Stratifications of Newton polygon strata and Traverso's conjectures for p-divisible groups

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dedicated to Thomas Zink, for his 60th anniversary

ABSTRACT. The isomorphism number (resp. isogeny cutoff) of a p-divisible group D over an algebraically closed field of characteristic p is the least positive integer m such that $D[p^m]$ determines D up to isomorphism (resp. up to isogeny). We show that these invariants are lower semicontinuous in families of p-divisible groups of constant Newton polygon. Thus they allow refinements of Newton polygon strata. In each isogeny class of p-divisible groups, we determine the maximal value of isogeny cutoffs and give an upper bound for isomorphism numbers, which is shown to be optimal in the isoclinic case. In particular, the latter disproves a conjecture of Traverso. As an application, we answer a question of Zink on the liftability of an endomorphism of $D[p^m]$ to D.

KEY WORDS: p-divisible groups, truncated Barsotti–Tate groups, displays, Dieudonné modules, Newton polygons, and stratifications.

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1 Introduction

Let k be an algebraically closed field of positive characteristic p. Let D be a p-divisible group over k. It is well-known that D is determined by some finite truncation $D[p^m]$ of sufficiently large level m. This allows to associate to D

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two numerical invariants: The *isomorphism number* n_D is the least level m such that $D[p^m]$ determines D up to isomorphism, and the *isogeny cutoff* b_D is the least level m such that $D[p^m]$ determines D up to isogeny. [1]

In this paper we study how these invariants of D behave in families and how large they can get. As it turns out, the following distance function on isogeny classes of p-divisible groups has closely related properties. The *distance* $q_{D,E}$ between two p-divisible groups D and E over k is the minimal non-negative integer m such that there exists an isogeny $D \to E$ with kernel annihilated by p^m, while $q_{D,E} = \infty$ if no such m exists. The *minimal height* of D is defined to be $q_D = q_{D,D_0}$ where D_0 is the unique (up to isomorphism) minimal p-divisible group in the isogeny class of D. We recall that a minimal p-divisible group is characterized by its isomorphism number being 1.

The numbers b_D, n_D, q_D, and $q_{D,E}$ are invariant under extensions of the algebraically closed base field k; see Lemma 2.8 and Corollary 4.9. For a p-divisible group Δ over an arbitrary field κ of characteristic p we define $b_\Delta = b_{\Delta_{\bar{\kappa}}}$, $n_\Delta = n_{\Delta_{\bar{\kappa}}}$, etc., where $\bar{\kappa}$ is an algebraic closure of κ.

Families

If $\mathscr{D}$ is a p-divisible group over an $\mathbb{F}_p$-scheme S, for each point $s \in S$ we denote by $b_{\mathscr{D}}(s)$, $n_{\mathscr{D}}(s)$, and $q_{\mathscr{D}}(s)$ the isogeny cutoff, the isomorphism number, and the minimal height (respectively) of the geometric fibre $\mathscr{D}_{\bar{s}}$ of $\mathscr{D}$ over s. If $\mathscr{E}$ is another p-divisible group over S we write $q_{\mathscr{D},\mathscr{E}}(s) = q_{\mathscr{D}_{\bar{s}},\mathscr{E}_{\bar{s}}}$.

Theorem 1.1. *Let $\mathscr{D}$ and $\mathscr{E}$ be two p-divisible groups with constant Newton polygon over an $\mathbb{F}_p$-scheme S and let m be a non-negative integer.*

(a) The set $U_{b_{\mathscr{D}}} = \{s \in S \mid b_{\mathscr{D}}(s) \leq m\}$ is closed in S.

(b) The set $U_{n_{\mathscr{D}}} = \{s \in S \mid n_{\mathscr{D}}(s) \leq m\}$ is closed in S.

(c) The set $U_{q_{\mathscr{D}}} = \{s \in S \mid q_{\mathscr{D}}(s) \leq m\}$ is closed in S.

(d) The set $U_{q_{\mathscr{D},\mathscr{E}}} = \{s \in S \mid q_{\mathscr{D},\mathscr{E}}(s) \leq m\}$ is closed in S.

It follows that S carries three natural stratifications into a finite number of reduced locally closed subschemes associated to $\mathscr{D}$: The strata of the b-stratification are the loci where the function $b_{\mathscr{D}} : S \to \mathbb{N}$ is constant; the n-stratification and the q-stratification are defined similarly.

Let us repeat in words part (d) for $m = 0$: *The set of points of S over which the geometric fibres of $\mathscr{D}$ and $\mathscr{E}$ are isomorphic, is a closed subset of S.* When either $\mathscr{D}$ or $\mathscr{E}$ is a constant p-divisible group, this is [Oo2, Thm. 2.2].

[1] This differs from the definitions in [NV2] and [Va3], which give $n_D = b_D = 0$ if either D or its dual is étale, while the definitions we use here give always $n_D, b_D \geq 1$.

The main step in the proof of Theorem 1.1 is the following going down principle: *For p-divisible groups over $k[[t]]$ with constant Newton polygon, the numbers b_D, n_D, q_D, and $q_{D,E}$ go down under specialization* (see Theorems 3.1, 3.8, 3.9, and Corollary 3.3). To deduce the general case of Theorem 1.1 we pass through a truncated variant of Theorem 1.1 presented in Theorems 4.14 and 4.15; this allows us to use the fact that truncated Barsotti–Tate groups have parameter spaces (locally) of finite type. Theorem 1.1 is proved at the end of Subsection 4.3.

Explicit upper bounds

In the following we fix a non-trivial p-divisible group D over k of dimension d, codimension c, and Newton polygon $\nu : [0, c+d] \to \mathbb{R}$. We are interested in upper bounds of b_D and n_D in terms of either ν or c and d. We recall that the a-number of D is $a_D = \dim_k(\mathrm{Hom}(\boldsymbol{\alpha}_p, D))$. Let us begin with the isogeny cutoff b_D and let

$$j(\nu) = \begin{cases} \nu(c) + 1 & \text{if } (c, \nu(c)) \text{ is a breakpoint of } \nu, \\ \lceil \nu(c) \rceil & \text{otherwise.} \end{cases}$$

Here endpoints of ν are considered as breakpoints.

Theorem 1.2. *We have $b_D \leq j(\nu)$ with equality when $a_D \leq 1$.*

This is proved in Section 6. Note that $\nu(c) \leq cd/(c+d)$ with equality precisely when ν is linear. Thus Theorem 1.2 refines the Traverso isogeny conjecture which is proved in [NV2] and which asserts that $b_D \leq \lceil cd/(c+d) \rceil$ when $cd > 0$ with equality for some D of dimension d and codimension c.

We have a similar upper bound for isomorphism numbers.

Theorem 1.3. *If D is not ordinary, then $n_D \leq \lfloor 2\nu(c) \rfloor$.*

If D is ordinary then $n_D = 1$ and $\nu(c) = 0$. We refer to Corollary 9.4 for the proof of Theorem 1.3. We expect that this upper bound of n_D is optimal for every Newton polygon ν, but in this paper this is proved only if ν is linear i.e., if D is isoclinic (see Proposition 9.16). We thus conclude that:

Corollary 1.4. *Assume that $cd > 0$. Then $n_D \leq \lfloor 2cd/(c+d) \rfloor$ with equality for some (isoclinic) p-divisible group D of dimension d and codimension c.*

Our search for upper bounds of n_D was guided by the Traverso truncation conjecture [Tr3, §40, Conj. 4] which predicts that $n_D \leq \min\{c, d\}$ if $cd > 0$. This estimate is well-known if $\min\{c, d\} = 1$. It is verified for supersingular

p-divisible groups in [NV1, Thm. 1.2] and for quasi-special p-divisible groups in [Va3, Thm. 1.5.2]; for $|d-c| \leq 2$ it follows indeed from Corollary 1.4. But Corollary 1.4 also shows that the original conjecture is wrong in general, even for isoclinic p-divisible groups; the first counterexamples show up when $\{c, d\} = \{2, 6\}$. For a fixed positive value of $t = \min\{c, d\}$, the natural number $\lfloor 2cd/(c+d) \rfloor$ can be any integer in the interval $[t, 2t-1]$.

Quantitative upper bounds of n_D in terms of c and d can be traced back to [Tr1, Thm. 3], where the inequality $n_D \leq cd + 1$ is established. A weaker upper bound with a simpler proof can be found in [Tr2, Thm. 1]. More recently, [GV, Cor. 1] shows that $n_D \leq cd$ if $cd > 0$ and that $n_D \leq cd + 1 - a_D^2$ if D is not ordinary.

A key result of [GV] characterizes n_D as the minimal positive integer such that the truncation homomorphism $\mathrm{End}(D[p^{n_D+1}]) \to \mathrm{End}(D[p])$ has finite image, cf. [GV, Cor. 2 (b)]. Using this characterization, we prove Theorem 1.3 in the case $a_D = 1$ by a detailed analysis of Dieudonné modules over k. The general case of Theorem 1.3 follows by the going down principle, using the fact that D is the specialization of a p-divisible group over $k((t))$ with a-number at most one and with Newton polygon ν by [Oo1].

Assume that D is equipped with a principal quasi-polarisation λ; thus $c = d > 0$. The isomorphism number $n_{D,\lambda}$ is the least level m such that $(D[p^m], \lambda[p^m])$ determines (D, λ) up to isomorphism. Based on [GV], [NV1], and Theorem 1.3 we prove in Subsection 9.4 that $n_{D,\lambda} \leq d$; this bound is optimal.

Relation with minimal heights

Another approach to bound n_D and b_D from above is based on the fact that an isogeny of D with kernel annihilated by p changes b_D at most by one and n_D at most by two, see Lemma 6.3 and Proposition 9.8. It turns out that the results on Dieudonné modules used in the proof of Theorem 1.3 also give the following upper bound of minimal heights (see end of Subsection 5.4):

Theorem 1.5. *We have $q_D \leq \lfloor \nu(c) \rfloor$ with equality when $a_D = 1$. In other words, if D_0 is the unique (up to isomorphism) minimal p-divisible group over k of Newton polygon ν, then there exists an isogeny $D \to D_0$ whose kernel is annihilated by $p^{\lfloor \nu(c) \rfloor}$, and this exponent is optimal when $a_D = 1$.*

By the new approach, this result gives the inequalities $b_D \leq 1 + \lfloor \nu(c) \rfloor$ and $n_D \leq 1 + 2\lfloor \nu(c) \rfloor$ because $b_{D_0} = n_{D_0} = 1$. The first estimate coincides with the upper bound in Theorem 1.2 except when $\nu(c)$ is an integer and ν is linear at c, in which case it is off by 1. When $\lfloor 2\nu(c) \rfloor$ is odd, the second estimate is precisely Theorem 1.3, while it is again off by 1 otherwise.

We remark that the existence of some upper bound of q_D is already proved in [Ma, p. 44]; see also [Oo2, p. 270]. The supersingular case of Theorem 1.5 follows from [NV1, Rmk. 2.6 and Cor. 3.2].

Lifting of endomorphisms

We apply the explicit upper bounds of n_D to the lifting of endomorphisms of truncations of D and to the level torsion ℓ_D of D defined in [Va3].

There exists a non-negative integer e_D, which we call the *endomorphism number* of D, characterized by the following property: For positive integers $m \geq n$, the two restriction homomorphisms

$$\mathrm{End}(D) \xrightarrow{\tau_{\infty,n}} \mathrm{End}(D[p^n]) \xleftarrow{\tau_{m,n}} \mathrm{End}(D[p^m])$$

have equal images if and only if $m \geq n + e_D$. In other words: An endomorphism of $D[p^n]$ lifts to an endomorphism of D if and only if it lifts to an endomorphism of $D[p^{n+e_D}]$, and e_D is minimal with this property for each positive integer n separately; see Lemma 2.1.

Let (M, F, V) be the covariant Dieudonné module of D. In Subsection 8.1 we recall the definition of the *level module* $O \subseteq \mathrm{End}_{W(k)}(M)$ introduced in [Va3]. The *level torsion* ℓ_D is the smallest non-negative integer such that $p^{\ell_D} \mathrm{End}_{W(k)}(M) \subseteq O$. [2]

These new invariants are related to n_D as follows (see Subsection 8.4):

Theorem 1.6. *If D is a non-ordinary p-divisible group over an algebraically closed field k, then we have* $n_D = \ell_D = e_D$.

If D is ordinary we have $n_D = 1$ and $\ell_D = e_D = 0$. We assume now that D is not ordinary. In [Va3] it was shown that $n_D \leq \ell_D$ and that the equality holds if D is a direct sum of isoclinic p-divisible groups; the equality was also expected to hold in general therein. In this paper, we prove the inequalities

$$n_D \leq e_D \leq \ell_D \leq n_D.$$

The second inequality $e_D \leq \ell_D$ is not too difficult. The other two inequalities use again the [GV] characterization of n_D mentioned above. Then $n_D \leq e_D$ is immediate, but the inequality $\ell_D \leq n_D$ is a lot more involved.

Together with the upper bound of Theorem 1.3 we obtain the following effective lifting of endomorphisms, which answers a question of Th. Zink.

[2]This differs from the definition in [Va3], which gives $\ell_D = 1$ if D is ordinary and $cd > 0$, while the definition we use here gives $\ell_D = 0$ when D is ordinary.

Corollary 1.7. *Let ν and c be the Newton polygon and the codimension of D. Let n be a positive integer. An endomorphism of $D[p^n]$ lifts to an endomorphism of D if and only if it lifts to an endomorphism of $D[p^{n+\lfloor 2\nu(c)\rfloor}]$.*

For similar results on homomorphisms we refer to Subsections 8.4 and 9.1. As a special case, for each $h \in \mathbb{N}$ we compute the minimal number N_h such that for every two p-divisible groups D and E over k of height at most h, a homomorphism $D[p^n] \to E[p^n]$ lifts to a homomorphism $D \to E$ if and only if it lifts to a homomorphism $D[p^{n+N_h}] \to E[p^{n+N_h}]$: By Proposition 9.20 we have $N_h = \lfloor h/2 \rfloor$.

Terminology. A *BT group of level* n is a truncated Barsotti–Tate group of level n. We denote by $\mathbb{N}^*$ the set of positive integers.

2 Preliminaries

We begin with a lemma on homomorphisms.

Lemma 2.1. *Let D and E be p-divisible groups over k.*

(a) For each positive integer n there exists a non-negative integer $e_{D,E}(n)$ with the following property: For $e \in \mathbb{N}$, the two restriction maps

$$\mathrm{Hom}(D, E) \to \mathrm{Hom}(D[p^n], E[p^n]) \leftarrow \mathrm{Hom}(D[p^{n+e}], E[p^{n+e}])$$

have equal images if and only if $e \geq e_{D,E}(n)$.

(b) There exists an upper bound of $e_{D,E}(n)$ in terms of the heights of D and E.

(c) The number $e_{D,E}(n)$ does not depend on n.

Proof. For (a) and (b) we refer to [Oo2, Prop. 1.6] or [Va1, Thm. 5.1.1 (c)]. We prove (c). Let $H_\infty = \mathrm{Hom}(D, E)$. For $n \in \mathbb{N}^*$ let $H_n = \mathrm{Hom}(D[p^n], E[p^n])$. Following [GV, Subsect. 2.1], we have two exact sequences

$$0 \to H_\infty \xrightarrow{p} H_\infty \to H_1, \qquad 0 \to H_n \xrightarrow{\iota_n} H_{n+1} \to H_1,$$

where ι_n maps $u \in H_n$ to the obvious composition

$$D[p^{n+1}] \to D[p^n] \xrightarrow{u} E[p^n] \to E[p^{n+1}].$$

For $m \in \mathbb{N} \cup \{\infty\}$ with $m \geq n$ let $H_{m,n}$ be the image of $H_m \to H_n$. One deduces that for all $e \in \mathbb{N}$ we have a homomorphism of exact sequences with vertical injections:

$$
\begin{array}{lccccccccc}
(2.1) & 0 & \longrightarrow & H_{\infty,n} & \xrightarrow{\iota_n} & H_{\infty,n+1} & \longrightarrow & H_{\infty,1} & \longrightarrow & 0 \\
& & & \downarrow & & \downarrow & & \downarrow & & \\
& 0 & \longrightarrow & H_{n+e,n} & \xrightarrow{\iota_n} & H_{n+1+e,n+1} & \longrightarrow & H_{1+e,1}. & &
\end{array}
$$

The snake lemma implies that $e_{D,E}(n) \leq e_{D,E}(n+1) \leq \max\{e_{D,E}(n), e_{D,E}(1)\}$. By induction on $n \in \mathbb{N}^*$ we get that $e_{D,E}(n) = e_{D,E}(1)$. Thus (c) holds. $\square$

Lemma 2.1 (c) allows to define:

Definition 2.2. The *homomorphism number* $e_{D,E} \in \mathbb{N}$ of D and E is the constant value of $e_{D,E}(n)$ for $n \in \mathbb{N}^*$. The *endomorphism number* of D is defined as $e_D = e_{D,D}$.

By Lemma 2.1 (b), for $h \in \mathbb{N}$ there exists a minimal number $N_h \in \mathbb{N}$ such that for every two p-divisible groups D and E over k of height at most h we have $e_{D,E} \leq N_h$. In Proposition 9.20, we will see that $N_h = \lfloor h/2 \rfloor$.

We continue with a formal definition of the numerical invariants n_D and b_D.

Definition 2.3. Let D be a p-divisible group over k. The *isomorphism number* n_D (resp. the *isogeny cutoff* b_D) is the smallest non-negative integer m with the following property: If E is a p-divisible group over k such that $E[p^m]$ is isomorphic to $D[p^m]$, then E is isomorphic (resp. isogenous) to D.

Such integers m exist, for example $m = 1 + N_h$ if D has height h. Thus b_D and n_D are well-defined. The following properties are easily established.

