into an element of K_Z.

There is a similar moduli interpretation of $\mathcal{A}_{K3,d,pp,K_Z,h_Z}$.

3.3. Some functions. Let f and g be the functions defined in 1.3.6. The proof of 3.2 shows that they are well defined (i.e. the role of K is irrelevant). From 3.2.2 and 3.3 we deduce:

3.3.1. Theorem. *$\mathcal{A}_{K3,d,p,K_{O[\frac{1}{6dlf(d)}]}}$ and $\mathcal{A}_{K3,d,pp,K_{O[\frac{1}{6dlf(d)}]}}$ are integral canonical models of their generic fibres.*

3.4. Remarks. 1) It is natural to expect that 3.2-3 (as well as the greatest part of §4-5) extends to other classes of polarized hyperkählerian varieties who have companion Shimura varieties (as in 2.5) (see [An] for such classes). What made us to refrain from approaching these classes as well, is the lack of a reference to a deformation theory in positive characteristic, similar to the one of [De4] (mainly [De4, 1.6 needs to be extended). The lack of a global Torelli theorem for many classes of polarized hyperkählerian varieties (see [An, 3.3.1 and 3.3.4]: in general we have local Torelli theorems with finite to one morphisms in the global context) is another impediment: it is not a serious one (we just have to move from open subschemes to étale schemes). Moreover two other things are worth being mentioned. First the different second cohomology groups with integral coefficients (like $\mathbb{Z}$, $\mathbb{Z}_p$, $W(k)$) of these varieties might not be free over the rings involved; so it might be more convenient to work with the notion of polarizations as in [An, 1.3], and to "put aside" the primes involved in the torsion parts obtained for these groups using $\mathbb{Z}$-coefficients. Second the understanding of what should be polarized generalized hyperkählerian varieties is much less.

2) We refer to 3.2. We do not know what would be the best integral models extending $\mathcal{A}_{K3,d,*,p,K}$ to a scheme over $\mathrm{Spec}(O_E)$, and if such "best" extensions can be interpreted as nice moduli schemes (like open subschemes of a similar "natural" extension of $\mathcal{N}_O$ over $\mathrm{Spec}(O_E)$), or are smooth (over O_E).

§4. Applications

The basic result 3.2 has many consequences. Here we list some of them, which are obtainable by combining 3.2 with ideas (results) of [Va1-7]. We use the notations of 3.2. We assume that K is open.

4.1*. Compactifications. In [Va6] it is shown that the schemes $\mathcal{N}$ and $\mathcal{N}_1$ of 3.2 have plenty of smooth projective toroidal compactifications. What we mean by this for $\mathcal{N}$ is: there are plenty of smooth projective schemes $\mathcal{N}^c$ over $\mathrm{Spec}(O)$ such that:

i) they contain $\mathcal{N}$ as an open subscheme;

ii) the complement of $\mathcal{N}$ in $\mathcal{N}^c$ is a divisor with normal crossing;

iii) $\mathcal{N}^c_{\mathbb{Q}}$ is constructed as in [Har].

The proof of existence of such schemes $\mathcal{N}^c$ is not at all difficult: it follows the pattern of [FC], as explained in [Va1, 1.8]. We deduce:

Theorem. *$\mathcal{A}_{K3,d,pp,K}$ (and so also $\mathcal{A}_{K3,d,p,K}$) has smooth projective compactifications over Spec(O). Moreover we can choose such compactifications over Spec$(O\big[\frac{1}{f(d)}\big]$ such that their complements are divisors with normal crossings.*

4.2*. Connectivity. Let p be a prime relatively prime to $6d$. From 2.6. B) and 4.1.1 we get directly:

Theorem. *The stacks $\mathcal{A}_{K3,d,p\mathbb{F}_p}$ and $\mathcal{A}_{K3,d,pp\mathbb{F}_p}$ are geometrically connected.*

4.3. Shafarevich conjecture. For the Shafarevich conjecture for polarized K3 surfaces over number fields (resp. over function fields in characteristic zero) we refer to [An, 1.3.1] (resp. to [JT1]). From the very definition of the function g of 3.3, as in [JT1] we get the Theorem 2 of 1.4.

The theorems of 4.1-3 have variants in the contexts of 3.2.4 and of 3.2.5. We will not stop to restate them.