Lemma 2.4. *For a p-divisible group D over k let $D^\vee$ be the dual of D and let $D = D^{\text{ét}} \times D^\circ$ be the canonical product decomposition, where $D^{\text{ét}}$ is étale and D° is connected. We have:*

(a) $1 \leq b_D \leq n_D$;

(b) $b_D = b_{D^\vee}$ *and* $n_D = n_{D^\vee}$;

(c) $b_D = b_{D^\circ}$ *and* $n_D = n_{D^\circ}$.

In particular, if D is ordinary, then $b_D = n_D = 1$. Similarly, if $\min\{c, d\} = 1$, then $b_D = n_D = 1$ because all one-dimensional connected p-divisible groups of given height are isomorphic (see [De, Ch. IV, Sect. 8, Prop.]). Thus the invariants b_D and n_D are interesting only when $\min\{c, d\} \geq 2$.

We consider the following distance function on isogeny classes.

Definition 2.5. Let D and E be two p-divisible groups over k. If D and E are isogenous, then their distance $q_{D,E}$ is the smallest non-negative integer m such that there exists an isogeny $\rho : D \to E$ with $\mathrm{Ker}(\rho) \subseteq D[p^m]$. If D and E are not isogenous we define $q_{D,E} = \infty$.

Again, the next lemma is easily checked.

Lemma 2.6. *The following three properties hold:*

(a) *We have $q_{D,E} = 0$ if and only if D and E are isomorphic.*

(b) *We have $q_{D,E} = q_{E,D} = q_{D^\vee,E^\vee}$.*

(c) *If $q_{D,E} < \infty$, then $q_{D,E} = q_{D^\circ,E^\circ}$.*

Recall that a p-divisible group D over k is called *minimal* if $\mathrm{End}(D)$ is a maximal order in $\mathrm{End}(D) \otimes_{\mathbb{Z}} \mathbb{Q}$; this is equivalent to the condition $n_D = 1$. For each p-divisible group D over k, there exists a unique (up to isomorphism) minimal p-divisible group D_0 over k isogenous to D. See [Oo3, Subsects. 1.1-1.2] and [Va3, Thm. 1.6] for these facts. Thus following [NV2] we define:

Definition 2.7. The *minimal height* of D is $q_D = q_{D,D_0}$.

As $(D_0)^\vee$ and $(D_0)^\circ$ are also minimal, Lemma 2.6 implies that

$$q_D = q_{D^\vee} = q_{D^\circ}.$$

We have the following permanence properties.

Lemma 2.8. *Let $k \subseteq \kappa$ be an extension of algebraically closed fields. For p-divisible groups D and E over k we have:*

(a) $\mathrm{Hom}(D, E) \cong \mathrm{Hom}(D_\kappa, E_\kappa)$*;*

(b) $q_{D,E} = q_{D_\kappa,E_\kappa}$ *and* $q_D = q_{D_\kappa}$*;*

(c) $e_{D,E} = e_{D_\kappa,E_\kappa}$.

Proof. Part (a) is well-known, and (b) follows from (a). For positive integers $m \geq n$, let $\underline{\mathrm{H}}_{m,n}$ be the scheme theoretic image of the reduction homomorphism $\underline{\mathrm{Hom}}(D[p^m], E[p^m]) \to \underline{\mathrm{Hom}}(D[p^n], E[p^n])$. If $l \geq m$ is an integer, then $\underline{\mathrm{H}}_{l,n}$ is a subgroup scheme of $\underline{\mathrm{H}}_{m,n}$. We have $m - n \geq e_{D,E}$ if and only if $\underline{\mathrm{H}}_{m,n}(k) = \underline{\mathrm{H}}_{l,n}(k)$ for all integers $l \geq m$. As $\underline{\mathrm{H}}_{m,n}$ is of finite type over k and its definition is compatible with the base change from k to κ, we get (c). $\square$

The identities $n_D = n_{D_\kappa}$ and $b_D = b_{D_\kappa}$ also hold, cf. Corollary 4.9 below.

3 The going down principle

In this section, we prove Theorem 1.1 in the case $S = \operatorname{Spec} k[[t]]$. By standard arguments this implies that the functions $n_{\mathscr{D}}$, $b_{\mathscr{D}}$, $q_{\mathscr{D}}$, and $q_{\mathscr{D},\mathscr{E}}$ go down under specialization when the base scheme is noetherian.

3.1 Distances and minimal heights

Going down of distance numbers follows from the constancy results of [OZ]:

Theorem 3.1. *Let $R = k[[t]]$, let $K = k((t))$, and let $\bar{K}$ be an algebraic closure of K. If $\mathscr{D}$ and $\mathscr{E}$ are p-divisible groups over R with constant Newton polygon ν, then*

$$q_{\mathscr{D}_k,\mathscr{E}_k} \leq q_{\mathscr{D}_{\bar{K}},\mathscr{E}_{\bar{K}}}.$$

Proof. Let $\bar{R}$ be the normalization of R in $\bar{K}$. By [OZ, Cor. 3.2] there exist p-divisible groups D and E over k and isogenies

$$\rho : D_{\bar{R}} \to \mathscr{D}_{\bar{R}} \qquad \text{and} \qquad \omega : E_{\bar{R}} \to \mathscr{E}_{\bar{R}}.$$

Let $m = q_{\mathscr{D}_{\bar{K}},\mathscr{E}_{\bar{K}}}$. Note that $m < \infty$. By definition there exists an isogeny $\xi_{\bar{K}} : \mathscr{D}_{\bar{K}} \to \mathscr{E}_{\bar{K}}$ with kernel annihilated by p^m. As $\operatorname{Hom}(D, E)$ is equal to $\operatorname{Hom}(D_{\bar{K}}, E_{\bar{K}})$, the composite quasi-isogeny over $\bar{K}$

$$\chi_{\bar{K}} : D_{\bar{K}} \xrightarrow{\rho_{\bar{K}}} \mathscr{D}_{\bar{K}} \xrightarrow{\xi_{\bar{K}}} \mathscr{E}_{\bar{K}} \xleftarrow{\omega_{\bar{K}}} E_{\bar{K}}$$

is defined over k i.e., it arises from a quasi-isogeny $\chi : D \to E$. Hence $\xi_{\bar{K}}$ extends to a quasi-isogeny over $\bar{R}$

$$\xi : \mathscr{D}_{\bar{R}} \xleftarrow{\rho} D_{\bar{R}} \xrightarrow{\chi_{\bar{R}}} E_{\bar{R}} \xrightarrow{\omega} \mathscr{E}_{\bar{R}}.$$

As ξ and $p^m\xi^{-1}$ are isogenies over $\bar{K}$, by the well-known Lemma 3.2 below the same is true over $\bar{R}$. As the residue field of the local ring $\bar{R}$ is k, the special fibre of ξ is an isogeny $\xi_k : \mathscr{D}_k \to \mathscr{E}_k$ with kernel annihilated by p^m, which implies that $q_{\mathscr{D}_k,\mathscr{E}_k} \leq m$ as required. $\square$

Lemma 3.2 ([RZ, Prop. 2.9])**.** *For a quasi-isogeny $\chi : \mathscr{D} \to \mathscr{E}$ of p-divisible groups over a scheme T of characteristic p, there exists a unique closed subscheme $T_0 \subseteq T$ such that a morphism $T' \to T$ factors through T_0 if and only if $\chi_{T'}$ is an isogeny.*

Corollary 3.3. *With R and $\bar{K}$ as in Theorem 3.1, for a p-divisible group $\mathscr{D}$ over R with constant Newton polygon ν we have*

$$q_{\mathscr{D}_k} \leq q_{\mathscr{D}_{\bar{K}}}.$$

Proof. Let D_0 be the minimal p-divisible group over k with Newton polygon ν and apply Theorem 3.1 with $\mathscr{E} = D_0 \times_{\operatorname{Spec} k} \operatorname{Spec} R$. $\square$

Remark 3.4. The proof of Theorem 3.1 could also be formulated in terms of Dieudonné modules over the perfect ring $\bar{R}$. Then the reference to [OZ] can be replaced by [Ka, Thm. 2.7.4]. Here one only needs to know that to a p-divisible group over $\bar{R}$ one can associate a covariant Dieudonné module (by [Be, Cor. 3.4.3] this association is actually an equivalence of categories).

3.2 An auxiliary family of p-divisible groups

The following general construction is used in the proof of the going down principle for isogeny cutoffs, but we prove more than is actually needed.

Lemma 3.5. *Let $\mathscr{D}$ be an infinitesimal p-divisible group of height h over an affine $\mathbb{F}_p$-scheme X and let $m \geq 1$ be an integer. Then there exists a vector bundle $Y \to X$ of rank h^2 and an infinitesimal p-divisible group $\mathscr{E}$ over Y such that the following three properties hold:*

(i) If $o : X \to Y$ denotes the zero section, then we have $o^*\mathscr{E} \cong \mathscr{D}$.

(ii) If $\pi : Y \to X$ denotes the projection, then we have $\mathscr{E}[p^m] \cong \pi^*\mathscr{D}[p^m]$.

(iii) For each geometric point $x : \operatorname{Spec} \kappa \to X$, every BT group B of level $m+1$ over κ with $B[p^m] \cong x^*\mathscr{D}[p^m]$ is isomorphic to $y^*\mathscr{E}[p^{m+1}]$ for some geometric point $y : \operatorname{Spec} \kappa \to Y$ with $\pi \circ y = x$.

The proof uses displays. For a commutative ring R with unit, let $W(R)$ be its p-typical Witt ring, and let I_R be the kernel of the projection $W(R) \to R$. The Frobenius endomorphism of $W(R)$ is denoted by σ.

Let us first recall how the display of a p-divisible group over k is related to the covariant Dieudonné modules of its truncations. For a positive integer n, let $W_n = W_n(k) = W(k)/(p^n)$ be the truncated Witt ring, and let $W_\infty = W(k)$. A Dieudonné module over k of level $n \in \mathbb{N}^* \cup \{\infty\}$ is a triple (P, F, V) where P is a free W_n-module of finite rank, $F : P \to P$ is σ-linear, $V : P \to P$ is σ^{-1}-linear, and we have $FV = VF = p$. If $n = 1$, we require that $\operatorname{Ker}(F) = \operatorname{Im}(V)$. A different way to represent these objects is as follows.

Definition 3.6. Let $n \in \mathbb{N}^* \cup \{\infty\}$. A display over k of level n is a collection $(P, Q, \iota, \varepsilon, F_1)$, where P and Q are free W_n-modules of the same finite rank, where $\iota : Q \to P$ and $\varepsilon : P \to Q$ are W_n-linear maps with $\iota\varepsilon = p$ and $\varepsilon\iota = p$, and where $F_1 : Q \to P$ is a σ-linear isomorphism. For $n = 1$, we require that $\operatorname{Ker}(\varepsilon) = \operatorname{Im}(\iota)$.

For $n \in \mathbb{N}^* \cup \{\infty\}$, Dieudonné modules over k of level n are equivalent to displays over k of level n via the association $(P, Q, \iota, \varepsilon, F_1) \mapsto (P, F, V)$ with $F = F_1\varepsilon$ and $V = \iota F_1^{-1}$. On the other hand, Dieudonné modules over k of level n are equivalent to BT groups of level n over k. These equivalences are compatible with the natural truncation operations on all sides. Displays over k of level ∞ are equivalent to (not necessarily nilpotent) displays over k in the sense of [Zi1] because for $n = \infty$ we can identify Q with the W_n-submodule $\iota(Q)$ of P.

The essence of the proof of Lemma 3.5 is the following. Assume that for $n \in \mathbb{N}^*$ we want to lift a display $(P, Q, \iota, \varepsilon, F_1)$ over k of level n to a display $(P', Q', \iota', \varepsilon', F_1')$ over k of level $n + 1$. It is easy to see that the quadruple $(P', Q', \iota', \varepsilon')$ is uniquely determined up to isomorphism and thus we can fix it. The set of lifts F_1' of F_1 is a principally homogeneous space under the k-vector space of σ-linear maps $Q \to p^n P' \cong P/pP$.

Proof of Lemma 3.5. Let $X = \operatorname{Spec} R$. We recall that the category of infinitesimal p-divisible groups over R is equivalent to the category of nilpotent displays over R, cf. [Zi1] and [La]. Let $\mathscr{P} = (P, Q, F, F_1)$ be the nilpotent display over R associated to $\mathscr{D}$ i.e., $\mathscr{D} = BT(\mathscr{P})$ in the sense of [Zi1, Thm. 81]. Here $F : P \to P$ and $F_1 : Q \to P$ are σ-linear maps of $W(R)$-modules. We choose a normal decomposition $P = J \oplus T$ of finitely generated projective $W(R)$-modules such that $Q = J \oplus I_R T$. Let $\Psi : P \to P$ be the σ-linear automorphism given by F_1 on J and by F on T, cf. [Zi1, Lemma 9].

Let $E = \operatorname{Hom}_{W(R)}(P^{(\sigma)}, P)$ where $P^{(\sigma)} = W(R) \otimes_{\sigma, W(R)} P$. We define $Y = \operatorname{Spec} R'$ to be the vector bundle over X associated to the projective R-module $\bar{E} = E/I_R E$, in other words $R' = \operatorname{Sym}^*(\bar{E}^\vee)$ with $\bar{E}^\vee = \operatorname{Hom}_R(\bar{E}, R)$. For a $W(R)$-module M we write $M_{R'} = W(R') \otimes_{W(R)} M$.

Let $u = \mathrm{id} \in \bar{E} \otimes_R \bar{E}^\vee \subset \bar{E} \otimes_R R'$ be the tautological section. Let $\tilde{u} \in E_{R'}$ be a lift of u that maps to zero under the $W(R)$-linear map $E_{R'} \to E$ induced naturally by the zero section of Y. Let $\tilde{\mathscr{P}} = (P_{R'}, J_{R'} \oplus I_{R'} T_{R'}, \tilde{F}, \tilde{F}_1)$ be the display over R' such that the σ-linear automorphism $\tilde{\Psi} : P_{R'} \to P_{R'}$ defined by $\tilde{F}_1$ on $J_{R'}$ and by $\tilde{F}$ on $T_{R'}$ is equal to $\sigma \otimes \Psi + p^m \tilde{u}$; here we view $\tilde{u} \in E_{R'}$ as a σ-linear endomorphism of $P_{R'}$. Then $\tilde{\mathscr{P}}$ is again a nilpotent display because the nilpotency condition depends only on $\tilde{\Psi}$ modulo p. Finally let $\mathscr{E} = BT(\tilde{\mathscr{P}})$ be the associated infinitesimal p-divisible group over Y.

We have to verify that properties (i) to (iii) hold. Condition (i) follows from the construction. The discussion preceding this proof implies that (iii) and the following weakening of (ii) hold (they suffice for our application).

(ii)$'$ If $y_1, y_2 : \operatorname{Spec} \kappa \to Y$ are geometric points with equal image in $X(\kappa)$, then there exists an isomorphism $y_1^* \mathscr{E}[p^m] \cong y_2^* \mathscr{E}[p^m]$.

Condition (ii) follows from the following general lemma. $\square$

Lemma 3.7. *Let R be a ring in which p is nilpotent and let*

$$\mathscr{P} = (P, Q, F, F_1) \qquad \text{and} \qquad \mathscr{P}' = (P, Q, F', F_1')$$

be two displays over R with the same $W(R)$-modules. For a given normal decomposition $P = J \oplus T$ i.e., $Q = J \oplus I_R T$, let $\Psi : P \to P$ (resp. $\Psi' : P \to P$) be the associated σ-linear automorphism given by F_1 (resp. F_1') on J and by F (resp. F') on T. If $\Psi' - \Psi = p^n \Omega$ for a σ-linear endomorphism $\Omega : P \to P$, then there exists an isomorphism of fppf sheaves over R

$$BT(\mathscr{P})[p^n] \cong BT(\mathscr{P}')[p^n].$$

Here $BT(\mathscr{P})$ is the formal group over R associated to $\mathscr{P}$ by [Zi1, Thm. 81], which is a p-divisible group when $\mathscr{P}$ is nilpotent.

Proof. Let $P_2 = P \oplus P$ and $Q_2 = Q \oplus Q$ be endowed with the induced normal decompositions $P_2 = J \oplus T \oplus J \oplus T$ and $Q_2 = J \oplus I_R T \oplus J \oplus I_R T$. We define two displays $\mathscr{K}$ and $\mathscr{K}'$ with the same underlying $W(R)$-modules $Q_2 \subseteq P_2$ such that the associated operators Ψ with respect to these normal decomposition are

$$\Psi_{\mathscr{K}} = \begin{pmatrix} \Psi & -\Omega \\ 0 & \Psi' \end{pmatrix} \qquad \text{and} \qquad \Psi_{\mathscr{K}'} = \begin{pmatrix} \Psi & 0 \\ \Omega & \Psi' \end{pmatrix}$$

as block matrices for $P_2 = P \oplus P$. The matrix $\Gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & p^n \end{smallmatrix}\right)$ defines an isomorphism $\Gamma : \mathscr{K}' \cong \mathscr{K}$. Indeed, as Γ respects the normal decompositions, this is equivalent to the relation $\Gamma \Psi_{\mathscr{K}'} = \Psi_{\mathscr{K}} \Gamma$, which is easily checked. Moreover, there exist homomorphisms of displays

$$\Theta : \mathscr{K} \to \mathscr{P} \oplus \mathscr{P}' \qquad \text{and} \qquad \Theta' : \mathscr{K}' \to \mathscr{P} \oplus \mathscr{P}'$$

given by the matrices $\Theta = \left(\begin{smallmatrix} p^n & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\Theta' = \left(\begin{smallmatrix} 1 & 0 \\ 1 & p^n \end{smallmatrix}\right)$; the required relations $\Theta \Psi_{\mathscr{K}} = (\Psi \oplus \Psi')\Theta$ and $\Theta' \Psi_{\mathscr{K}'} = (\Psi \oplus \Psi')\Theta'$ are easily verified. We have $\Theta\Gamma = \Theta'$. Consider the following homomorphisms of complexes of displays:

$$[\mathscr{P}' \xrightarrow{p^n} \mathscr{P}'] \xrightarrow{u'} [\mathscr{K}' \xrightarrow{\Theta'} \mathscr{P} \oplus \mathscr{P}'] \cong [\mathscr{K} \xrightarrow{\Theta} \mathscr{P} \oplus \mathscr{P}'] \xleftarrow{u} [\mathscr{P} \xrightarrow{p^n} \mathscr{P}].$$

The middle isomorphism is given by $\Gamma : \mathscr{K}' \cong \mathscr{K}$ and the identity of $\mathscr{P} \oplus \mathscr{P}'$. The homomorphism u is $x \mapsto (x, 0)$ in both degrees, while u' is $x \mapsto (0, x)$ in both degrees. We have an exact sequence of complexes of displays

$$0 \to [\mathscr{P} \xrightarrow{p^n} \mathscr{P}] \xrightarrow{u} [\mathscr{K} \xrightarrow{\Theta} \mathscr{P} \oplus \mathscr{P}'] \to [\mathscr{P}' \xrightarrow{1} \mathscr{P}'] \to 0$$

and a similar one with the roles of u and Θ taken by u' and Θ'. As the functor BT preserves exact sequences, it follows that $BT(u)$ and $BT(u')$ are quasi-isomorphisms of complexes of fppf sheaves over R. Thus the complex $[p^n : BT(\mathscr{P}) \to BT(\mathscr{P})]$ is quasi-isomorphic to $[p^n : BT(\mathscr{P}') \to BT(\mathscr{P}')]$, and the lemma holds. □

3.3 Isogeny cutoffs

Theorem 3.8. *Let $R = k[[t]]$ and let $\bar{K}$ be an algebraic closure of $K = k((t))$. For every p-divisible group $\mathscr{D}$ over R with constant Newton polygon ν we have*

$$b_{\mathscr{D}_k} \leq b_{\mathscr{D}_{\bar{K}}}.$$

Proof. We can assume that $\mathscr{D}$ is infinitesimal because there exists an exact sequence of p-divisible groups $0 \to \mathscr{D}^\circ \to \mathscr{D} \to \mathscr{D}^{\text{ét}} \to 0$ with $\mathscr{D}^{\text{ét}}$ étale and $\mathscr{D}_k^\circ$ connected. Then $\mathscr{D}^\circ$ is infinitesimal because its Newton polygon is constant. By Lemma 2.4, we can replace $\mathscr{D}$ by $\mathscr{D}^\circ$.

We show that the assumption $b_{\mathscr{D}_k} > b_{\mathscr{D}_{\bar{K}}}$ leads to a contradiction. We can assume that $\mathscr{D}$ has the following maximality property: For each p-divisible group $\mathscr{D}'$ over R of constant Newton polygon ν with $b_{\mathscr{D}_{\bar{K}}} = b_{\mathscr{D}'_{\bar{K}}}$, we have $b_{\mathscr{D}_k} \geq b_{\mathscr{D}'_k}$. Let $m = b_{\mathscr{D}_k} - 1 > 0$. We consider a vector bundle $\pi : Y \to X$ and an infinitesimal p-divisible group $\mathscr{E}$ over Y such that properties (i) to (iii) of Lemma 3.5 hold. Note that $Y \cong \operatorname{Spec} R[t_1, \ldots, t_{h^2}]$, as R is local.

We claim that $\mathscr{E}$ has constant Newton polygon ν. Let $Y_\nu \subseteq Y$ be the ν-stratum of the Newton polygon stratification on Y defined by $\mathscr{E}$. As $\mathscr{E}[p^m] \cong \pi^*\mathscr{D}[p^m]$ and as $m \geq b_{\mathscr{D}_{\bar{K}}}$ by assumption, Y_ν contains all the $\bar{K}$-valued points of the generic fibre Y_K and thus it contains Y_K. In particular, Y_ν is the unique open stratum. Moreover, Y_ν contains the image of the zero section $o : X \to Y$ as $o^*\mathscr{E} \cong \mathscr{D}$. Hence the complement of Y_ν in Y has codimension ≥ 2. By the (weak) purity theorem for Newton polygon strata (see [dJO, Thm. 4.1], [Va1, Thm. 1.6], or [Zi2]) we get that $Y = Y_\nu$ as claimed.

Next we consider the function $b_{\mathscr{E}} : Y \to \mathbb{N}$. As $\mathscr{E}[p^m] \cong \pi^*\mathscr{D}[p^m]$, the relation $b_{\mathscr{D}_{\bar{K}}} \leq m$ implies that $b_{\mathscr{E}}(y) = b_{\mathscr{D}_{\bar{K}}}$ for all closed points $y \in Y_K$, and the relation $b_{\mathscr{D}_k} > m$ implies that $b_{\mathscr{E}}(y) > m$ for all closed points $y \in Y_k$. By the maximality property of $\mathscr{D}$ it follows that $b_{\mathscr{E}}(y) = m + 1$ for all closed points $y \in Y_k$. Thus, as $\mathscr{E}$ has constant Newton polygon, from the property (iii) of Lemma 3.5 we get that $b_{\mathscr{D}_k} \leq m$. Contradiction. □

3.4 Isomorphism numbers

Theorem 3.9. *Let R, K, and $\bar{K}$ be as in the previous theorem. For every p-divisible group $\mathscr{D}$ over R with constant Newton polygon ν we have*

$$n_{\mathscr{D}_k} \leq n_{\mathscr{D}_{\bar{K}}}.$$

Proof. As in the proof of Theorem 3.8 we can assume that $\mathscr{D}$ is infinitesimal. Let $m = n_{\mathscr{D}_{\bar{K}}}$. Let E be a p-divisible group over k such that $E[p^m]$ is isomorphic to $\mathscr{D}_k[p^m]$. By [Ill, Thm. 4.4 f)] there exists a p-divisible group

$\mathscr{E}$ over R such that $\mathscr{E}_k \cong E$ and $\mathscr{E}[p^m] \cong \mathscr{D}[p^m]$. By the choice of m, the last isomorphism implies that $\mathscr{E}_{\bar{K}}$ and $\mathscr{D}_{\bar{K}}$ are isomorphic. Thus $q_{\mathscr{D}_{\bar{K}},\mathscr{E}_{\bar{K}}} = 0$. By Theorem 3.8 and Lemma 2.4 (a) we have $b_{\mathscr{D}_k} \leq b_{\mathscr{D}_{\bar{K}}} \leq m$. Therefore $\mathscr{E}_k$ and $\mathscr{D}_k$ have the same Newton polygon, which implies that $\mathscr{E}$ has constant Newton polygon ν. As $q_{\mathscr{D}_{\bar{K}},\mathscr{E}_{\bar{K}}} = 0$, from Theorem 3.1 we get that $q_{\mathscr{D}_k,E} = 0$ i.e., E is isomorphic to $\mathscr{D}_k$. Thus $n_{\mathscr{D}_k} \leq m$. $\square$

4 Semicontinuity

In this section, we prove Theorem 1.1 and truncated variants of it.

4.1 Lifting the level of truncated BT groups

Assume that $n \geq m \geq 1$ are integers. Let $\mathscr{X}_n$ be the algebraic stack of BT groups of level n and let $\tau : \mathscr{X}_n \to \mathscr{X}_m$ be the truncation morphism.

Definition 4.1. Let $\mathscr{B}$ be a BT group of level m over a scheme X. An *exhaustive extension* of $\mathscr{B}$ to level n is a BT group $\mathscr{C}$ of level n over an X-scheme Y together with an isomorphism $\mathscr{B}_Y \cong \mathscr{C}[p^m]$ such that the induced morphism $Y \to X \times_{\mathscr{X}_m} \mathscr{X}_n$ is surjective on geometric points.

It is easy to see that each $\mathscr{B}$ over X as above has an exhaustive extension to level n over an affine X-scheme Y of finite type. Namely, if $Z \to \mathscr{X}_n$ is an affine smooth presentation, we can take $Y = X \times_{\mathscr{X}_m} Z$. As τ is smooth and surjective by [Ill, Thm. 4.4], $Y \to X$ is as well affine smooth and surjective.

With more effort, one can arrange that the universal BT group of level n over Z (and thus also the exhaustive extension $\mathscr{C}$ over Y) comes from a p-divisible group (cf. [NVW, Prop. 2.3] via a natural passage to an affine open cover). Here we need only the following consequence of this fact, which can be deduced from the previous paragraph by a limit argument.

Lemma 4.2. *For a BT group $\mathscr{B}$ of level m over a scheme X there exists a faithfully flat affine morphism $Y \to X$ and a p-divisible group $\mathscr{D}$ over Y such that $\mathscr{B}_Y \cong \mathscr{D}[p^m]$.*

Next we show that having a unique lift (up to isomorphism or up to isogeny) is a constructible property of truncated BT groups.

Proposition 4.3. *Let $\mathscr{B}$ be a BT group of level m over a scheme X and let $n \geq m$ be an integer. There exists a constructible subset U of X such that a geometric point $\bar{x} : \operatorname{Spec} \kappa \to X$ lies in $U(\kappa)$ if and only if all BT groups of level n over κ which extend the geometric fibre $\mathscr{B}_{\bar{x}}$ are isomorphic.*

This includes the following invariance under field extensions.

Corollary 4.4. *Let $k \subseteq \kappa$ be an extension of algebraically closed fields. A BT group B of level m over k extends uniquely to level n (up to isomorphism) if and only if B_κ extends uniquely to level n (up to isomorphism).* $\square$

Proof of Proposition 4.3. We can assume that X is of finite type over $\operatorname{Spec}\mathbb{Z}$. Let $Y \to X$ be a morphism of finite type such that over Y there exists an exhaustive extension $\mathscr{C}$ of $\mathscr{B}$ to level n. Let $Y' = Y \times_X Y$, let $p_1, p_2 : Y' \to Y$ be the two projections, and consider

$$Z = \underline{\mathrm{Isom}}(p_1^*\mathscr{C}, p_2^*\mathscr{C}) \xrightarrow{\psi} Y'.$$

Let $x \in X$ be the image of $\bar{x} : \operatorname{Spec}\kappa \to X$. The geometric fibre $\mathscr{B}_{\bar{x}}$ extends uniquely to level n (up to isomorphism) if and only if the geometric fibre $\psi_{\bar{x}} : Z_{\bar{x}} \to Y'_{\bar{x}}$ is surjective on κ-valued points, which is equivalent to the fibre $\psi_x : Z_x \to Y'_x$ being surjective. Hence U is a well-defined subset of X, and $X \setminus U$ is the image of $Y' \setminus \mathrm{Im}(\psi) \to X$. As X, Y', Z are of finite type, U is constructible by Chevalley's theorem. $\square$

Definition 4.5. We say that a BT group B of level m over k has well-defined Newton polygon ν if each p-divisible group D over k with $D[p^m]$ isomorphic to B has Newton polygon ν. In this case we let $b_B = b_D$. Similarly, if all p-divisible groups D with $D[p^m] \cong B$ are isomorphic, we let $n_B = n_D$, while n_B is undefined otherwise.

Proposition 4.6. *Let $\mathscr{B}$ be a BT group of level m over an $\mathbb{F}_p$-scheme X. There exists a constructible subset U of X such that a geometric point $\bar{x}$: $\operatorname{Spec}\kappa \to X$ lies in $U(\kappa)$ if and only if the geometric fibre $\mathscr{B}_{\bar{x}}$ has a well-defined Newton polygon.*

Proposition 4.6 includes the following invariance under field extensions.

Corollary 4.7. *Let $k \subset \kappa$ be an extension of algebraically closed fields. A BT group B of level m over k has a well-defined Newton polygon if and only if B_κ has a well-defined Newton polygon.* $\square$

The proof of Proposition 4.6 uses the following standard fact.

Lemma 4.8. *Let $\mathscr{B}$ be a BT group of level n over an $\mathbb{F}_p$-scheme X such that for each geometric point $\bar{x} : \operatorname{Spec}\kappa \to X$, the geometric fibre $\mathscr{B}_{\bar{x}}$ has a well-defined Newton polygon $\nu(\bar{x})$. If $x \in X$ is the image of $\bar{x}$, then $\nu(x) := \nu(\bar{x})$ is well-defined. Moreover, for each Newton polygon ν the set $X_\nu = \{x \in X \mid \nu(x) \preceq \nu\}$ is closed in X.*

We recall that $\nu' \preceq \nu$ if and only if the polygons ν' and ν share the same endpoints and all points of ν' lie on or above ν.

Proof. As Newton polygons of p-divisible groups are preserved under extensions of the base field, $\nu(x)$ is well-defined. By Lemma 4.2 there exists a faithfully flat affine morphism $f : Y \to X$ such that $\mathscr{B}_Y$ extends to a p-divisible group $\mathscr{D}$ over Y. Then $\pi^{-1}(X_\nu)$ is the set of points where the Newton polygon of $\mathscr{D}$ is $\preceq \nu$, which is closed in Y by [Ka, Thm. 2.3.1] applied to the Dieudonné F-crystal of $\mathscr{D}$. It follows that X_ν is closed in X as f is faithfully flat. □

Proof of Proposition 4.6. We can assume that X is of finite type over $\operatorname{Spec} \mathbb{F}_p$. Choose $n \in \mathbb{N}^*$ such that for each geometric point $\bar{x} : \operatorname{Spec} \kappa \to X$ and every p-divisible group D over κ that extends $\mathscr{B}_{\bar{x}}$, we have $b_D \leq n$. This is possible because $\mathscr{B}$ has bounded height. Let $Y \to X$ be a morphism of finite type such that over Y there exists an exhaustive extension $\mathscr{C}$ of $\mathscr{B}$ to level n. For $y \in Y$ let $\nu(y)$ be the well-defined Newton polygon of the fibre $\mathscr{C}_y$. By Lemma 4.8 the set $Y^\nu = \{y \in Y \mid \nu(y) \neq \nu\}$ is locally closed in Y. Let $U_\nu \subseteq X$ be the complement of the image of $Y^\nu \to X$. This is a constructible set by Chevalley's theorem. The required subset U of X is the union of all U_ν, which is constructible because only finitely many U_ν's are non-empty. □

Corollary 4.9. *Let $k \subseteq \kappa$ be an extension of algebraically closed fields. For a p-divisible group D over k we have $n_D = n_{D_\kappa}$ and $b_D = b_{D_\kappa}$.*

Proof. Corollary 4.4 applied to $n = \max\{n_D, n_{D_\kappa}\}$ and $B = D[p^m]$ with $m = \min\{n_D, n_{D_\kappa}\}$ gives $n_D = n_{D_\kappa}$. Corollary 4.7 applied to $B = D[p^m]$ with $m = \min\{b_D, b_{D_\kappa}\}$ gives $b_D = b_{D_\kappa}$. □

4.2 Distances of truncated BT groups

In order to deduce semicontinuity of distance numbers from their going down property we need to define distance numbers also for truncated BT groups.

Definition 4.10. Let B and C be truncated BT groups of level n over k The distance $q_{B,C}$ is the smallest non-negative integer m such that there exist homomorphisms $B \to C$ and $C \to B$ whose kernels are annihilated by p^m.

If κ is an algebraically closed extension of k, we have $q_{B,C} = q_{B_\kappa, C_\kappa}$. This can be viewed as a consequence of the following proposition.

Proposition 4.11. *Let $\mathscr{B}$ and $\mathscr{C}$ be truncated BT groups of level n over an $\mathbb{F}_p$-scheme X. There exists a constructible subset U of X such that a geometric point $\bar{x} : \operatorname{Spec} \kappa \to X$ lies in $U(\kappa)$ if and only if $q_{\mathscr{B}_{\bar{x}}, \mathscr{C}_{\bar{x}}} \leq m$.*

Proof. We can assume that X is of finite type over $\operatorname{Spec}\mathbb{F}_p$. Let

$$Z \subseteq \underline{\mathrm{Hom}}(\mathscr{B},\mathscr{C}) \times \underline{\mathrm{Hom}}(\mathscr{C},\mathscr{B})$$

be the subscheme of all pairs of homomorphisms (f, g) such that p^m annihilates both $\mathrm{Ker}(f)$ and $\mathrm{Ker}(g)$. This is a closed subscheme. The set U is the image of $Z \to X$, which is constructible by Chevalley's theorem. $\square$

Proposition 4.12. *Let $h \in \mathbb{N}$. Let D and E be p-divisible groups over k of height at most h. Let $B = D[p^n]$ and $C = E[p^n]$ for some $n \in \mathbb{N}^*$.*

(a) We have $q_{B,C} \leq q_{D,E}$.

(b) If $q_{B,C} < n - N_h$, then $q_{B,C} = q_{D,E}$.

The number N_h was defined in Section 2.

Proof. (a) Let $m = q_{D,E}$ and let $\rho : D \to E$ be an isogeny such that $\mathrm{Ker}(\rho)$ is contained in $D[p^m]$. The isogenies ρ and $p^m\rho^{-1}$ induce homomorphisms $B \to C$ and $C \to B$ with kernels annihilated by p^m; thus $q_{B,C} \leq m$.