4.4*. Estimates. Let v be a prime of O, and let p be the rational prime it divides. Let $m \in \mathbb{N}$ such that the residue field $k(v)$ of v is $\mathbb{F}_{p^m}$. We denote by a right lower index $k(v)$ the different fibres over v to be considered. Let $r \in \mathbb{N}$ with $r \geq m$. We assume that $\mathcal{A}_{K3,d,pp,K}$ has two connected components; their fibres over E are geometrically connected. Let $\mathcal{N}^0$ be the disjoint union of two connected components of $\mathcal{N}_O$ such that $\mathcal{A}_{K3,d,pp,K}$ is dense in $\mathcal{N}^0$. From 4.1* we deduce that $\mathcal{N}^0_{k(v)}$ has two connected components, which are geomtrically connected. So $\mathcal{A}_{K3,d,p,K\,k(v)}$ is dense in $\mathcal{N}^0_{k(v)}$.

Let $N(K3,d,p,K,v,r)$ be the number of elements of the set

$$\mathcal{S}_{K3,d,p,K,v,r} := \mathcal{A}_{K3,d,p,K\,k(v)}(\mathbb{F}_{p^r}).$$

Similarly we define $N(K3,d,pp,K,v,r)$ and $\mathcal{S}_{K3,d,p,K,v,r}$. Let $N(\mathcal{N},v,r)$ be the number of elements of the set

$$\mathcal{N}_{k(v)}(\mathbb{F}_{p^r}).$$

Similarly we define $N(\mathcal{N}^0,v,r)$. Let C be the number of connected components of $\mathcal{N}_O$. Warning: not always we have $N(\mathcal{N}^0,v,r) = \frac{2}{C}N(\mathcal{N},v,r)$, as the connected components of $\mathcal{N}_O$ are not always isomorphic as O-schemes. Obviously we have:

$$N(K3,d,p,K,v,r) \leq N(K3,d,pp,K,v,r) \leq N(\mathcal{N}^0,v,r). \tag{3}$$

Let Φ_v^r be the Frobenius of $\mathbb{F}_{p^r}$ fixing $k(v)$ and generating $\mathrm{Gal}(\mathbb{F}_{p^r}/k(v))$. It acts on the three sets $\mathcal{S}_{K3,d,p,K,v,r}$, $\mathcal{S}_{K3,d,pp,K,v,r}$ and $\mathcal{N}_{k(v)}(\mathbb{F}_{p^r})$. In [Mi4, 5.1] a conjectural description of the set $\mathcal{N}_{k(v)}(\mathbb{F}_{p^r})$ acted upon by Φ_v^r is made. In [Mi4, 5.6] it is shown that [Mi4, 5.1] is implied by the Main conjecture (the Langlands-Rapoport conjecture) 4.4 of [Mi4] for the Shimura quadruple (G,X,K^p,v) (see [Va1, 3.2.6] for def.). From [Mi4, 4.9] we deduce that the Langlands-Rapoport conjecture for (G,X,K^p,p) is implied by the Langlands-Rapoport conjecture for (G_1,X_1,K_1^p,p). But this is proved in [Va7]. Using this and [Mi4, 6.13] we obtain a formula for $N(\mathcal{N},v,r)$ in terms of sums of products of twisted orbital integrals; this formula implies easily a similar formula for $N(\mathcal{N}^0,v,r)$, as in loc. cit. is easy to trace back the connected components. This shows the utility of (3). In particular, if moreover p does not divide $f(d)$, then $N(K3,d,pp,K,v,r) = N(\mathcal{N}^0,v,r)$, and so we get a very precise formula for $N(K3,d,pp,K,v,r)$. See [Va7] for extra details.

The same applies to the relative context of 3.2.5.

4.5. Extension properties. Another form of 3.3.1 is:

Theorem (the extension property). *Let $n \geq 3$. Let $(\mathcal{Y},\mathcal{U})$ be an extensible pair with $\mathcal{Y}$ a healthy regular scheme. We assume that $6df(d)$ (resp. $6dnf(d)$) is invertible in $\mathcal{Y}$. Then any polarized K3 surface (resp. level-n marked pseudo-polarized) K3 surface of degree d over $\mathcal{U}$ extends to a polarized (resp. level-n marked pseudo-polarized) K3 surface over $\mathcal{Y}$.*

Proof: Using the classical purity theorem and [Va1, B) of 3.2.2 4)], in the polarized context, we can assume that Y is local and that we do have a level-n primitively marked structure. So 3.2.3 B) and 3.3.1 apply.

4.6. Milne's conjecture for K3 surfaces. In the whole of 4.6-7 by k we denote an arbitrary perfect field of characteristic p; here p is an arbitrary prime. Let V be a DVR faithfully flat over $\mathbb{Z}_{(p)}$. Let $K_V := V\left[\frac{1}{p}\right]$. Let Z_V be a K3 surface over V. For simplicity we assume that all the Hodge cycles of $Z_{\overline{K_V}}$ are defined over K_V. Let $(t_\alpha)_{\alpha\in\mathcal{J}^Z}$ be this family of Hodge cycles. Following [Va3, 2.2.0] we define:

4.6.1. Definition. We say that Z_V has the MC (Milne's conjecture) property if there is a faithfully flat p-adically complete V-algebra R such that there is an isomorphism

$$\rho\colon H^2(Z_{\overline{K_V}},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} R \tilde{\to} H^2_{dR}(Z_V/V)\otimes_V R$$

taking the p-étale component of t_α into the de Rham component of t_α, $\forall\alpha\in\mathcal{J}^Z$.

We assume now that V is complete having $\bar{k}$ as its residue field. We also assume that Z_V has a polarization L_V of degree d, where $(2d,p)=1$. It is trivial to see that the existence of ρ is equivalent to the existence of a similar isomorphism involving the primitive parts of the H^2's. Let (Z,L) be a complex model of (Z_V,L_V). Let G_Z be the Mumford–Tate group of (Z,L) or of (Z_V,L_V) (cf. also the review of 5.2.1). We also assume, with the notations of 2.4-5, that the closure of G_Z in $\mathcal{G}_{\mathbb{Z}_{(p)}}$ is a reductive group $G_{Z\mathbb{Z}_{(p)}}$. So the closure of G_{1Z} in $\mathcal{G}_{1\mathbb{Z}_{(p)}}$ is a reductive group as well. The assumptions on V imply that (Z_V,L_V) has r-marked structure for any prime r relatively prime to p. From 3.2 and 3.2.5 we deduce that V contains E_Z (of 2.5.4 F)). Let v_Z be the prime of E_Z over which the maximal ideal of V sits. We also assume that the triple $(f_{1Z},L_W\otimes\mathbb{Z}_{(p)},v_Z)$ is a SHS. In [Va7] we will see that this is automatically so (cf. [Va2, 2.3.8 2)]; here we will just remark that this is automatically so if $p\geq 2^9$ (cf. [Va1, 5.8.6] and [Va2, 2.3.6]).

From [Va2, d) of 4.4.1 3)] (its proof is achieved in [Va3]; see V3 of [Va3, 3.2]) and 2.7.2.1 A and B we get:

4.6.2. Theorem. *Z_V has the MC property.*

As in [Va4, 4.2], it is worth pointing out that the value of d_Z is irrelevant in 4.6.2.

4.6.3. Corollary. *The closure in $GL(H^2_{dR}(Z_V/V))$ of the subgroup of $GL(H^2_{dR}(Z_V/V)\left[\frac{1}{p}\right])$ fixing the de Rham component of t_α, $\forall\alpha\in\mathcal{J}^Z$, is a reductive group isomorphic to G_{ZV}.*

4.6.4. We assume now that $V=W(k)$ and do not assume anymore that the Hodge cycles of $Z_{\overline{K_V}}$ are defined over K_V. We know that $Z_{W(\bar{k})}$ has the MC property. Let H_{pr}, B_{crys}, $\mathfrak{g}$ (and its filtrations) and ϕ_Z have the same meaning as in 2.8.1, but working with (Z_V,L_V). As the Hodge cycles of $Z_{W(\bar{k})}$ are permuted by the Galois action of $\mathrm{Gal}(W(\bar{k})/W(k))$, we get that the subgroup $G_{ZW(\bar{k})}$ of $GL(H^2_{dR}(Z_V/V))_{W(\bar{k})}$ is obtained by extension of scalars from a reductive (cf. 4.6.3) subgroup $\tilde{G}_{ZW(k)}$ of $GL(H^2(Z_V/V))$. $\tilde{G}_{ZW(k)}$ is an inner form of $G_{ZW(k)}$, cf. 4.6.2. Let $\mathfrak{g}_Z := \mathrm{Lie}(\tilde{G}_{ZW(k)})$. We deduce that the Shimura filtered Lie σ-crystal

$$(\mathfrak{g},F^0(\mathfrak{g}),F^1(\mathfrak{g}),\phi_Z)$$

associated to (Z_V,L_V) (cf. 2.8.2) has naturally a Shimura Lie σ-subcrystal

$$(\mathfrak{g}_Z,F^0(\mathfrak{g}_Z),F^1(\mathfrak{g}_Z),\phi_Z)$$

called the Shimura filtered Lie σ-crystal associated to (Z_V, L_V) and the Hodge cycles of $Z_{W(\bar{k})}$. Ignoring the filtration we obtain the Shimura Lie σ-crystal

$$(\mathfrak{g}_Z, \phi_Z)$$

associated to (Z_k, L_k) with respect to (let us say) $\mathcal{A}_{K3,d,p,K_Z(2^\infty):=2-marked,h_Z}$.