(b) Let $m = q_{B,C}$ and let $f : B \to C$ and $g : C \to B$ be homomorphisms with kernels annihilated by p^m. We must show that $q_{D,E} \leq m$. By the choice of N_h, there exist homomorphisms $f' : D \to E$ and $g' : E \to D$ which coincide with f and g on $D[p^{n-N_h}]$ and $E[p^{n-N_h}]$ (respectively). As $m < n - N_h$, by Lemma 4.13 below it follows that $\mathrm{Ker}(f') = \mathrm{Ker}(f)$ and $\mathrm{Ker}(g') = \mathrm{Ker}(g)$ are annihilated by p^m. Hence $q_{D,E} \leq m$ as required. $\square$

Lemma 4.13. *For a positive integer l, let B and C be truncated BT groups over k of level $l + 1$. Let $f : B \to C$ be a homomorphism with restriction $f_l : B[p^l] \to C[p^l]$. If $\mathrm{Ker}(f_l) \subseteq B[p^{l-1}]$, then we have $\mathrm{Ker}(f) = \mathrm{Ker}(f_l)$.*

Proof. Let S be a k-scheme. If $x \in \mathrm{Ker}(f)(S)$, then $px \in \mathrm{Ker}(f_l)(S)$, thus $px \in B[p^{l-1}](S)$ by the assumption. Hence $x \in B[p^l](S)$, which implies that $x \in \mathrm{Ker}(f_l)(S)$ as $x \in \mathrm{Ker}(f)(S)$. $\square$

4.3 Proof of the semicontinuity results

If $\mathscr{B}$ is a truncated BT group over an $\mathbb{F}_p$-scheme S, for each point $s \in S$ we write $b_{\mathscr{B}}(s) = b_{\mathscr{B}_{\bar{s}}}$ and $n_{\mathscr{B}}(s) = n_{\mathscr{B}_{\bar{s}}}$, where $\mathscr{B}_{\bar{s}}$ is the geometric fibre of $\mathscr{B}$ at s; see Definition 4.5.

Theorem 4.14. *Let $m \leq l$ be positive integers. Let $\mathscr{B}$ be a BT group of level l over an $\mathbb{F}_p$-scheme S with well-defined and constant Newton polygon ν i.e., all geometric fibres of $\mathscr{B}$ have well-defined Newton polygon ν. Then the following two properties hold:*

(a) The set $U_{b_{\mathscr{B}}} = \{s \in S \mid b_{\mathscr{B}}(s) \leq m\}$ is closed in S.

(b) The set $U_{n_{\mathscr{B}}} = \{s \in S \mid n_{\mathscr{B}}(s)$ is defined and $\leq m\}$ is closed in S.

Proof. Let $\square \in \{n, b\}$. The set $U_{\square_{\mathscr{B}}}$ is functorial in the sense that for a morphism $\pi : S' \to S$ we have $\pi^{-1}U_{\square_{\mathscr{B}}} = U_{\square_{\pi^*(\mathscr{B})}}$; here we use Corollary 4.9. Thus we can assume that S is of finite type over $\operatorname{Spec}\mathbb{F}_p$. The set $U_{b_{\mathscr{B}}}$ (resp. $U_{n_{\mathscr{B}}}$) is constructible because it coincides with the set U associated to $\mathscr{B}[p^m]$ in Proposition 4.6 (resp. in Proposition 4.3 applied with sufficiently large n). Hence it suffices to show that $U_{\square_{\mathscr{B}}}$ is stable under specialization. To prove this we can assume that $S = \operatorname{Spec} k[[t]]$ for some algebraically closed field k; here we use functoriality again. By [Ill, Thm. 4.4] there exists a p-divisible group $\mathscr{D}$ over S that extends $\mathscr{B}$. As for $s \in S$ we have $b_{\mathscr{B}_{\bar{s}}} = b_{\mathscr{D}_{\bar{s}}}$, and $n_{\mathscr{B}_{\bar{s}}} = n_{\mathscr{D}_{\bar{s}}}$ when $n_{\mathscr{D}_{\bar{s}}} \leq l$ while $n_{\mathscr{B}_{\bar{s}}}$ is undefined otherwise, Theorem 4.14 follows from Theorems 3.8 and 3.9. $\square$

We note that the results of [GV] and [Va2] also allow a short and different proof of Theorem 4.14 (b) (see Remark 7.12). For distance numbers we have the following analogue of Theorem 4.14.

Theorem 4.15. *Let $\mathscr{B}$ and $\mathscr{C}$ be truncated BT groups of level n over an $\mathbb{F}_p$-scheme S with well-defined and constant Newton polygons and height $\leq h$. Then for each non-negative integer $m < n - N_h$ the set*

$$U_{q_{\mathscr{B},\mathscr{C}}} = \{s \in S \mid q_{\mathscr{B},\mathscr{C}}(s) \leq m\}$$

is closed in S. Here we write $q_{\mathscr{B},\mathscr{C}}(s) = q_{\mathscr{B}_{\bar{s}},\mathscr{C}_{\bar{s}}}$ as above.

Proof. We can assume that S is of finite type over $\operatorname{Spec}\mathbb{F}_p$. The set $U_{q_{\mathscr{B},\mathscr{C}}}$ is constructible, cf. Proposition 4.11. Thus, as in the previous proof, we can assume that $S = \operatorname{Spec} k[[t]]$ and that $\mathscr{B} = \mathscr{D}[p^n]$ and $\mathscr{C} = \mathscr{E}[p^n]$ for p-divisible groups $\mathscr{D}$ and $\mathscr{E}$ over S. By Proposition 4.12, for $s \in S$ we have $s \in U_{q_{\mathscr{B},\mathscr{C}}}$ if and only if $q_{\mathscr{D}_{\bar{s}},\mathscr{E}_{\bar{s}}} \leq m$. Thus $U_{q_{\mathscr{B},\mathscr{C}}}$ is stable under specialization by Theorem 3.1. We conclude that $U_{q_{\mathscr{B},\mathscr{C}}}$ is closed in S. $\square$

Proof of Theorem 1.1. We can assume that S is quasi-compact. Then $\mathscr{D}$ and $\mathscr{E}$ have height $\leq h$ for some integer h. We choose an integer $l \geq m$ such that $\mathscr{D}[p^l] = \mathscr{B}$ and $\mathscr{E}[p^l] = \mathscr{C}$ have well-defined (necessarily constant) Newton polygons. Then the sets $U_{b_{\mathscr{D}}}$ and $U_{n_{\mathscr{D}}}$ coincide with the sets $U_{b_{\mathscr{B}}}$ and $U_{n_{\mathscr{B}}}$ (respectively) considered in Theorem 4.14, which implies that $U_{b_{\mathscr{D}}}$ and $U_{n_{\mathscr{D}}}$ are closed. Clearly (c) follows from (d). To prove (d) we assume in addition that $l > m + N_h$. By Proposition 4.12 it follows that $U_{q_{\mathscr{D},\mathscr{E}}}$ coincides with the set $U_{q_{\mathscr{B},\mathscr{C}}}$ in Theorem 4.15, which implies that $U_{q_{\mathscr{D},\mathscr{E}}}$ is closed. $\square$

5 Complements on Dieudonné modules

In this section we collect several properties of Dieudonné modules which will be used later on to study n_D and b_D. The results of Subsections 5.1 and 5.2 are either well-known or trivial. Subsections 5.3 and 5.4 are more involved and contain new material.

In this section, to be short we write $W = W(k)$ and $W_{\mathbb{Q}} = W(k)[1/p]$. Let $\sigma : W \to W$ be the Frobenius automorphism and let $v : W_{\mathbb{Q}} \to \mathbb{Z} \cup \{\infty\}$ be the p-adic valuation. Let $W_{\mathbb{Q}}\{F, F^{-1}\}$ be the non-commutative Laurent polynomial ring. We consider the quotient ring

$$\mathbb{D} = W_{\mathbb{Q}}\{F, F^{-1}\}/I$$

where I is the two-sided ideal generated by all elements $Fa - \sigma(a)F$ with $a \in W_{\mathbb{Q}}$. Let $\mathbb{E} \subset \mathbb{D}$ be the W-subalgebra generated by F and $V = pF^{-1}$.

5.1 Valuations on Frobenius modules

A *valuation* on a W-module M is a map $w : M \to \mathbb{R} \cup \{\infty\}$ that has the following two properties:

(i) $w(ax) = v(a) + w(x)$ for all $a \in W$ and $x \in M$;

(ii) $w(x + y) \geq \min\{w(x), w(y)\}$ for all $x, y \in M$.

The valuation is called non-degenerate if $w(x) = \infty$ implies that $x = 0$. It is called non-trivial if $w(x) \neq \infty$ for some $x \in M$. If M_{tors} is the maximal torsion W-submodule of M, then w always factors through M/M_{tors}. Denoting $M_{\mathbb{Q}} = M \otimes_W W_{\mathbb{Q}}$, valuations on M extend uniquely to valuations on $M_{\mathbb{Q}}$. If M is a $W_{\mathbb{Q}}$-vector space, then (i) holds for all $a \in W_{\mathbb{Q}}$.

Definition 5.1. Let w be a valuation on a W-module M. A direct sum decomposition $M = \bigoplus_{i \in I} M_i$ is called *valuative* if we have

$$w\Big(\sum_{i \in I} m_i\Big) = \min\{w(m_i) \mid i \in I\}$$

whenever $m_i \in M_i$ with almost all $m_i = 0$. A W-basis or $W_{\mathbb{Q}}$-basis of M is called valuative if the associated direct sum decomposition into modules of rank one is valuative.

A direct sum decomposition $M = \bigoplus_{i \in I} M_i$ is valuative if and only if w is minimal among all valuations of M that coincide with w on each M_i. A direct sum decomposition of M is valuative if and only if the induced direct sum decomposition of $M_{\mathbb{Q}}$ is valuative.

Definition 5.2. A pair (M, F), where M is a W-module and F is a σ-linear endomorphism of M, is called a Frobenius module (over k). A valuation w on M is called an *F-valuation of slope* $\lambda \in \mathbb{R}$ if for all $x \in M$ we have

$$w(Fx) = w(x) + \lambda.$$

An F-valuation on M extends uniquely to an F-valuation on $M_{\mathbb{Q}}$. We view each left $\mathbb{D}$-module as a Frobenius module. For each $\lambda \in \mathbb{R}$ there exists a unique F-valuation $\mathbb{w}_\lambda$ on $\mathbb{D}$ of slope λ such that the $W_{\mathbb{Q}}$-basis $(F^i)_{i\in\mathbb{Z}}$ of $\mathbb{D}$ is valuative and we have $\mathbb{w}_\lambda(1) = 0$. It is given by the formula

$$\mathbb{w}_\lambda\Big(\sum_{i\in\mathbb{Z}} e_i F^i\Big) = \min\{v(e_i) + i\lambda \mid i \in \mathbb{Z}\}$$

whenever $e_i \in W_{\mathbb{Q}}$ with almost all $e_i = 0$. This can also be expressed in terms of Newton polygons. For a non-zero element $\Phi \in \mathbb{D}$ we define its Newton polygon ν_Φ as follows. Write $\Phi = \sum_{i=n}^{m} a_i F^{-i}$ with $n \leq m$ and $a_i \in W_{\mathbb{Q}}$ such that $a_n \neq 0$ and $a_m \neq 0$. Then $\nu_\Phi : [n, m] \to \mathbb{R}$ is the maximal upper convex function such that $v(a_i) \geq \nu_\Phi(i)$ for each integer $i \in [n, m]$. For $\lambda \in \mathbb{R}$ let $\nu_{\Phi,\lambda} : \mathbb{R} \to \mathbb{R}$ be the maximal linear function of slope λ such that $v(a_i) \geq \nu_{\Phi,\lambda}(i)$ for each integer $i \in [n, m]$. For $t \in [n, m]$ we have

$$\nu_\Phi(t) = \max\{\nu_{\Phi,\lambda}(t) \mid \lambda \in \mathbb{R}\},$$

and the functions $\nu_{\Phi,\lambda}$ are maximal with this property. We have

$$\mathbb{w}_\lambda(\Phi) = \nu_{\Phi,\lambda}(0). \tag{5.1}$$

In the following let N be a left $\mathbb{D}$-module of finite dimension over $W_{\mathbb{Q}}$.

Lemma 5.3. *There exists a non-degenerate F-valuation of slope λ on N if and only if N is isoclinic of slope λ. When N is simple of slope λ, then any two non-trivial F-valuations on N differ by the addition of a constant.*

Sketch of proof. Use the facts that N has a $W_{\mathbb{Q}}$-basis consisting of elements x with $F^{s_x}(x) = p^{r_x}x$ for some integers $s_x \neq 0$ and r_x, and that N is isoclinic of slope λ if and only if we have $r_x = \lambda s_x$ for all x in the $W_{\mathbb{Q}}$-basis. $\square$

Lemma 5.4. *Let N be as above and let $\lambda \in \mathbb{R}$. We consider a free W-submodule $M \subset N$ with $M_{\mathbb{Q}} = N$. Let $\mathscr{W}$ be the set of all F-valuations w of slope λ on N with $w(x) \geq 0$ for all $x \in M$. Then $\mathscr{W}$ has a minimal element $w_\circ$ i.e., we have $w_\circ(x) \leq w(x)$ for all $w \in \mathscr{W}$ and $x \in N$. The valuation $w_\circ$ is non-degenerate if and only if N is isoclinic of slope λ.*

Proof. For $x \in N$ let $w_\circ(x) = \inf\{w(x) \mid w \in \mathscr{W}\}$. Then $w_\circ$ is an F-valuation on N of slope λ. The last assertion follows from Lemma 5.3. $\square$

Let N be as above. For an F-valuation w on N of slope λ we consider the W-submodules of N defined for each $\alpha \in \mathbb{R}$ by:

$$N^{w\geq\alpha} = \{x \in N \mid w(x) \geq \alpha\} \qquad \text{and} \qquad N^{w>\alpha} = \{x \in N \mid w(x) > \alpha\}.$$

Let $gr^\alpha N$ be the k-vector space $N^{w\geq\alpha}/N^{w>\alpha}$.

Lemma 5.5. *Let $\Phi \in \mathbb{D} \setminus \{0\}$ be a sum $\Phi = \sum_{i\in\mathbb{Z}} e_i F^i$ with each $e_i \in W_\mathbb{Q}$ (only a finite number of the e_i's are non-zero). Let $\delta = \mathrm{w}_\lambda(\Phi)$. Then the multiplication by Φ induces a group homomorphism*

$$\bar{\Phi}_\alpha : gr^\alpha N \to gr^{\alpha+\delta} N$$

which is surjective with finite kernel.

Proof. Clearly $\Phi(N^{w\geq\alpha}) \subseteq N^{w\geq\alpha+\delta}$ and $\Phi(N^{w>\alpha}) \subseteq N^{w>\alpha+\delta}$. The induced homomorphism $\bar{\Phi}_\alpha$ does not change if we omit from Φ all terms e_iF^i with $v(e_i) + i\lambda > \delta$. As the validity of the lemma is invariant under multiplying Φ with integral powers of F or with non-zero elements of $W_\mathbb{Q}$, we can assume that $\delta = 0$ and that $\Phi = 1 + \sum_{i\in\mathbb{N}^*} e_iF^i$ with $e_i \in W_\mathbb{Q}$ such that only a finite number of them are non-zero and we have $v(e_i) = -i\lambda$ for all non-zero e_i. Then $\bar{\Phi}_\alpha$ can be viewed naturally as an étale endomorphism of the vector group scheme over k defined by $gr^\alpha N$, and the assertion follows. $\square$

For later use we record the following elementary result.

Lemma 5.6. *Let $\alpha, \alpha' \in \mathbb{R}$ be such that $gr^\alpha N \neq 0$. The minimal non-negative integer m such that $p^m N^{w\geq\alpha} \subseteq N^{w>\alpha'}$ is equal to $\lfloor \alpha' - \alpha \rfloor + 1$.* $\square$

5.2 Presentations of cyclic Dieudonné modules

In this subsection we fix a non-zero bi-nilpotent Dieudonné module M over k i.e., M is an $\mathbb{E}$-module which is a free W-module of finite positive rank, and F and V are nilpotent on $\bar{M} = M/pM$. Let $d = \dim_k(M/FM)$ and $c = \dim_k(M/VM)$ and $h = c + d$. We have $cd > 0$, and h is the rank of M.

The a-number $a(M) = \dim_k(M/(FM + VM))$ is positive. An element $z \in M$ generates M as an $\mathbb{E}$-module if and only if z generates $M/(FM+VM)$ as a k-vector space. For completeness we prove the following well-known lemma.

Lemma 5.7. *Assume that z generates M as an $\mathbb{E}$-module. Then the following three properties hold:*

(a) The h-tuple $\Upsilon = (F^i z)_{1\leq i\leq c} \sqcup (V^i z)_{0\leq i\leq d-1}$ is a W-basis of M.

(b) There exists an element $\Psi \in \mathbb{E}$ for which we have $\Psi z = 0$ and which is of the form

$$\Psi = \sum_{i=0}^{c} a_i F^{c-i} + \sum_{i=1}^{d} b_i V^i,$$

with a_0 and b_d as units in W and with $a_i \in pW$ for $i \in \{1, \dots, c\}$ and $b_i \in pW$ for $i \in \{1, \dots, d-1\}$.

(c) We have an $\mathbb{E}$-linear isomorphism $\mathbb{E}/\mathbb{E}\Psi \cong M$ given by $1 + \mathbb{E}\Psi \mapsto z$.

Proof. As V is nilpotent on M/FM and as M/FM is a k-vector space of dimension d generated by the iterates of z under V, we get that $(V^i z)_{0\leq i<d}$ is a k-basis of M/FM. Similarly we argue that $(F^i z)_{0\leq i<c}$ is a k-basis of M/VM, which implies that $(F^i z)_{1\leq i\leq c}$ is a k-basis of FM/pM. We conclude that Υ is a k-basis of M/pM. From this (a) follows.

By (a) there exists a relation $\Psi z = 0$ with $\Psi = \sum_{i=0}^{c} a_i F^{c-i} + \sum_{i=1}^{d} b_i V^i$ and $b_d = 1$. As $V^d z \in FM$ we have $a_c \in pW$ and $b_i \in pW$ for $i \in \{1, \dots, d-1\}$. By interchanging the roles of F and V in (a), there exists also a relation $\Psi' z = 0$ with $\Psi' = \sum_{i=0}^{c} a'_i F^{c-i} + \sum_{i=1}^{d} b'_i V^i$ and $a'_0 = 1$ and such that $a'_i \in pW$ for $i \in \{1, \dots, c\}$. The element $\Psi' - b'_d \Psi$ must be zero by (a), which implies that $a'_i = b'_d a_i$ for all $i \in \{0, \dots, c\}$. As $a'_0 = 1$, we get that b'_d and a_0 are units of W and that we have $a_i \in pW$ for all $i \in \{1, \dots, c-1\}$. Therefore (b) holds.

We have an $\mathbb{E}$-linear epimorphism $\mathbb{E}/\mathbb{E}\Psi \to M$ that maps $1 + \mathbb{E}\Psi$ to z. It is easy to see that $\{F^i \mid 1 \leq i \leq c\} \cup \{V^i \mid 0 \leq i \leq d-1\}$ generates $\mathbb{E}/\Psi\mathbb{E}$ over W, which proves (c). $\square$

Lemma 5.8. *Let $M = \mathbb{E}/\mathbb{E}\Psi$ be as in Lemma 5.7. Let $\nu_M : [0, h] \to \mathbb{R}$ be the Newton polygon of M and let $\nu_\Psi : [-c, d] \to \mathbb{R}$ be the Newton polygon of Ψ defined above. Then $\nu_M(t) = \nu_\Psi(t - c)$ for $t \in [0, h]$.*

Proof. This is proved in [De, Lemma 2 on p. 82]. $\square$

Remark 5.9. In view of Lemma 5.8 it would be natural to shift the Newton polygon of M so that its domain is $[-c, d]$. This would cause c to be replaced by 0 in many formulas, including the assertions of Theorems 1.2, 1.3, and 1.5. We keep the traditional notation in order to avoid confusion.

5.3 Valuations on cyclic Dieudonné modules

We assume now that M is a non-zero bi-nilpotent Dieudonné module over k generated as an $\mathbb{E}$-module by a fixed element $z \in M$. Thus $M \cong \mathbb{E}/\Psi\mathbb{E}$ with Ψ as in Lemma 5.7 (b) and we have $a(M) = 1$.

Lemma 5.10. *Assume that M is isoclinic of slope λ. Let w be the minimal F-valuation on M of slope λ with $w(x) \geq 0$ for all $x \in M$, cf. Lemma 5.4. Then the W-basis Υ of M introduced in Lemma 5.7 (a) is valuative for w.*

Proof. We write $\Psi = \sum_{i=0}^{h} a_i F^{c-i}$ with $a_i \in W$; thus for $i \in \{1, \ldots, d\}$ we have $a_{c+i} = p^i b_i$. For $x = \sum_{i=0}^{h-1} e_i F^{c-i} z \in M$ with $e_i \in W$ let

$$w_1(x) = \min\{v(e_i) + (c-i)\lambda \mid 0 \leq i \leq h-1\}.$$

Then w_1 is a valuation for which the W-basis Υ is valuative. We claim that $w = w_1$. It is easy to see that $w(x) \geq w_1(x) \geq 0$ for all $x \in M$. Hence we must show that w_1 is an F-valuation of slope λ i.e., that we have $w_1(Fx) = w_1(x) + \lambda$. This is a straightforward computation based on the relation $v(a_i) \geq i\lambda$ for all $0 \leq i \leq h$ with equality for $i = 0$ and $i = h$. The details are left to the reader. $\square$

For the general (non-isoclinic) case we need some additional notations.

Notation 5.11. Let $N = M_{\mathbb{Q}}$ and let $N = N_1 \oplus \cdots \oplus N_r$ be the direct sum decomposition into isoclinic components, ordered such that each N_j with $j \in \{1, \ldots, r\}$ is isoclinic of slope λ_j and $0 < \lambda_1 < \cdots < \lambda_r < 1$. Let h_j be the dimension of N_j i.e., the multiplicity of λ_j in ν. We write $h_j = c_j + d_j$ such that $\lambda_j = d_j/h_j$. Let $M_j \subseteq N_j$ be the image of M. Then $a(M_j) = 1$. Let w_j be the minimal F-valuation on N_j of slope λ_j such that $w_j(M_j) \geq 0$, see Lemma 5.4. For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$ let

$$N^{\alpha} = \bigoplus_{j=1}^{r} N_j^{w_j \geq \alpha_j}, \qquad N^{\alpha+} = \bigoplus_{j=1}^{r} N_j^{w_j > \alpha_j},$$

and $gr^{\alpha} N = N^{\alpha}/N^{\alpha+}$. Let $\nu : [0, h] \to \mathbb{R}$ be the Newton polygon of M. For each $j \in \{1, \ldots, r\}$ let $\nu_j : \mathbb{R} \to \mathbb{R}$ be the unique linear function of slope λ_j such that for all $t \in [0, h]$ we have

$$\nu(t) = \max\{\nu_j(t) \mid 1 \leq j \leq r\}.$$

We define $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{R}^r$ by $\beta_j = \nu_j(c)$.

The following result will be used in the proof of Theorem 9.1. Another application of it, to minimal Dieudonné modules, is given in Subsection 5.4.

Proposition 5.12. *We have $N^{\beta+} \subseteq pM$ as W-submodules of N.*

We will deduce this from a description of $N^{\beta+}$ in terms of certain auxiliary valuations $\tilde{w}_j$ on N. Let $\mathcal{I} = \{0, 1, \dots, h-1\}$. We recall that $(F^{c-i}z)_{i\in\mathcal{I}}$ is a $W_{\mathbb{Q}}$-basis of N; see Lemma 5.7 (a). For $j \in \{1, \dots, r\}$ let $\tilde{w}_j$ be the valuation on N which for $x = \sum_{i\in\mathcal{I}} e_i F^{c-i}z \in N$ with $e_i \in W_{\mathbb{Q}}$ is given by

$$\tilde{w}_j(x) = \min\{v(e_i) + (c-i)\lambda_j \mid i \in \mathcal{I}\}.$$

This is an F-valuation only when M is isoclinic. A different way to look at $\tilde{w}_j$ is the following one. Let $\nu_{x,j} : \mathbb{R} \to \mathbb{R}$ be the maximal linear function of slope λ_j such that we have $\nu_{x,j}(i) \leq v(e_i)$ for all $i \in \mathcal{I}$. Then

$$\tilde{w}_j(x) = \nu_{x,j}(c).$$

Let $\nu_x : [0, h-1] \to \mathbb{R}$ be defined by

$$\nu_x(t) = \max\{\nu_{x,j}(t) \mid 1 \leq j \leq r\}.$$

For $\alpha \in \mathbb{R}^r$ let

$$\tilde{N}^{\alpha+} = \{x \in N \mid \tilde{w}_j(x) > \alpha_j \text{ for } 1 \leq j \leq r\}.$$

Lemma 5.13. *We have $\tilde{N}^{\beta+} \subseteq pM$ as W-submodules of N.*

Proof. Let $x \in N$. As $\beta_j = \nu_j(c)$ we see that $x \in \tilde{N}^{\beta+}$ if and only if we have $\nu_{x,j} > \nu_j$ for all $1 \leq j \leq r$. In other words, $x \in \tilde{N}^{\beta+}$ if and only if $v(e_i) > \nu(i)$ for all $i \in \mathcal{I}$. The last condition implies that $x \in pM$. $\square$

By Lemma 5.13 the following description of $N^{\beta+}$ implies Proposition 5.12.

Proposition 5.14. *We have $N^{\beta+} = \tilde{N}^{\beta+}$ as W-submodules of N.*

As the following lemma shows, the inclusion $\tilde{N}^{\beta+} \subseteq N^{\beta+}$ does not depend on the specific choice of β.

Lemma 5.15. *The following two properties hold:*

(a) Let $j \in \{1, \dots, r\}$. For $x \in N$ with projection $x_j \in N_j$ we have $w_j(x_j) \geq \tilde{w}_j(x)$.

(b) For $\alpha \in \mathbb{R}^r$ we have $\tilde{N}^{\alpha+} \subseteq N^{\alpha+}$.

Proof. If $x = F^{c-i}z$ with $i \in \mathcal{I}$, then $w_j(x_j) = (c-i)\lambda_j = \tilde{w}_j(x)$. As w_j is a valuation and as the $W_{\mathbb{Q}}$-basis $(F^{c-i}z)_{i\in\mathcal{I}}$ of N is valuative for $\tilde{w}_j$ we get (a). Clearly (a) implies (b). $\square$

The opposite inclusion $N^{\alpha+} \subseteq \tilde{N}^{\alpha+}$ does not hold in general. To prove Proposition 5.14 we need conditions on x under which equality holds in Lemma 5.15 (a). For $x = \sum_{i\in\mathcal{I}} e_i F^{c-i} z$ with $e_i \in W_{\mathbb{Q}}$ let $\mathcal{I}_j(x) \subseteq \mathcal{I}$ be the (non-empty) set of those indices i with $v(e_i) = \nu_{x,j}(i)$ and let $s_j(x)$ be the difference between the maximal and minimal elements of $\mathcal{I}_j(x)$. Similarly let $\Lambda_j(x) \subseteq [0, h-1]$ be the closed interval of all t with $\nu_{x,j}(t) = \nu_x(t)$ and let $s'_j(x)$ be the length of $\Lambda_j(x)$. We have $\mathcal{I}_j(x) \subseteq \Lambda_j(x)$; thus $s_j(x) \leq s'_j(x)$.

Lemma 5.16. *Let $j \in \{1, \ldots, r\}$. Let $x \in N$ and let $x_j \in N_j$ be its projection. If $s_j(x) < h_j$, then $w_j(x_j) = \tilde{w}_j(x)$.*

Proof. As $w_j(x_j) \geq \tilde{w}_j(x)$, to prove that $w_j(x_j) = \tilde{w}_j(x)$ we can replace x by an arbitrary element $x' \in N$ with $\tilde{w}_j(x - x') > \tilde{w}_j(x)$. Using this and the inequality $s_j(x) < h_j$, we can assume that for some integer i_0 the element x lies in the $W_{\mathbb{Q}}$-vector subspace of N spanned by the finite set $\Upsilon_{i_0,j} = \{F^{c-i} z \mid i_0 \leq i \leq i_0 + h_j - 1\}$. If $x \in \Upsilon_{i_0,j}$, then $w_j(x_j) = \tilde{w}_j(x)$. Thus it suffices to show that $\Upsilon_{i_0,j}$ projects to a valuative $W_{\mathbb{Q}}$-basis of N_j for w_j. For $i_0 = c - c_j$ this is true by Lemma 5.10. The general case follows because the operators F and F^{-1} on N_j preserve valuative $W_{\mathbb{Q}}$-bases as w_j is an F-valuation. □

Proof of Proposition 5.14. We know that $\tilde{N}^{\beta+} \subseteq N^{\beta+}$, see Lemma 5.13 (b). Thus to prove Proposition 5.14, it suffices to show that the assumption that there exists an element $x \in N^{\beta+} \setminus \tilde{N}^{\beta+}$ leads to a contradiction.

As $x \notin \tilde{N}^{\beta+}$ there exists at least one index $j \in \{1, \ldots, r\}$ such that $\tilde{w}_j(x) \leq \beta_j$. Choose a maximal chain

$$\mathcal{J} = \{j_1, j_1 + 1, \ldots, j_2\} \subseteq \{1, \ldots, r\}$$

such that $\tilde{w}_j(x) \leq \beta_j$ for all $j \in \mathcal{J}$. Let $[a, b] \subseteq [0, h]$ be the closed interval of all t with $\nu(t) \in \{\nu_j(t) \mid j \in \mathcal{J}\}$, and let $[a', b'] \subseteq [0, h-1]$ be the closed interval of all t with $\nu_x(t) \in \{\nu_{x,j}(t) \mid j \in \mathcal{J}\}$. In other words, $[a, b]$ (resp. $[a', b']$) is the maximal interval where the slopes of ν (resp. of ν_x) lie in the set $\{\lambda_j \mid j \in \mathcal{J}\}$. We claim that the following implications hold:

$$\begin{aligned} j_1 > 1 &\Rightarrow a' > a \\ j_2 < r &\Rightarrow b' < b. \end{aligned}$$

As the proofs are similar, we will only check here the first implication. Assume that the implication does not hold i.e., we have $j_1 > 1$ and $a' \leq a$. Then $j_0 = j_1 - 1$ lies in $\{1, \ldots, r\} \setminus \mathcal{J}$. We have $\nu_{j_0}(a) = \nu_{j_1}(a)$ and

$\nu_{x,j_0}(a') = \nu_{x,j_1}(a')$, which implies that $\nu_{x,j_0}(a) \leq \nu_{x,j_1}(a)$ as $a' \leq a$. Thus we compute

$$\begin{aligned}\tilde{w}_{j_0}(x) - \beta_{j_0} &= (\nu_{x,j_0} - \nu_{j_0})(c) = (\nu_{x,j_0} - \nu_{j_0})(a) \\ &\leq (\nu_{x,j_1} - \nu_{j_1})(a) = (\nu_{x,j_1} - \nu_{j_1})(c) = \tilde{w}_{j_1}(x) - \beta_{j_1} \leq 0\end{aligned}$$

which contradicts the maximality of $\mathcal{J}$. This proves our claim.

We note that $j_1 = 1$ implies $a' = a = 0$, while $j_2 = r$ implies $b' = h - 1$ and $b = h$. Thus in all cases we have $a' \geq a$ and $b' < b$, and hence

$$b - a > b' - a'.$$

On the other hand, as $x \in N^{\beta+}$, for $j \in \mathcal{J}$ we have

$$\tilde{w}_j(x) \leq \beta_j < w_j(x_j).$$

From this and Lemma 5.16 we get that $s'_j(x) \geq s_j(x) \geq h_j$. Thus we get

$$b - a = \sum_{j \in \mathcal{J}} h_j \leq \sum_{j \in \mathcal{J}} s'_j(x) = b' - a'.$$

Contradiction. This ends the proof of Proposition 5.14 (and thus also of Proposition 5.12). $\square$

5.4 Minimal Dieudonné modules

Following Oort [Oo3], a Dieudonné module M over k is called minimal if $\mathrm{End}_{\mathbb{E}}(M)$ is a maximal order in $\mathrm{End}_{\mathbb{D}}(M_{\mathbb{Q}})$ i.e., M is the Dieudonné module of a minimal p-divisible group in the sense recalled in Section 2. A Dieudonné module is minimal if and only if it is a direct sum of isoclinic minimal Dieudonné modules. In the isoclinic case we have the following characterization of minimality; see also [Yu, Sect. 3].

Proposition 5.17. *Let M be an isoclinic Dieudonné module of slope λ. Then the following three statements are equivalent.*

(a) M is minimal.

(b) If $\Phi \in \mathbb{D}$ satisfies $\mathrm{w}_\lambda(\Phi) \geq 0$, then $\Phi(M) \subseteq M$.

(c) For some F-valuation w on $N = M_{\mathbb{Q}}$ of slope λ we have $M = N^{w \geq 0}$.

We begin with a special case; see also [dJO, Subsects. 5.3-5.6].

Lemma 5.18. *If N is a simple $\mathbb{D}$-module of slope λ, then the statements (a) and (c) of Proposition 5.17 are equivalent.*

Proof. Let $\Gamma = \mathrm{End}_{\mathbb{D}}(N)$. Let w be an F-valuation of slope λ on N. We note that w is unique up to adding a constant, see Lemma 5.3. Thus for each $\varphi \in \Gamma$ there exists a $\tilde{v}(\varphi) \in \mathbb{R} \cup \{\infty\}$ such that $w(\varphi x) = w(x) + \tilde{v}(\varphi)$ for all $x \in N$. Then $\tilde{v}$ is the unique valuation on the division algebra Γ that extends the p-adic valuation on $\mathbb{Q}_p$. The maximal order in Γ is $\Gamma_0 = \{\varphi \in \Gamma \mid \tilde{v}(\varphi) \geq 0\}$.

In view of these remarks, (c)$\Rightarrow$(a) is clear. We prove (a)$\Rightarrow$(c). Let $h = \dim(N)$. We choose w such that $\mathbb{Z} \subseteq w(N)$. The k-vector space $gr^{\alpha}N$ is 1-dimensional if $h\alpha \in \mathbb{Z}$, and it is 0 otherwise. In particular, $w(N) = (1/h)\mathbb{Z}$. Let $\pi \in \Gamma_0$ be a generator of the maximal ideal, which means that $\tilde{v}(\pi) = 1/h$. As M is stable under π, the subset $w(M)$ of $w(N)$ takes the form $\{i/h \mid i \in \mathbb{Z},\ i \geq i_0\}$ for some integer i_0. By replacing w with $w - i_0/h$ we can assume that $i_0 = 0$. It follows easily that $M = N^{w \geq 0}$. □

Proof of Proposition 5.17. Let $\lambda = l/n$ with coprime integers l, n and $n \geq 1$. Let $\Phi_0 = F^n p^{-l} \in \mathbb{D}$ and choose $\Phi = F^a p^m \in \mathbb{D}$ such that $\mathrm{w}_\lambda(\Phi) = 1/n$. The element Φ is unique up to multiplication by an integral power of Φ_0. Let $q = p^n$. Let $\mu = \dim(N)/n$ be the multiplicity of the $\mathbb{D}$-module N. First we show that (b) implies the existence of a W-basis of M of the form

$$\Upsilon_1 = (\Phi^i x_j)_{0 \leq i < n,\ 1 \leq j \leq \mu}$$

such that each $x_j \in M$ satisfies the equation $\Phi_0(x_j) = x_j$. Indeed, let $\Pi = \{x \in M \mid \Phi_0(x) = x\}$. This is a $W(\mathbb{F}_q)$-submodule of M. As Φ_0 has slope zero and preserves M by the assumption (b), we have $M = \Pi \otimes_{W(\mathbb{F}_q)} W$. As $\Phi^n : \Pi \to \Pi$ is multiplication by p, the quotient $\Pi/\Phi\Pi$ is an $\mathbb{F}_q$-vector space of dimension μ. We choose elements $x_1, \ldots, x_\mu \in \Pi$ which project to an $\mathbb{F}_q$-basis of $\Pi/\Phi\Pi$. Then for each $i \geq 0$, $(\Phi^i x_j)_{1 \leq j \leq \mu}$ projects to an $\mathbb{F}_q$-basis of $\Phi^i\Pi/\Phi^{i+1}\Pi$. We conclude that Υ_1 is a W-basis of M.