4.6.5. Exercise 4. Show that 4.6.2-4 remains true for K3 surfaces which have a pseudo-polarization of degree d. Hint: Using [Hart, 7.10 of p. 161], reduce the situation to the polarized context.

4.7. Specializations. Here we state some of the implications of [Va2] for the context of (pseudo-) polarized K3 surfaces.

A) Stratifications.

All the stratifications to be mentioned are finite and are formed by locally closed subschemes. We assume that K_1 is small enough so that we can speak as in the proof of 3.2 about the principally polarized abelian scheme $(\mathcal{A}_1, \mathcal{P}_{\mathcal{A}_1})$ sitting naturally over $\mathcal{N}_1$. Let p be a prime relatively prime to $6dl$. $\mathcal{N}_{\mathbb{F}_p}$ has a natural Newton-polygon stratification $\mathcal{ST}$ which we call the refined canonical Lie stratification (see [Va2, 4.9.8]). It is the quotient (see the proof of [Va2, 4.9.8]) stratification of the refined canonical Lie stratification of $\mathcal{N}_{1\mathbb{F}_p}$. [Va2, 4.5.6.1] allows us to drop the word refined. The canonical Lie stratification of $\mathcal{N}_{1\mathbb{F}_p}$ can be defined (cf. loc. cit.) either starting from the Newton polygons of the Shimura σ-crystals attached to points of $\mathcal{N}_{1\mathbb{F}_p}$, or of their Shimura Lie σ-crystals (see [Va2, 2.3.10] for terminology). So $\mathcal{A}_{K3,d,*,K\,k(v)}$ gets naturally a Newton-polygon stratification by pulling back $\mathcal{ST}$. From the proof of 3.2 we deduce that it is the same as the one attached (see 2.8.1) to Newton-polygons of the F-crystals attached to K3 surfaces with a pseudo-polarization of degree d (over perfect $\mathbb{F}_p$-fields). From [Va1, 4.2.1 and P1 of 4.6] we deduce that its generic stratum (called the ordinary locus) is Zariski dense. We got:

Fact. *Any pseudo-polarized K3 surface of degree d over k is the specialization of an ordinary polarized K3 surface.*

B) Shimura-ordinary loci.

We refer to 3.2.5. We assume that $(p, 6dl_Z) = 1$. Let v_Z be the prime of $E(G_Z, X_Z)$ divided by v. We denote by f_{1Z} the composite of the natural inclusion $(G_{1Z}, X_{1Z}) \hookrightarrow (G_1, X_1)$ with f_1. We assume that the triple $(f_{1Z}, L_W \otimes \mathbb{Z}_{(p)}, v_Z)$ is a SHS (cf. 4.6). To it, it is associated naturally in [Va1, 4.1.1.2] two formal isogeny types τ and $\mathrm{Lie}_{G_1}(\tau)$. As in A) above we deduce that $\mathcal{N}_{1Z\,k(v_Z)}$, $\mathcal{N}_{Z\,k(v_Z)}$, and $\mathcal{A}_{K3,d,*,K_Z,h_Z\,k(v)}$ are naturally and compatibly endowed with refined canonical Lie stratifcations. If $G^{\mathrm{ad}}_{Z\mathbb{R}}$ has only one non-compact factor (this is the case if (G_Z, X_Z) is as in 5.2.1 b)), then we can drop the word refined (the arguments of [Va2, 4.5.6.1] apply as well). From A) and [Va2, 4.2.1] we deduce that $\mathrm{Lie}_{G_1}(\tau)$ is the formal isogeny type associated to the F-crystals defined by the points of the Zariski dense open stratum (called the G_Z-ordinary or the Shimura-ordinary locus) of $\mathcal{A}_{K3,d,*,K_Z,h_Z\,k(v)}$. A pseudo-polarized K3 surface obtained through a point of this Shimura-ordinary locus with values in fields will be called G_Z-ordinary. We got:

Theorem. *Any pseudo-polarized K3 surface defined by a point of $\mathcal{A}_{K3,d,*,K_Z,h_Z\,k(v)}$ with values in fields, is the specialization of a pseudo-polarized K3 surface which is G_Z-ordinary.*

C) Shimura-canonical lifts for pseudo-polarized K3 surfaces.

From [Va2, 4.4.1 and 4.9.8 d)] we deduce that any k-valued point y of the Shimura-ordinary locus of $\mathcal{A}_{K3,d,*,K_Z,h_Z\,k(v)}$, has a (uniquely determined) G_Z-canonical lift to $W(k)$. At the level of filtrations it is defined as follows. For simplicity we restrict to the polarized context. Let $(\mathfrak{g}_Z,\phi_y)$ be the Shimura Lie σ-crystal associated to the polarized K3 surface defined by y (cf. 4.6.5 and the Claim 2 of the proof of 3.2); then there is a unique $W(k)$-valued lift of y to $\mathcal{A}_{K3,d,*,K_Z,h_Z}$ (cf. [Va2, a) 4.4.1 2)]) such that the resulting Shimura filtered Lie σ-crystal $(\mathfrak{g}_Z,F^0(\mathfrak{g}_Z),F^1(\mathfrak{g}_Z),\phi_y)$ is such that the $W(k)$-submodule of $\mathfrak{g}_Z$ generated by the non-negative slopes of $(\mathfrak{g}_Z,\phi_y)$, is contained in $F^0(\mathfrak{g}_Z)$.

The (pseudo-) polarized K3 surfaces defined by such G_Z-canonical lifts, will be called G_Z-canonical or Shimura-canonical. We have:

Theorem C1. *The G_Z-ordinary locus of $\mathcal{A}_{K3,d,*,K_Z,h_Z\,k(v)}$ is included in the ordinary locus of $\mathcal{N}_{Z\,k(v)}$ iff $k(v_Z)=\mathbb{F}_p$.*

Proof: As $E(G_{1Z},X_{1Z})=E(G_Z,X_Z)$, this is a consequence of property P1 of [Va2, 4.6].

From C1 and property P2 of [Va2, 4.6] we get directly:

Theorem C2. *If $k(v_Z)=\mathbb{F}_p$, then any G_Z-canonical lift is a usual canonical lift.*

From 3.2, 3.2.5 and [MB, 5.2 p. 237] we get directly:

Theorem C3. *If $k(v_Z)=\mathbb{F}_p$, then the ordinary locus of $\mathcal{N}_{Z\,k(v)}$ is a quasi-affine scheme.*

Proof: We can assume that K contains $K(n)$ for some $n\geq 3$ and relatively prime to p. It is known (see [MB, 5.2 of p. 237]) that the ordinary locus of $\mathcal{A}_{g_W,1,n_{\mathbb{F}_p}}$ is quasi-affine. So (cf. 2.5.4 D)) the ordinary locus of $\mathcal{N}_{1\,k(v)}$ is a quasi-affine scheme. As $\mathcal{N}_1$ is a pro-étale cover of $\mathcal{N}$ (see 2.5.4 D)), from the definition of refined canonical stratifications, we get that the ordinary locus of $\mathcal{N}_{k(v)}$ is a quasi-affine scheme. So the theorem follows from 3.2.5.

In [Va7] we will see that if k is a finite field than these G_Z-canonical lifts, when viewed just as K3 surfaces, are of CM type (see 5.2.1 for the meaning of this). It seems to us that the answer to the following question is (at least in general) yes.

Q. *Is it true that under the process of taking Shimura-canonical lifts, the number of singularities is preserved?*

D) Local deformations.

In what follows p is an arbitrary prime relatively prime to $2d$. Let (Z,L) be a polarized K3 surface over $W(k)$ of degree d. Let $\mathbb{H}:=(H_{\mathrm{pr}},F^0(H_{\mathrm{pr}}),F^1(H_{\mathrm{pr}}),\phi_Z,B_{\mathrm{crys}})$ be its polarized p-divisible object of $\mathcal{MF}_{[-1,1]}(W(k))$. Let $R:=W(k)[[x_1,...,x_m]]$, for some $m\in\mathbb{N}$. Let Φ_R be a Frobenius lift of R such that it leaves invariant the ideal $I:=(x_1,...,x_m)$ of R. Let (Z_R,L_R) be a deformation of (Z,L) over $\mathrm{Spec}(R)$ (cf. 2.8). As in 2.8.1 we can define its polarized p-divisible object $\mathbb{H}_R$ of $\mathcal{MF}_{[-1,1]}(R)$. As in [Va2, 3.4.18] we can write

$$\mathbb{H}_R=(H_{\mathrm{pr}}\otimes R,F^0(H_{\mathrm{pr}}),F^1(H_{\mathrm{pr}}),g\phi_Z),$$

for some $g\in SO(H_{\mathrm{pr}},B_{\mathrm{crys}})(R)$ which is trivial mod I. A natural question arises:

Q. *What are the possible values of* g*?*

It can be easily checked starting from [Va2, 3.4.18.3] that not always g can take any possible value.