The implication (c)$\Rightarrow$(b) is clear. We prove (b)$\Rightarrow$(c). Let Υ_1 be as above. There exists a unique F-valuation w of slope λ on N such that $w(x_j) = 0$ for $1 \leq j \leq \mu$ and such that the W-basis Υ_1 is valuative for w. As $w(\Phi^i x_j) = i/n$ lies in the interval $[0, 1)$ when $0 \leq i < n$, property (c) follows easily.

We prove (b)$\Rightarrow$(a). Let Υ_1 be as above and let $M_j = \mathbb{D}^{\mathrm{w}_\lambda \geq 0} x_j$ for $1 \leq j \leq \mu$. Then as $\Phi_0(x_j) = x_j$, it follows that $M = M_1 \oplus \cdots \oplus M_\mu$ is a direct sum decomposition into pairwise isomorphic simple Dieudonné modules. Hence $\mathrm{End}_{\mathbb{E}}(M)$ is a matrix algebra over $\mathrm{End}_{\mathbb{E}}(M_1)$, and (a) follows by Lemma 5.18.

Finally, we prove (a)$\Rightarrow$(c). Let N_1 be a simple constituent of N and let $\Gamma = \mathrm{End}_{\mathbb{D}}(N_1)$. As each maximal order in $\mathrm{End}_{\mathbb{D}}(N)$ is isomorphic to the matrix algebra over the maximal order Γ_0 of Γ we see that M is the direct sum of simple Dieudonné modules. Hence (c) follows from Lemma 5.18 again. □

The next proposition is proved as well in [Yu, Lemma 4.2].

Proposition 5.19. *Let M be a Dieudonné module and let $N = M_{\mathbb{Q}}$. There exists a minimal minimal Dieudonné module M_+ with $M \subseteq M_+ \subset N$ i.e., for every minimal Dieudonné module M' with $M \subseteq M' \subset N$ we have $M_+ \subseteq M'$.*

Proof. If N is isoclinic, then by Proposition 5.17 every minimal M' as above takes the form $M' = N^{w \geq 0}$ for some F-valuation w on N of slope λ with $w(M) \geq 0$. Hence in the isoclinic case the assertion follows from Lemma 5.4. In general, let $N = \bigoplus_{j=1}^{r} N_j$ be the direct sum decomposition into isoclinic components and let M_j be the image of $M \to N_j$. As each M' as above is a direct sum $M' = \bigoplus_{j=1}^{r} M'_j$ with $M_j \subseteq M'_j \subset N_j$, we have $M_+ = \bigoplus_{j=1}^{r} (M_j)_+$. □

By duality there exists also a maximal minimal Dieudonné module $M_- \subseteq M$.

Lemma 5.20. *A homomorphism of Dieudonné modules $f : M' \to M$ induces homomorphisms $f_- : M'_- \to M_-$ and $f_+ : M'_+ \to M_+$.*

Proof. By duality it suffices to show that $f(M'_-) \subseteq M_-$. We can assume that M' and M are isoclinic of the same slope λ. Using Proposition 5.17 it is easy to see that M_- is the set of all elements $x \in M_{\mathbb{Q}}$ such that we have $\Phi x \in M$ for each $\Phi \in \mathbb{D}$ with $\mathrm{w}_\lambda(\Phi) \geq 0$, and the analogous statement holds for M'_-. As $f_{\mathbb{Q}} : M_{\mathbb{Q}} \to M'_{\mathbb{Q}}$ is a $\mathbb{D}$-linear map it follows that f maps M'_- to M_-. □

If $M = M_b \oplus M_o$ is the unique decomposition such that M_b is bi-nilpotent and M_o has integral slopes, then $M_{\pm} = M_{b\pm} \oplus M_o$. We have the following explicit descriptions of M_+ and M_- in the case when $a(M) = a(M_b) = 1$.

Theorem 5.21. *Let M be a bi-nilpotent Dieudonné module and let $M_- \subseteq M \subseteq M_+$ be the minimal Dieudonné modules considered above. We assume that $a(M) = 1$ and we use Notation 5.11.*

(a) We have $M_+ = N^{\underline{0}}$, where $\underline{0} = (0, \ldots, 0) \in \mathbb{R}^r$.

(b) We have $M_- = p^{-1} N^{\beta+}$.

By Proposition 5.14 the W-module $N^{\beta+}$ has a W-basis consisting of easily computable W-multiples of $F^{c-i}z$ for $i \in \mathcal{I} = \{0, \ldots, h-1\}$, which makes it more explicit than the W-module $N^{\underline{0}}$.

Proof of Theorem 5.21. Let M_j be the image of $M \to N_j$. We recall that w_j is the minimal F-valuation of slope λ_j on N_j such that $w_j(M_j) \geq 0$. This implies (a); see the proof of Proposition 5.19.

The Dieudonné module $p^{-1}N^{\beta+}$ is minimal by Proposition 5.17 and contained in M by Proposition 5.12. Thus we have inclusions

$$p^{-1}N^{\beta+} \subseteq M_{-} \subseteq M \subseteq M^{+} = N^{\underline{0}}.$$

For a W-module A, let $A^\vee = \mathrm{Hom}_W(A, W)$. If A has finite length, let $\ell(A)$ be its length. Consider

$$\ell_1 = \ell(M/p^{-1}N^{\beta+}) \qquad \text{and} \qquad \ell_2 = \ell(N^{\underline{0}}/p^{-1}N^{\beta+}).$$

We claim that $\ell_2 = 2\ell_1$. This implies that $\ell(M_+/M) \geq \ell(M/M_-)$ with equality if and only if $M_- = p^{-1}N^{\beta+}$. The same reasoning applied to the dual Dieudonné module $M^\vee$ gives the opposite inequality $\ell(M/M_-) \geq \ell(M_+/M)$. Here we use that $a(M^\vee) = 1$ and $(M^\vee)_+ = (M_-)^\vee$ and $(M^\vee)_- = (M_+)^\vee$. Thus (b) follows from our claim.

It remains to show that $\ell_2 = 2\ell_1$. Let s be the multiplicity of N (i.e., the sum of the multiplicities of the isoclinic direct factors N_j of N). Then

$$\ell_2 + h - s = \ell(pN^{\underline{0}}/N^{\beta+}) + \ell(N^{\underline{0}}/pN^{\underline{0}}) - \ell(N^{\beta}/N^{\beta+}) = \ell(N^{\underline{0}}/N^{\beta}) = \sum_{j=1}^{r} \beta_j h_j.$$

On the other hand, let ρ be the ordinary Newton polygon with the same endpoints as ν and let $\varOmega \subseteq \mathbb{R}^2$ be the compact set enclosed by ν and ρ. A W-basis of pM is formed by the elements $p^{n_i}F^{c-i}z$ for $0 \leq i < h$ where n_i is the minimal integer such that $n_i > \rho(i)$. A W-basis of $\tilde{N}^{\beta+}$ is formed by the elements $p^{m_i}F^{c-i}z$ for $0 \leq i < h$ where m_i is the minimal integer with $m_i > \nu(i)$. Thus $\ell(pM/\tilde{N}^{\beta+})$ is the number of elements of $\mathbb{Z}^2$ which lie strictly above ρ and on or below ν. Hence the cardinality of the finite set $\mathrm{int}(\varOmega) \cap \mathbb{Z}^2$ is equal to $\ell(pM/\tilde{N}^{\beta+}) - s + 1$, which is equal to $\ell_1 - s + 1$ by Proposition 5.14. The set $\partial\varOmega \cap \mathbb{Z}^2$ has $s + h$ elements. Hence the area of $\varOmega$ can be expressed in two ways as follows.

$$\int_0^h \nu(t)\,dt - d^2/2 = \ell_1 - s + 1 + \frac{s+h}{2} - 1 = \ell_1 + \frac{h-s}{2}.$$

Finally, the function $g(t) = \nu(t) - \nu'(t)(t - c)$ is well-defined for those $t \in [0, h]$ where ν is linear. Its value is $g(t) = \beta_j$ if $\nu'(t) = \lambda_j$. Thus we get

$$\sum_{j=1}^{r} \beta_j h_j = \int_0^h (\nu(t) - \nu'(t)(t-c))\,dt = 2\int_0^h \nu(t)\,dt - d^2$$

by integration by parts. The last three displayed equations give $\ell_2 = 2\ell_1$. $\square$

The p-exponent of a finitely generated torsion W-module $\bar{M}$ is the smallest non-negative integer m such that we have $p^m x = 0$ for all $x \in \bar{M}$.

Lemma 5.22. *Let M be a Dieudonné module. Then the three W-modules M_+/M_-, M/M_-, and M_+/M have the same p-exponent.*

Proof. If p^m annihilates M_+/M, then $p^m M_+ \subseteq M$ and thus we have $p^m M_+ \subseteq M_-$ as $p^m M_+$ is minimal. Similarly, if p^m annihilates M/M_-, then we have $M \subseteq p^{-m} M_-$ which implies that $M_+ \subseteq p^{-m} M_-$. □

Corollary 5.23. *Let M be a Dieudonné module of Newton polygon ν. Let m be the p-exponent of M_+/M_-. We have $m \le \lfloor \nu(c) \rfloor$ with equality if $a(M) = 1$.*

Proof. We can assume that M is bi-nilpotent and non-zero. If $a(M) = 1$, then m is the p-exponent of $p^{-1} N^{\beta+}/N^{\underline{0}}$ by Theorem 5.21 and thus we have $m = \max\{\lfloor \nu_j(c) \rfloor \mid 1 \le j \le r\} = \lfloor \nu(c) \rfloor$ (cf. Lemma 5.6). If $a(M) \ge 2$, then by Lemma 5.22 it suffices to show that for every non-zero element $x \in M$ we have $p^{\lfloor \nu(c) \rfloor} x \in M_-$. Let $M' = \mathbb{E}x$ be the Dieudonné module generated by x. Let ν' be its Newton polygon and let c' be the dimension of M'/FM'. As $a(M') = 1$, we know that $p^{\lfloor \nu'(c') \rfloor} x \in M'_-$. By Lemma 5.20, M'_- is contained in M_-. Thus $\nu'(c') \le \nu(c)$ by Lemma 5.24 below. Hence $p^{\lfloor \nu(c) \rfloor} x \in M_-$ as desired. □

Lemma 5.24. *Let ν and ν' be Newton polygons with endpoints $(c+d, d)$ and $(c' + d', d')$ (respectively). If the slopes of ν' form a subset of the slopes of ν (counted with multiplicities), then $\nu'(c') \le \nu(c)$.*

Proof. It is easy to see that the number $\nu(c)$ is invariant under duality in the sense that $\nu(c) = \nu^\vee(d)$ if $\nu^\vee$ is defined by $\nu^\vee(x) = \nu(c + d - x) + x - d$ for $x \in [0, c + d]$. To prove the lemma, by induction it suffices to consider the case where ν' arises from ν by deleting a line (i.e., an isoclinic part) of some slope λ. Then at least one of the following holds:

(1) The slopes of $\nu'(x)$ for $x \le c'$ are less or equal to λ;

(2) The slopes of $\nu'(x)$ for $x \ge c'$ are greater or equal to λ.

The passage to the duals of ν and ν' interchanges (1) and (2) and therefore we can assume that (1) holds. Then $\nu(x) = \nu'(x)$ for $x \le c$ and thus $\nu'(c') = \nu(c') \le \nu(c)$ as $c' \le c$ and the function ν is increasing. □

Proof of Theorem 1.5. Let M be the covariant Dieudonné module of the given p-divisible group D over k. Isogenies $f : D \to D_0$ with minimal D_0 correspond to minimal Dieudonné modules M_0 with $M \subseteq M_0 \subseteq M_{\mathbb{Q}}$ in such a way that the p-exponents of $\mathrm{Ker}(f)$ and of M_0/M coincide. Hence Theorem 1.5 follows from Corollary 5.23 together with Lemma 5.22. □

6 Values of isogeny cutoffs

In this section we fix a p-divisible group D over k of dimension d and codimension c with Newton polygon ν. We will prove Theorem 1.2 and list all possible values of the isogeny cutoff b_D of D.

Lemma 6.1. *To prove Theorem 1.2 we can assume that D is connected with connected dual and that $a_D = 1$.*

Proof. There exists a p-divisible group over $k[[t]]$ whose special fibre is D and whose geometric generic fibre has a-number at most 1, cf. [Oo1, Prop. 2.8]. By Theorem 3.8, after replacing if needed k by an algebraic closure of $k((t))$, we can assume that $a_D \leq 1$. Let $D = D^\circ \times D^{\mathrm{ord}}$, where D° is connected with connected dual and D^{ord} is ordinary. Let ν_0 be the Newton polygon of D° and let c_0 be the codimension of D°. We have $\nu(c) = \nu_0(c_0)$, $b_D = b_{D^\circ}$ (by Lemma 2.4 (c)), and $a_{D^\circ} = a_D \leq 1$. If $a_{D^\circ} = 0$ then D° is trivial. Thus to prove Theorem 1.2 we can assume that $D = D^\circ$ and $a_D = 1$. $\square$

Proof of Theorem 1.2. Let M be the covariant Dieudonné module of D. By Lemma 6.1 we can assume that M is bi-nilpotent and that $a(M) = 1$. We write $M = \mathbb{E}/\mathbb{E}\Psi$ as in Lemma 5.7 with $a_0 = 1$.

Let D' be another p-divisible group over k with Dieudonné module M' and with Newton polygon ν'.

First we show that $b_D \leq j(\nu)$. Assume that $D[p^{j(\nu)}] \cong D'[p^{j(\nu)}]$; we must show that $\nu = \nu'$. As $j(\nu) \geq 1$, the p-divisible group D' and its dual are connected and we have $a_{D'} = 1$. Choose an element $z' \in M'$ such that the class of z maps to the class of z' under the isomorphism $M/p^{j(\nu)}M \cong M'/p^{j(\nu)}M'$, and let $\Psi' z' = 0$ be the associated relation given by Lemma 5.7 (b) with $a_0 = 1$. Then $\Psi' - \Psi \in p^{j(\nu)}\mathbb{E}$. Let us write $\Psi - \Psi' = \sum_{i=1}^{h} e_i F^{c-i}$. By the definition of $j(\nu)$, as $\Psi' - \Psi \in p^{j(\nu)}\mathbb{E}$ we have $v(e_i) \geq \nu(i)$ always and $v(e_i) > \nu(i)$ if $(i, \nu(i))$ is a breakpoint of ν. From this and Lemma 5.8 we get that $\nu' = \nu$.

Next we show that $b_D \geq j(\nu)$. If $j(\nu) = 1$ this is clear. Thus we can assume that $m = j(\nu) - 1 > 0$. If $m < \nu(c)$, we take $\Psi' = \Psi + p^m$. If $m \geq \nu(c)$, then we have $m = \nu(c)$ and $(c, \nu(c))$ is a breakpoint of ν, in particular $v(a_c) = \nu(c) = m$. In this case, we take $\Psi' = \Psi - a_c$. We define D' by $M' = \mathbb{E}/\mathbb{E}\Psi'$. Then $D[p^m] \cong D'[p^m]$. From Lemma 5.8 we get that $\nu' \neq \nu$. Thus $m < b_D$. $\square$

Proposition 6.2. *If m is an integer with $1 \leq m \leq j(\nu)$, then there exists a p-divisible group D over k with Newton polygon ν and $b_D = m$.*

Proof. The assertion is true for $m = j(\nu)$, cf. Theorem 1.2. It is also true for $m = 1$, as for a minimal p-divisible group D_0 we have $n_{D_0} = 1$ (see either [Oo3, Thm. 1.2] or [Va3, Thm. 1.6]) and thus also $b_{D_0} = 1$. As two p-divisible groups over k of the same Newton polygon ν can be linked by a chain of isogenies with kernels annihilated by p, the proposition follows from the next lemma. $\square$

Lemma 6.3. *Let $g : D \to E$ be an isogeny of p-divisible groups over k such that the kernel of g is annihilated by p. Then $|b_D - b_E| \leq 1$.*

Proof. As pg^{-1} is an isogeny $E \to D$ with kernel annihilated by p, by symmetry it suffices to show that $b_E \geq b_D - 1$. Let m be an integer with $0 < m < b_D$. This means that there exists a p-divisible group D' over k which is not isogenous to D and for which there exists an isomorphism $u : D[p^m] \cong D'[p^m]$. Let $E' = D'/u(\mathrm{Ker}\, g)$. Then E' is not isogenous to E and u induces an isomorphism $E[p^{m-1}] \cong E'[p^{m-1}]$. Thus $m - 1 < b_E$. Therefore $b_E \geq b_D - 1$. $\square$

Remark 6.4. Here is another approach to bound b_D from above. By Theorem 1.5 there exists an isogeny $D \to D_0$ with kernel annihilated by $p^{\lfloor \nu(c) \rfloor}$ and with D_0 minimal. As $b_{D_0} = n_{D_0} = 1$, either Lemma 6.3 or [NV2, Lemma 2.9] gives $b_D \leq 1 + \lfloor \nu(c) \rfloor$. This estimate is equivalent to the upper bound in Theorem 1.2 except when $\nu(c) \in \mathbb{Z}$ and ν is linear at c; then it is off by 1.

Remark 6.5. We assume that either $\nu(c) \notin \mathbb{Z}$ or ν is not linear at c; equivalently, we have $j(\nu) = \lfloor \nu(c) \rfloor + 1$. In this case we have the following refinement of Proposition 6.2. By Theorem 1.5 there exists a chain $D_1 \leftarrow \cdots \leftarrow D_{j(\nu)}$ of isogenies of p-divisible groups of Newton polygon ν, where D_1 is minimal, where $a_{D_{j(\nu)}} = 1$ and thus $b_{D_{j(\nu)}} = j(\nu)$ by Theorem 1.2, and where the kernel of each $D_i \leftarrow D_{i+1}$ with $i \in \{1, \ldots, j(\nu) - 1\}$ is annihilated by p. Lemma 6.3 implies that we have $b_{D_i} = i$ for all $i \in \{1, \ldots, j(\nu)\}$.