Theorem (the inducing property for polarized K3 surfaces). *We assume that one of the following three conditions is satisfied:*

a) $k = \bar{k}$*;*

b) Φ_R *is essentially of additive type (in the sense of [Va, 3.4.18.0.1]);*

c) There is a Shimura filtered σ*-crystal* $(M_0, F_0^1, \phi_0, \tilde{G}_{1W(k)}, (u_\alpha)_{\alpha \in \mathcal{J}_1})$ *such that:*

i) its attached Shimura filtered Lie σ*-crystal is isomorphic to the one attached to* (Z, L)*;*

ii) (M_0, ϕ_0) *does not have either the slope 0 or the slope 1;*

iii) $\tilde{G}_{1W(k)}$ *is an inner form of* $G_{1W(k)}$ *and the faithful representation* $\tilde{G}^{\mathrm{der}}_{1W(\bar{k})} \hookrightarrow GL(M_0 \otimes W(\bar{k}))$ *is the spin representation.*

Then the answer to **Q** *above is: any* g *as above shows up.*

Proof: This is a consequence of [Va2, 3.4.18.5 and 3.4.18.7], once we remark that we can always shift things from $SO(H_{\mathrm{pr}}, B_{\mathrm{crys}})$ to $\tilde{G}_{1W(k)}$. In other words, $\mathbb{H}$ is always obtainable from a p-divisible object with cycles

$$\mathbb{H}_1 := (M_0 \otimes R, F_0^1 \otimes R, \tilde{g}\phi_0, (u_\alpha)_{\alpha \in \mathcal{J}_1})$$

of $\mathcal{MF}_{[0,1]}(R)$ in the standard way (of taking the adjoint of $\tilde{G}_{1R}$); here

$$\mathbb{H}_0 := (M_0, F_0^1, \phi_0, \tilde{G}_{1W(k)}, (u_\alpha)_{\alpha \in \mathcal{J}_1})$$

is a Shimura filtered σ-crystal satisfying i) and iii) of c), while $\tilde{g} \in \tilde{G}_{1W(k)}(R)$ is congruent to 1 mod I. Moreover if $m = 19$, $\Phi_R(x_i) = x_i^p$, for any $i \in \{1, ..., 19\}$, and (Z_R, L_R) is the versal deformation of (Z, L) of 2.8, then $\mathbb{H}_1$ is a Shimura filtered F-crystal, which is a versal deformation of $\mathbb{H}_0$ (in the sense of [Va2, 3.4.19 B]). So [Fa2, th. 10 and the remarks after] applies as in [Va2, 3.4.18.5 and 3.4.18.7]. This ends the proof of the theorem.

Of course the same result can be stated in the relative situation of 3.2.5 (cf. 4.6), provided we still assume (we repeat that in [Va7] we will see that this is implied by the fact that the closure of G_Z in $\mathcal{G}_{\mathbb{Z}_{(p)}}$ is a reductive group over $\mathbb{Z}_{(p)}$) as in 4.6 that the triple $(f_{1Z}, L_W \otimes \mathbb{Z}_{(p)}, v_Z)$ is a SHS: the same proof applies.

4.8. Remarks. 1) Condition ii) of 4.7 D) is equivalent to: the Shimura filtered Lie σ-crystal associated to $\mathbb{H}$ does not have the slope -1. For instance, it is automatically satisfied if Z is supersingular.

2) There are other parts of [Va2] which can be entirely transferred to the context of K3 surfaces. We mainly have in mind: the part of [Va2, 3.4] involving connections, and the part of [Va2, 4.7.30] involving crystalline coordinates. In particular for the existence of crystalline coordinates for ordinary K3 surfaces over an algebraically closed field of characteristic 2, we refer to [Va2, b) of 4.7.30 2)].

3) See [Va2, 2.3.9] for an abstract discussion on the inner form $\tilde{G}_{1W(k)}$ of $G_{1W(k)}$. In fact, from [Va4, 4.4 2)] we deduce that in fact we always have $\tilde{G}_{1W(k)} = G_{1W(k)}$.

4) For the Mumford–Tate (resp. the ordinary reduction) conjecture for K3 surfaces over number fields see [An, 1.6.1 3)] and [Va3, 5.1.3.2 2)] (resp. see [Va3, 5.4 2)] and [Va7]).

§5. Some open problems

We quickly present here some (to our knowledge) open problems pertaining to polarized K3 which are inspired from "chapters" pertaining to abelian varieties dear to us.

5.1. Arithmetics à la Tate of (some) polarized K3 surfaces. Starting from the Weierstrass equation $Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$ describing a general elliptic curve E, one defines (see [Ta] and [Si, p. 46]) other algebraic quantities b_2, b_4, b_6, b_8, c_4, c_6, Δ, j, etc., meant for a refined analysis of the arithmetics of E. We feel that it would be very much desirable to have a similar theory (arithmetic à la Tate) for subclasses of polarized K3 surfaces. We have in mind especially subclasses of the four (or at least the first three) classes of K3 surfaces mentioned in 2.3. Of course the situation is much more complicated. Even for the class of smooth surfaces in $\mathbb{P}^3$, is not at all easy to define a similar Weierstrass form (involving the minimum number of variables). Moreover one would get not just one j invariant but more than 19. Still we do believe in the possibility of (useful) such theories for different subclasses of polarized K3 surfaces.