7 A variant of homomorphism numbers

Let D and E be p-divisible groups over the algebraically closed field k. As suggested by the results of [GV] (see Theorem 7.3 below) we consider the following variant of the homomorphism numbers $e_{D,E}$.

Lemma 7.1 ([GV, Subsect. 6.1])**.** *There exists a non-negative integer $f_{D,E}$ such that for positive integers $m \geq n$ the restriction homomorphism*

$$\tau_{m,n} : \mathrm{Hom}(D[p^m], E[p^m]) \to \mathrm{Hom}(D[p^n], E[p^n])$$

has finite image if and only if $m \geq n + f_{D,E}$.

Proof. As $\mathrm{Hom}(D, E)$ is a finitely generated $\mathbb{Z}_p$-module, its image in the p^n-torsion group $\mathrm{Hom}(D[p^n], E[p^n])$ is finite for each n. By Lemma 2.1 it follows that for each n there exists an m such that $\tau_{m,n}$ has finite image. We have to show that the minimal such m takes the form $m = n + f$ where f does not depend on n. This follows from the lower exact sequence in (2.1) in the proof of Lemma 2.1 as for the numbers $e_{D,E}(n)$. $\square$

Definition 7.2. We call $f_{D,E}$ the *coarse homomorphism number* of D and E. The *coarse endomorphism number* of D is $f_D = f_{D,D}$.

Using this notion we can state [GV, Cor. 2 (b)] as follows.

Theorem 7.3 ([GV]). *If D is not ordinary, then we have $f_D = n_D$.*

Remark 7.4. We note that [GV] also gives a similar interpretation of $f_{D,E}$ in general. Namely, let $n_{D,E}$ be the minimal non-negative integer m such that the truncation map $\mathrm{Ext}^1(D, E) \to \mathrm{Ext}^1(D[p^m], E[p^m])$ is injective. Then we have $f_{D,E} = n_{D,E}$, cf. [GV, Subsect. 6.1 (iii)]. The resulting equality $n_D = n_{D,D}$ if D is not ordinary seems not to be obvious from the definitions.

The following is the first and easiest of three related inequalities.

Proposition 7.5. *We have $f_{D,E} \leq e_{D,E}$.*

Proof. This is immediate from the definitions and the finiteness of the image of the reduction homomorphism $\mathrm{Hom}(D, E) \to \mathrm{Hom}(D[p^n], E[p^n])$. $\square$

The following analogue of Lemma 2.4 is easily checked.

Lemma 7.6. *Let $D = D^{ord} \times D^\circ$ and $E = E^{ord} \times E^\circ$ be the canonical decompositions such that D^{ord} and E^{ord} are the maximal ordinary subgroups of D and E (respectively). Then $f_{D,E} = f_{D^\circ,E^\circ}$.* $\square$

Lemma 7.7. *For each algebraically closed field $\kappa \supseteq k$ we have $f_{D,E} = f_{D_\kappa,E_\kappa}$.*

Proof. For positive integers $m \geq n$ we have $f_{D,E} > m - n$ if and only if the image of $\underline{\mathrm{Hom}}(D[p^m], E[p^m]) \to \underline{\mathrm{Hom}}(D[p^n], E[p^n])$ has positive dimension. This property is invariant under the base change from k to κ. $\square$

7.1 Semicontinuity of coarse homomorphism numbers

We will show that the coarse homomorphism numbers $f_{D,E}$ are lower semicontinuous in families of p-divisible groups with constant Newton polygon. The following characterization of $f_{D,E}$ is taken from [GV].

Lemma 7.8 ([GV, Subsect. 6.1 (i) and (ii)]). *For $m \in \mathbb{N}$ let*

$$\gamma_{D,E}(m) = \dim(\underline{\mathrm{Hom}}(D[p^m]), E[p^m]).$$

We have $\gamma_{D,E}(m) \leq \gamma_{D,E}(m+1)$ with equality if and only if $m \geq f_{D,E}$.

Proof. Let $\underline{\mathrm{H}}_m = \underline{\mathrm{Hom}}(D[p^m]), E[p^m])$. We have an exact sequence of algebraic groups $0 \to \underline{\mathrm{H}}_m \xrightarrow{\iota_m} \underline{\mathrm{H}}_{m+1} \xrightarrow{\tau_{m+1,1}} \underline{\mathrm{H}}_1$, cf. the proof of Lemma 2.1. Thus the lemma follows from the definition of $f_{D,E}$, see Lemma 7.1. □

Definition 7.9. The stable value of $\gamma_{D,E}$ is denoted $s_{D,E} = \gamma_{D,E}(f_{D,E})$. If $D = E$ we write $\gamma_{D,D}(m) = \gamma_D(m)$ and $s_{D,D} = s_D = \gamma_D(f_D)$.

The semicontinuity of $f_{D,E}$ relies on the following result of [Va2].

Theorem 7.10 ([Va2, Thm. 1.2 (f)] and [GV, Rm. 4.5]). *If D and D' are isogenous p-divisible groups over k, then we have $s_D = s_{D'}$.*

As earlier, if $\mathscr{D}$ and $\mathscr{E}$ are p-divisible groups over an $\mathbb{F}_p$-scheme S, we define functions $f_{\mathscr{D},\mathscr{E}}, \gamma_{\mathscr{D},\mathscr{E}}(m), s_{\mathscr{D},\mathscr{E}}, s_{\mathscr{D}} = s_{\mathscr{D},\mathscr{D}} : S \to \mathbb{N}$ by $f_{\mathscr{D},\mathscr{E}}(s) = f_{\mathscr{D}_{\bar{s}},\mathscr{E}_{\bar{s}}}$, etc., where $\mathscr{D}_{\bar{s}}$ and $\mathscr{E}_{\bar{s}}$ are the geometric fibres of $\mathscr{D}$ and $\mathscr{E}$ over $s \in S$.

Theorem 7.11. *Let $\mathscr{D}$ and $\mathscr{E}$ be p-divisible groups of constant Newton polygon over an $\mathbb{F}_p$-scheme S.*

(a) The function $s_{\mathscr{D},\mathscr{E}}$ is locally constant on S.

(b) For $m \in \mathbb{N}$, the set $U_{f_{\mathscr{D},\mathscr{E}}} = \{s \in S \mid f_{\mathscr{D},\mathscr{E}}(s) \leq m\}$ is closed in S.

Proof. We have $s_{\mathscr{D}\oplus\mathscr{E}} = s_{\mathscr{D}} + s_{\mathscr{E}} + s_{\mathscr{D},\mathscr{E}} + s_{\mathscr{E},\mathscr{D}}$ as functions on S. The functions $s_{\mathscr{D}\oplus\mathscr{E}}$, $s_{\mathscr{E}}$, and $s_{\mathscr{D}}$ are constant by Theorem 7.10. Locally in S, for sufficiently large m we have $s_{\mathscr{D},\mathscr{E}} = \gamma_{\mathscr{D},\mathscr{E}}(m)$ and $s_{\mathscr{E},\mathscr{D}} = \gamma_{\mathscr{E},\mathscr{D}}(m)$. These functions are upper semicontinuous. Hence $s_{\mathscr{D},\mathscr{E}} = s_{\mathscr{D}\oplus\mathscr{E}} - s_{\mathscr{D}} - s_{\mathscr{E}} - s_{\mathscr{E},\mathscr{D}}$ is upper and lower semicontinuous, thus locally constant, which proves (a).

To prove (b) we note that for $s \in S$ we have $(s_{\mathscr{D},\mathscr{E}} - \gamma_{\mathscr{D},\mathscr{E}}(m))(s) \geq 0$ with equality if and only if $s \in U_{f_{\mathscr{D},\mathscr{E}}}$, see Lemma 7.8. As $s_{\mathscr{D},\mathscr{E}}$ is locally constant and $\gamma_{\mathscr{D},\mathscr{E}}(m)$ is upper semicontinuous we get that $U_{f_{\mathscr{D},\mathscr{E}}}$ is closed. □

Remark 7.12. We have $U_{n_{\mathscr{D}}} = U_{f_{\mathscr{D},\mathscr{D}}}$, cf. Theorem 7.3. Thus Theorem 1.1 (b) also follows from Theorem 7.11 (b).

8 Homomorphisms of truncated BT groups

In this section, we prove Theorem 1.6, but we work more generally with homomorphisms instead of endomorphisms.

8.1 The level torsion $\ell_{D,E}$

Let $W(k)$ and σ be as in Section 5. Let D and E be two p-divisible groups over k. Let (M, F, V) and (L, F, V) be the covariant Dieudonné modules of D and E (respectively). Let $H = H_{D,E} = \mathrm{Hom}_{W(k)}(M, L)$. Let $(H[\frac{1}{p}], F)$ be the F-isocrystal defined by the rule

$$F(\flat) = F \circ \flat \circ F^{-1} = V^{-1} \circ \flat \circ V.$$

We consider the direct sum decomposition into $W(k)[1/p]$-vector spaces

$$H[\tfrac{1}{p}] = N_+ \oplus N_0 \oplus N_-$$

which is invariant under F and such that all slopes of (N_+, F) are positive, all slopes of (N_0, F) are 0, and all slopes of (N_-, F) are negative.

Definition 8.1. Let $O_+ \subseteq H \cap N_+$, $O_0 \subseteq H \cap N_0$, and $O_- \subseteq H \cap N_-$ be the maximal $W(k)$-submodules such that $F(O_+) \subseteq O_+$, $F(O_0) = O_0$, and $F^{-1}(O_-) \subseteq O_-$. The *level module* of D and E is defined as

$$O = O_{D,E} = O_+ \oplus O_0 \oplus O_-.$$

If $D = E$ we write $O_{D,D} = O_D$.

For a more explicit description, let $A_0 = \{z \in H \mid F(z) = z\}$, which we identify with the free $\mathbb{Z}_p$-module $\mathrm{Hom}(D, E)$. We have

$$O_+ = \{z \in N_+ \mid F^t(z) \in H \quad \forall t \in \mathbb{N}\},$$
$$O_0 = A_0 \otimes_{\mathbb{Z}_p} W(k) = \bigcap_{t\in\mathbb{N}} F^t(H \cap N_0) = \bigcap_{t\in\mathbb{N}} F^{-t}(H \cap N_0),$$
$$O_- = \{z \in N_- \mid F^{-t}(z) \in H \quad \forall t \in \mathbb{N}\}.$$

Lemma 8.2. *The $W(k)$-module O is a lattice of $H[\frac{1}{p}]$.*

Proof. As all slopes of (N_+, F) are positive, for each $z \in N_+$ the sequence $(F^t(z))_{t\in\mathbb{N}}$ of elements of N_+ converges to 0 in the p-adic topology. Therefore there exists $s \in \mathbb{N}$ such that $p^s z \in O_+$. Thus we have $O_+[\frac{1}{p}] = N_+$. As O_+ is a $W(k)$-submodule of the finitely generated $W(k)$-module H, we conclude that O_+ is a lattice of N_+. A similar argument shows that O_0 and O_- are lattices of N_0 and N_- (respectively). Thus O is a lattice of $H[\frac{1}{p}]$. □

Definition 8.3. The *level torsion* of D and E is the smallest non-negative integer $\ell_{D,E}$ such that we have

$$p^{\ell_{D,E}} \operatorname{Hom}_{W(k)}(M, L) \subseteq O \subseteq \operatorname{Hom}_{W(k)}(M, L).$$

If $D = E$ we write $\ell_{D,D} = \ell_D$.

If D and E are isoclinic, then this definition of $\ell_{D,E}$ coincides with the one in [Va3, Def. 4.1 (c)]. The level torsion is symmetric:

Lemma 8.4. *We have $\ell_{D,E} = \ell_{E,D} = \ell_{D^\vee,E^\vee}$.*

Proof. Let $\tilde{O} = \tilde{O}_{D,E} \subseteq H[\frac{1}{p}]$ be the minimal $W(k)$-submodule which contains H and which takes the form $\tilde{O} = \tilde{O}_+ \oplus \tilde{O}_0 \oplus \tilde{O}_-$ with $\tilde{O}_+ \subseteq N_+$, $\tilde{O}_0 \subseteq N_0$, and $\tilde{O}_- \subseteq N_-$, such that $F(\tilde{O}_+) \subseteq \tilde{O}_+$, $F(\tilde{O}_0) = \tilde{O}_0$, and $F^{-1}(\tilde{O}_-) \subseteq \tilde{O}_-$. Let $\tilde{\ell}_{D,E} \in \mathbb{N}$ be the minimal number such that $p^{\tilde{\ell}_{D,E}} \tilde{O} \subseteq H$. First we show that $\ell_{D,E} = \tilde{\ell}_{D,E}$. We have $H \subseteq p^{-\ell_{D,E}} O$, thus $O \subseteq H \subseteq \tilde{O} \subseteq p^{-\ell_{D,E}} O$ by the minimality of $\tilde{O}$, and therefore $\tilde{\ell}_{D,E} \leq \ell_{D,E}$. Similarly we have $p^{\tilde{\ell}_{D,E}} \tilde{O} \subseteq O \subseteq H \subseteq \tilde{O}$ by the maximality of O, and therefore $\ell_{D,E} \leq \tilde{\ell}_{D,E}$.

Let $F_{D,E} = F$ be the Frobenius of $H_{D,E}[\frac{1}{p}]$. For a $W(k)$-module A we write $A^\vee = \operatorname{Hom}_{W(k)}(A, W(k))$. The Dieudonné modules of $D^\vee$ and $E^\vee$ can be identified with $(M^\vee, V^\vee, F^\vee)$ and $(L^\vee, V^\vee, F^\vee)$ (respectively). Hence we have natural isomorphisms $H_{E,D} \cong H_{D,E}^\vee \cong H_{D^\vee,E^\vee}$. In terms of these isomorphisms we have $F_{E,D} = (F_{D,E}^{-1})^\vee = F_{D^\vee,E^\vee}$, and they induce isomorphisms of $W(k)$-submodules $O_{E,D} \cong \tilde{O}_{D,E}^\vee \cong O_{D^\vee,E^\vee}$ because these $W(k)$-submodules are all defined by the same maximality property. Hence we have $\ell_{E,D} = \tilde{\ell}_{D,E} = \ell_{D^\vee,E^\vee}$. This proves the lemma as we have $\tilde{\ell}_{D,E} = \ell_{D,E}$. $\square$

Lemma 8.5. *For each algebraically closed field $\kappa \supseteq k$ we have $\ell_{D,E} = \ell_{D_\kappa,E_\kappa}$.*

Proof. Let H_κ and O_κ be the analogues of H and O defined with respect to D_κ and E_κ instead of D and E. One can check that $H_\kappa = H \otimes_{W(k)} W(\kappa)$ and $O_\kappa = O \otimes_{W(k)} W(\kappa)$. The assertion follows. $\square$

8.2 The inequality $e_{D,E} \leq \ell_{D,E}$

For $x \in O$ we write $x = x_+ + x_0 + x_-$, where $x_+ \in O_+$, $x_0 \in O_0$, and $x_- \in O_-$. We call x_+, x_0, x_- the components of x in O.

Lemma 8.6. *Let $x \in O$. The equation in X*

$$x = F(X) - X \tag{8.1}$$

has a solution in O which is unique up to the addition by an arbitrary element of A_0. If $x \in p^s O$ for some $s \in \mathbb{N}$, then there exists a solution in $p^s O$.

Proof. We define $y_+ = -\sum_{i=0}^{\infty} F^i(x_+)$ and $y_- = \sum_{i=1}^{\infty} F^{-i}(x_-)$. We have $x_+ = F(y_+) - y_+$ and $x_- = F(y_-) - y_-$. Let $\Upsilon_0 = (e_1, \dots, e_s)$ be a $\mathbb{Z}_p$-basis of A_0. Then Υ_0 is a $W(k)$-basis of O_0, and thus we can write $x_0 = \sum_{i=1}^{s} \gamma_i e_i$ with $\gamma_i \in W(k)$. For $y_0 = \sum_{i=1}^{s} z_i e_i$ with $z_i \in W(k)$ we have $x_0 = F(y_0) - y_0$ if and only if for each $i \in \{1, \dots, s\}$ we have

$$\sigma(z_i) - z_i = \gamma_i.$$

It is well-known that this equation in z_i has solutions in $W(k)$. Therefore $y = y_+ + y_0 + y_-$ is a solution of (8.1) in O. Two solutions of (8.1) in O differ by a solution of the equation $F(X) = X$ and thus they differ by an arbitrary element of A_0. If $x = p^s x'$ with $x' \in O$, then there exists an element $y' \in O$ with $x' = F(y') - y'$, and $p^s y'$ is a solution of (8.1) in $p^s O$. □

Lemma 8.7. *Let $m \in \mathbb{N}^*$. Each homomorphism of truncated Dieudonné modules*

$$\zeta_m : (M/p^m M, F, V) \to (L/p^m L, F, V)$$

can be lifted to a $W(k)$-linear map $\zeta : M \to L$ such that $F(\zeta) - \zeta \in p^m H$.

It is not true that every lift of ζ_m satisfies $F(\zeta) - \zeta \in p^m H$.