5.2. What is between? On one extreme we have K3 surfaces which are complete intersection. At the other extreme we have Kummer surfaces. The natural problem arises:

P. *Describe (as well as possible) what is "in between".*

We can not be very precise on what we mean by "describe". The description can be in terms of some embeddings in $\mathbb{P}^5$ (or other projective space) subject to conditions, or in terms of covers of $\mathbb{P}^2$ ramified above some specific (plane curve) branch locus, or in terms of Shimura varieties involved (like $\mathrm{Sh}(G_Z, X_Z)$ of 2.4.1), etc.

5.2.1. K3 surfaces of CM type. Let Z be a K3 surface over a field k of characteristic zero. It is known that any Hodge cycle of Z is an absolute Hodge cycle. This allows to define (see [De3]) the Mumford-Tate group G_Z of Z even if k is not embeddable in $\mathbb{C}$ (see also 2.4.1). Z is said to be of CM type (CM stands for complex multiplication) if G_Z is a torus. We would like very much to have a "description" (perhaps one similar to the one for abelian varieties of CM type; see [Mu, ch. 22] and [Sh]) of this subclass (of K3 surfaces CM type). We have:

Proposition. *If Z is of CM type, then, passing to a finite field extension of k, Z becomes definable over a number field.*

Proof: This is an immediate consequence of 3.2 and of the similar property for abelian varieties of CM type.

This proposition is not a trivial one: it is easy to check that it is equivalent to part a) of 2.7.2.1 A.

5.3. Hodge conjecture for correspondances. To our knowledge the Hodge conjecture for correspondances between complex K3 surfaces is not yet known. In positive characteristic, often such correspondances are replaced by ones imerging from Hecke operators.

5.3.1. Hecke orbits. Let $\mathbb{G}(6dl)$ be the subgroup of $G(\prod_{r \text{ is a prime dividing 6dl}} \mathbb{Q}_r)$ fixing the connected component of $\mathcal{K}3_{d,pp,marked}$ which projects onto $\mathcal{C}^0$. It acts naturally on the connected component of $\mathcal{N}_O$ dominated by $\mathcal{C}^0$, and (cf. [Va2, 6.4.4]) on the normalization $\tilde{\mathcal{N}}$ of $\mathcal{N}_O$ in the ring of fractions of $\mathrm{Sh}(G,X)$. We do expect that $\mathbb{G}(6dl)$ takes points of $\tilde{\mathcal{N}}$ mapping into points of $\mathcal{A}_{K3,d,pp,K}$ with values in fields of positive characteristic, into points of $\tilde{\mathcal{N}}$ mapping into points of $\mathcal{A}_{K3,d,pp,K}$; moreover would be very useful to have a full (i.e. constructive) direct description of the effect of this action on the level-K marked pseudo-polarized K3 surfaces of degree d.

5.4. The functions. It is very much desirable (cf. §4) to compute (or estimate the functions) f and g of 3.3. The expectation formulated in 1.3.6 is based on:

a) We use the notations of 4.4. Using Kummer surfaces attached to a product $E\times E$, with E a supersingular elliptic curve over $\mathbb{F}$, we deduce easily that the supersingular locus of $\mathcal{A}_{K3,d,pp,K_{k(v)}}$ is non-empty. So (cf. 3.2.2) $\mathcal{A}_{K3,d,p,K,v_{k(v)}}$ is non-empty.

b) As $p\geq 5$, 3.2.1 3) and Rudakov-Shafarevich theorem (see [RS2] and [Og, p. 384]) should imply that $\mathcal{A}_{K3,d,p,K,v_{k(v)}}$ contains a connected component of the supersingular locus of $\mathcal{N}_{k(v)}$.

c) The expectation of 5.3.1.

d) It should be possible to prove that $f(d)=1$, $\forall d\in\mathbb{N}$, by just using things similar to the degeneration results and ideas of [RS2].

e) There is (see [Har] and [Va6]) an arithmetic version of the Borel-Baily compactification (over $\mathbb{C}$) of the moduli space of pseudo-polarized K3 of degree d; so a great part of [JT1-2] should be redonable in this arithmetic setting.

5.5. Satake and Kuga–Satake construction in positive characteristic. Let p be a prime relatively prime to $6dl$. Let k be a perfect field of characteristic p. We refer to 3.2. We assume that either k is algebraically closed or that K_1 and K are small enough so that the principally polarized abelian scheme $(\mathcal{A}_1,\mathcal{P}_{\mathcal{A}_1})$ is obtained by pulling back a principally polarized abelian scheme over $\mathcal{N}_O$. Then to any k-valued point y of $\mathcal{A}_{K3,d,pp,K}$ (i.e. to any level-K marked pseudo-polarized K3 surface (Z,L,k_K) over k) it is associated a principaly polarized abelian variety (A_y,p_{A_y}) over k (we ignore here the different finite symplectic similitude structures it gets naturally). We refer to it as a (see 2.7.2 for a discussion on its unicity; it is uniquely determined if $d'=1$) Satake principally polarized abelian variety attached to (Z,L,k_K) or to (Z,L). We think it would be very useful and interesting to find a direct construction of it in terms of the triple (Z,L,k_K), which does not involve lifts to characteristic zero and specialization.

Of course we have variants (to which the same problem applies) when instead of the data (f_1,L_W) we work with another data (f_3,L_3') or (f_2,L_2) (resp. (f_3,L_3)) of 2.7. As above we obtain a modified p.p. Kuga–Satake or a generalized Kuga–Satake (resp. the Kuga–Satake) "construction" in positive characteristic. Moreover 2.7.2.1 A and B can be entirely transposed into the characteristic p situation (of course, for the part c) of 2.7.2.1 A we need to work with primes l different from p).

5.6. The integral Manin problem for polarized K3 surfaces. Let $d\in\mathbb{N}$. Let k be an algebraically closed field of characteristic $p>0$. In this section we rephrase part of [Ma] and of [Va2, 4.12] in the context of polarized K3 surfaces of degree d over k.

5.6.1. Definition. By a K3 Newton polygon, we mean a Newton polygon starting from $(0,0)$ and ending in $(21,0)$, symmetric with respect to the axis $x = 10.5$, and of whose slopes are $-\alpha$, 0 and α, for some $\alpha \in \{0, \frac{1}{10}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$.

So we have precisely 11 such K3 Newton polygons; they are all above the Newton polygon defined by the slopes -1 (multiplicity 1), 0 (multiplicity 19) and 1 (multiplicity 1).

5.6.2. Let (Z, L) be a polarized K3 surface of degree d over k. It is known (see [Og, (1.5)] for a quick discussion) that its Newton polygon is a K3 Newton polygon. A Manin problem (see [Ma] for the original Manin problem for abelian varieties) in this context can be formulated as follows:

P1. *Determine which K3 Newton polygons are associated to polarized K3 surfaces of degree d over k.*

Of course one expects that the answer to P1 is: all. We assume now that $(p, 6d) = 1$. An integral Manin problem (see [Va2, 4.12] for the context of Shimura varieties) in this context can be formulated as follows:

P2. *Determine which polarized F-crystals over k are associated to polarized K3 surfaces of degree d over k.*

P2 can be reformulated. Let $(W(k)^{21}, \phi_0)$ be the F-crystal over k which respect to the standard basis $E := \{e_1,...,e_{21}\}$ of $H_{\mathrm{pr}} := W(k)^{21}$, is defined by $\phi_0(e_1) = pe_1$, $\Phi(e_i) = e_i$ if $i = \overline{2, 20}$, and $\phi(e_{21}) = \frac{1}{p}e_{21}$. Let B be the perfect symmetric bilinear form on H_{pr} whose attached quadratic form w.r.t. E is $2x_1x_{21} + x_2^2 + ... + x_{20}^2$. From 4.6 we deduce that the polarized F-crystal over k associated to (Z, L) is of the form $(H_{\mathrm{pr}}, g\phi_0, B)$, with $g \in SO(H_{\mathrm{pr}}, B)(W(k))$. So P2 can be reformulated:

P3. *Determine the possible values of such $g \in SO(H, B)(W(k))$.*

Of course we have a relative situation: the problems P1-3 can be rephrased in the context of 3.2.5; this is achieved in the same manner as we passed from [Va2, 4.12.1] to [Va2, 4.12.1] (cf. 4.6-7). Also, once the problems P1-3 are solved, using 4.7 D), we get directly the solution of the general local integral Manin problems for polarized K3 surfaces of degree d relatively prime to the odd prime p (they can be formulated following the pattern of [Va2, 4.12.14-15]).

5.6.3. Remark. The above three problems (in the relative or not context) can be put in the abstract way: we just work with F-crystals, and we interpret them in terms of global deformations. As in the proof of the theorem of 4.7 D), such problems (following the pattern of the proof of the inducing property of 4.6 D)) are already solved by [Va2, Fact' of 3.15 and 4.12.12]; moreover [Va2, Fact' of 3.15] handles (quite successfully) the case when k is just perfect.

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