Proof. By passing to $M \oplus L$ it suffices to consider endomorphisms instead of homomorphisms, so we consider $\mathrm{End}_{W(k)}(M)$ instead of $\mathrm{Hom}_{W(k)}(M, L)$. Let $Q = VM$. Then V induces a bijective σ^{-1}-linear map $\bar{V}_m : M/p^m M \to Q/p^m Q$. Let $\zeta'_m = \bar{V}_m \zeta_m \bar{V}_m^{-1} : Q/p^m Q \to Q/p^m Q$. There exists an element $\zeta \in \mathrm{End}_{W(k)}(M)$ which lifts ζ_m and ζ'_m at the same time. Indeed, let $M = J \oplus T$ be such that $Q = J \oplus pT$. Choose $\zeta : T \to M$ to be any lift of the restriction of ζ_m to $T/p^m T$ and $\zeta : J \to Q$ to be any lift of the restriction of ζ'_m to $J/p^m J$. Then the resulting $W(k)$-linear map $\zeta : M \to M$ lifts ζ_m and ζ'_m. It follows that $\zeta V - V\zeta$ maps M to $p^m Q$, thus $V^{-1}\zeta V - \zeta$ maps M to $p^m M$, which means that $F(\zeta) - \zeta \in p^m \mathrm{End}_{W(k)}(M)$ as required. □

Proposition 8.8. *We have $e_{D,E} \leq \ell_{D,E}$.*

Proof. Let $m = \ell_{D,E} + 1$. We have to show that for each homomorphism

$$\zeta_m : (M/p^m M, F, V) \to (L/p^m L, F, V)$$

of truncated Dieudonné modules, its reduction $\zeta_1 : M/pM \to L/pL$ lifts to a homomorphism of Dieudonné modules $\zeta : (M, F, V) \to (L, F, V)$. By Lemma 8.7, ζ_m can be lifted to a $W(k)$-linear map $\zeta' : M \to L$ with $F(\zeta') - \zeta' \in p^m H$. As $p^m H \subseteq pO$, by Lemma 8.6 there exists an element $\xi \in pO$ such that $F(\xi) - \xi = F(\zeta') - \zeta'$. Thus $\zeta = \zeta' - \xi$ has the desired property. □

8.3 The inequality $\ell_{D,E} \leq f_{D,E}$

Let $\mathfrak{m}_{\mathcal{R}}$ be the maximal ideal of a local ring $\mathcal{R}$.

Lemma 8.9. *For $P \in k[[t]]$ and $x \in uk[[u]]$ the expression $P(x) \in k[[u]]$ is well-defined. Moreover we have $P(x) = 0$ only if either $P = 0$ or $x = 0$.*

Proof. The power series $P(x)$ converges u-adically, so $P(x)$ is well-defined. If $x \neq 0$ and $P \neq 0$, let $r, s \in \mathbb{N}$ be minimal such that $x \in u^r k[[u]]$ and $P \in t^s k[[t]]$. Then $r > 0$ and $P(x) \in u^{rs}k[[u]] \setminus u^{rs+1}k[[u]]$. Thus $P(x) \neq 0$. $\square$

Proposition 8.10. *We have $\ell_{D,E} \leq f_{D,E}$.*

Proof. We show that the assumption that $f_{D,E} < \ell_{D,E}$ leads to a contradiction. By Lemmas 7.7 and 8.5 we can replace the field k by some algebraically closed field extension of it. Therefore we can assume that there exists an algebraically closed field k' and an inclusion of rings $R = k'[[t]] \subset k$ such that D and E are the base change of p-divisible groups D' and E' over k'.

Let (M', F', V') and (L', F', V') be the Dieudonné modules over k' of D' and E' (respectively). We have $(M, F, V) = (M' \otimes_{W(k')} W(k), F' \otimes \sigma, V' \otimes \sigma^{-1})$ and similarly for L. Let H' and $O' = O'_+ \oplus O'_0 \oplus O'_-$ be the analogues of H and $O = O_+ \oplus O_0 \oplus O_-$ obtained working with D' and E' instead of D and E. We have $H = H' \otimes_{W(k')} W(k)$ and $O = O' \otimes_{W(k')} W(k)$.

Let $x \in p^{\ell_{D,E}} H' \cap (O' \setminus pO')$. Then $x \in O \setminus pO$. For each $\eta \in W(\mathfrak{m}_R)$ we define $y_\eta = y_{\eta,+} + y_{\eta,0} + y_{\eta,-} \in O$ by the two p-adically convergent series

$$y_{\eta,+} = -\sum_{i=0}^{\infty} F^i(\eta x_+) \in O_+, \qquad y_{\eta,-} = \sum_{i=1}^{\infty} F^{-i}(\eta x_-) \in O_-,$$

and by the t-adically convergent series in $O' \otimes_{W(k')} W(R) \subset O$

$$y_{\eta,0} = -\sum_{i=0}^{\infty} F^i(\eta x_0).$$

The last series converges because $\sigma^{p^i}(\eta) \to 0$ for $i \to \infty$, so for each $n \in \mathbb{N}$ the series maps to a finite sum in $W(R/\mathfrak{m}_R^n)$. We have

$$\eta x = F(y_\eta) - y_\eta.$$

We note that for every $\eta \in W(k)$ the last equation can be solved for y_η (see Lemma 8.6) and that the components $y_{\eta,\pm}$ are always given by the above p-adic series, but in general we do not have a natural choice for $y_{\eta,0}$ and in particular no explicit formula for it.

The reduction modulo $p^{\ell_{D,E}}$ of $y_\eta : M \to L$ is a homomorphism of truncated Dieudonné modules $(M/p^{\ell_{D,E}}M, F, V) \to (L/p^{\ell_{D,E}}L, F, V)$. Indeed, the relation $\eta x = F\circ y_\eta \circ F^{-1} - y_\eta = V^{-1}\circ y_\eta \circ V - y_\eta$ gives $\eta x \circ F = F\circ y_\eta - y_\eta \circ F$ and $V \circ \eta x = -V \circ y_\eta + y_\eta \circ V$. As $x(M) \subseteq p^{\ell_{D,E}}L$, we conclude that $F \circ y_\eta$ and $y_\eta \circ F$ (resp. $V \circ y_\eta$ and $y_\eta \circ V$) coincide modulo $p^{\ell_{D,E}}$.

By classical Dieudonné theory it follows that each y_η defines a homomorphism $D[p^{\ell_{D,E}}] \to E[p^{\ell_{D,E}}]$. The assumption $f_{D,E} < \ell_{D,E}$ implies that the resulting restrictions $D[p] \to E[p]$ take only finitely many values, which means that the reductions of y_η in H/pH take only finitely many values. As the assignment $\eta \mapsto y_\eta$ is additive and as $\mathfrak{m}_R$ is infinite it follows that there exists an element $\eta = (\eta_0, \eta_1, \ldots) \in W(\mathfrak{m}_R)$ with $\eta_0 \neq 0$ such that $y_\eta \in O \cap pH$.

From the relation $y_\eta \in O \cap pH$ we want to deduce the following relations.

Claim. *For each $n \in \mathbb{N}$ the elements $F^n(x_+)$, $F^n(x_0)$, $F^{-n}(x_-)$ lie in pH.*

By the definition of O this implies that $p^{-1}x = p^{-1}(x_+ + x_0 + x_-) \in O$, which contradicts the assumption that $x \notin pO$. Hence to end the proof of Proposition 8.10 it remains to prove the claim.

Let $\mathcal{V} = O \cap pH$. We have $pO \subseteq \mathcal{V} \subseteq O$. Thus $\bar{\mathcal{V}} = \mathcal{V}/pO$ is a k-vector subspace of O/pO. Similarly let $\mathcal{V}' = O' \cap pH'$ and $\bar{\mathcal{V}}' = \mathcal{V}'/pO'$. We have $\bar{\mathcal{V}} = \bar{\mathcal{V}}' \otimes_{k'} k$. For an element $b \in O$ (resp. $b \in W(k)$) we denote by $\bar{b} \in O/pO$ (resp. $\bar{b} \in k$) its reduction modulo p. We denote also by F (resp. F^{-1}) the σ-linear (resp. σ^{-1}-linear) endomorphism of O_+/pO_+ and O_0/pO_0 (resp. O_-/pO_-) induced by F (resp. F^{-1}). Let n_+ (resp. n_-) be the minimal non-negative integer such that $F^{n_+}(x_+) \in pO_+$ (resp. $F^{-n_-}(x_-) \in pO_-$).

An element $\bar{z} \in O/pO$ lies in $\bar{\mathcal{V}}$ if and only if for every k'-linear map $\varpi : O'/pO' \to k'$ with $\varpi(\bar{\mathcal{V}}') = 0$ we have $(\varpi \otimes 1_k)(\bar{z}) = 0$. For $n \in \mathbb{N}$ we consider the elements of k'

$$a_{+,n} = \varpi(F^n(\bar{x}_+)), \quad a_{0,n} = \varpi(F^n(\bar{x}_0)), \quad a_{-,n} = \varpi(F^{-n}(\bar{x}_-)).$$

For $n \geq n_+$ we have $a_{+,n} = 0$ and for $n \geq n_-$ we have $a_{-,n} = 0$. To prove the claim we have to show that $a_{+,n} = a_{0,n} = a_{-,n} = 0$ for all $n \geq 0$.

As $y_\eta \in \mathcal{V}$ we have $(\varpi \otimes 1_k)(\bar{y}_\eta) = 0$. From the definition of y_η we get

$$\sum_{n=0}^{n_+} a_{+,n}\eta_0^{p^n} + \sum_{n=0}^{\infty} a_{0,n}\eta_0^{p^n} - \sum_{n=1}^{n_-} a_{-,n}\eta_0^{p^{-n}} = 0.$$

This expression can be viewed as a power series in $R = k'[[t]]$ evaluated at the element $\eta_0^{p^{-n_-}}$ of the maximal ideal of $R' = k'[[t^{p^{-n_-}}]]$. By Lemma 8.9 it follows that we have $a_{-,n} = 0$ for $n \geq 1$ and $a_{+,n} + a_{0,n} = 0$ for $n \geq 0$; in particular we have $a_{0,n} = 0$ for $n \geq n_+$. As F is bijective on O_0/pO_0, the subspace of O_0/pO_0 generated by $\{F^n(\bar{x}_0) \mid n \geq 0\}$ is equal to the subspace generated by $\{F^n(\bar{x}_0) \mid n \geq n_+\}$. Thus we get $a_{0,n} = 0$ for $n \geq 0$, which gives also $a_{+,n} = 0$ for $n \geq 0$. As $x \in p^{\ell_{D,E}} H$ and as $0 \leq f_{D,E} < \ell_{D,E}$ we have $x \in pH$ and hence $x \in \mathcal{V}$. Thus $0 = \varpi(\bar{x}) = a_{+,0} + a_{0,0} + a_{-,0}$ and therefore $a_{-,0} = 0$. □

8.4 Conclusions

Theorem 8.11. *For two p-divisible groups D and E over k we have*

$$f_{D,E} = \ell_{D,E} = e_{D,E}.$$

Proof. This follows from Propositions 7.5, 8.8, and 8.10. □

Corollary 8.12. *We have $e_{D,E} = e_{E,D} = e_{D^\vee,E^\vee}$ and similarly for f.*

The equation $e_{D,E} = e_{E,D}$ seems not to be obvious from the definitions.

Proof. This follows from Theorem 8.11 and Lemma 8.4. □

Proof of Theorem 1.6. It follows from Theorems 8.11 and 7.3. □

Proposition 8.13. *We have $\ell_{D\oplus E} = \max\{\ell_D, \ell_E, \ell_{D,E}\}$ and $\ell_{D,E} \leq n_{D\oplus E}$.*

Proof. We have a direct sum decomposition $\mathrm{End}_{W(k)}(M\oplus L) = \mathrm{End}_{W(k)}(M)\oplus \mathrm{End}_{W(k)}(L) \oplus \mathrm{Hom}_{W(k)}(L, M) \oplus \mathrm{Hom}_{W(k)}(M, L)$ into $W(k)$-modules which is compatible with F and therefore which induces a direct sum decomposition $O_{D\oplus E} = O_D\oplus O_E\oplus O_{D,E}\oplus O_{E,D}$. It follows that $\ell_{D\oplus E} = \max\{\ell_D, \ell_E, \ell_{D,E}, \ell_{E,D}\}$. From this and Lemma 8.4 we get $\ell_{D\oplus E} = \max\{\ell_D, \ell_E, \ell_{D,E}\}$. By Theorem 1.6 and the subsequent remark we have $\ell_{D\oplus E} \leq n_{D\oplus E}$. Thus $\ell_{D,E} \leq n_{D\oplus E}$. □

Proposition 8.14. *If D and E are minimal then $\ell_{D,E} \leq 1$ with equality if and only if both D and E are non-ordinary minimal.*

Proof. This is proved in [Va3, Thm. 1.6]. We give here also a direct argument. The group $\mathrm{Hom}(D[p], E[p])$ is finite if and only if one of D and E is ordinary, so $\ell_{D,E} = f_{D,E}$ is zero if and only if one of D and E is ordinary. Thus it suffices to prove that $\ell_{D,E} \leq 1$ if D and E are minimal. We can assume that D and E are simple. By Proposition 5.17 there are F-valuations w on $M_{\mathbb{Q}}^\vee$ and u on $L_{\mathbb{Q}}$ such that $M^\vee = (M_{\mathbb{Q}}^\vee)^{w\geq 0}$ and $L = (L_{\mathbb{Q}})^{u\geq 0}$. Let $w \otimes u$ be the product valuation on the isoclinic $\mathbb{D}$-module $(M^\vee \otimes L)_{\mathbb{Q}} = \mathrm{Hom}_{W(k)}(M, L)_{\mathbb{Q}}$. This is an F-valuation. As $M^\vee$ and L have valuative $W(k)$-bases consisting of elements with valuation in $[0, 1)$, $M^\vee \otimes L$ has a valuative $W(k)$-basis consisting of elements with valuations in $[0, 2)$. Therefore

$$(M^\vee \otimes L)_{\mathbb{Q}}^{(w\otimes u)\geq 1} \subseteq \mathrm{Hom}_{W(k)}(M, L) \subseteq (M^\vee \otimes L)_{\mathbb{Q}}^{(w\otimes u)\geq 0}.$$

As the outer $W(k)$-modules are stable under either F or F^{-1} and as their quotient is annihilated by p, we conclude that $(M^\vee \otimes L)_{\mathbb{Q}}^{(w\otimes u)\geq 1} \subseteq O$ and $\ell_{D,E} \leq 1$. □

9 Values of homomorphism numbers

In this section we study possible values of the homomorphism numbers $e_{D,E}$ and the isomorphism numbers n_D; the latter is a special case of the former by Theorem 1.6. In particular, we prove Theorem 1.3. We actually work with $f_{D,E}$ and ℓ_D, which gives equivalent results by Theorems 1.6 and 8.11.

9.1 Upper bounds

We fix p-divisible groups D and D' over k. The dimension, codimension, height, and Newton polygon of D (resp. D') are denoted d, c, h, and ν (resp. d', c', h', and ν'). Recall that Theorem 1.3 claims that if D is not ordinary, then the isomorphism number n_D is at most $\lfloor 2\nu(c) \rfloor$. We will provide an upper bound for the coarse homomorphism number $f_{D',D}$, which will imply Theorem 1.3 by setting $D = D'$.

Let $d^+ = d + d'$ be the dimension, $c^+ = c + c'$ be the codimension, and $h^+ = h + h' = c^+ + d^+$ be the height of $D \oplus D'$. Let ν^+ be the Newton polygon of $D \oplus D'$. Let $\lambda_1^+ < \cdots < \lambda_{r^+}^+$ be the slopes of ν^+. For each $j^+ \in \{1, \ldots, r^+\}$ let $\nu_{j^+}^+ : \mathbb{R} \to \mathbb{R}$ be the unique linear function of slope $\lambda_{j^+}^+$ such that for all $t \in [0, h^+]$ we have $\nu^+(t) = \max\{\nu_{j^+}^+(t) \mid 1 \leq j^+ \leq r^+\}$. Let $\mathcal{J}_D \subseteq \{1, \ldots, r^+\}$ be the set of indices j^+ such that $\lambda_{j^+}^+$ is a slope of D.

Theorem 9.1. *We have* $f_{D',D} \leq \max\{\nu_{j^+}^+(c^+) \mid j^+ \in \mathcal{J}_D\}$.

As $f_{D',D}$ is symmetric (see Corollary 8.12), by interchanging D and D' we get another, possibly better upper bound of it. For example, if either D or D' is ordinary, in this way we get that $f_{D',D} = 0$, which is easily verified directly. Before proving Theorem 9.1 we deduce some corollaries of it.

Corollary 9.2. *The following inequalities hold*

$$f_{D',D} \leq \nu^+(c^+) \leq \nu(c) + \nu'(c') \leq cd/h + c'd'/h' \leq c^+d^+/h^+ \leq h^+/4.$$

Proof. This is elementary and left to the reader. □

Corollary 9.3. *We have the following relations*

$$n_{D',D} = e_{D',D} = \ell_{D',D} = f_{D',D} \le \max\{\nu^+_{j^+}(c^+) \mid j^+ \in \mathcal{J}_D\}.$$

Proof. This follows from Remark 7.4, Theorem 8.11, and Theorem 9.1. □

Corollary 9.4. *We have $n_D = f_D \le \lfloor 2\nu(c) \rfloor$.*

Proof. For $D' = D$ we have $r^+ = r$, $c^+ = 2c$. Thus from Theorem 9.1 we get that $f_D = f_{D,D} \le \max\{\nu^+_j(2c) \mid 1 \le j \le r\} = \nu^+(2c) = 2\nu(c)$. From this and Theorem 7.3 we get that the corollary holds. □

Proof of Theorem 1.3. It follows from Theorem 7.3 and Corollary 9.4. □

Lemma 9.5. *To prove Theorem 9.1 we can assume that D and D' are connected with connected duals and that $a_D = a_{D'} = 1$.*

Proof. This is similar to the proof of Lemma 6.1. One has to replace Lemma 2.4 (b) by Lemma 7.6 and Theorem 3.8 by Theorem 7.11 (b). If $D = D'$ one can also use Theorem 3.9 instead of Theorem 7.11 (b). □

Proof of Theorem 9.1. Let M and M' be the Dieudonné modules of D and D' (respectively). By Lemma 9.5 we can assume that M and M' are bi-nilpotent and that $a(M) = a(M') = 1$. Let $z \in M$ and $z' \in M'$ be generators as $\mathbb{E}$-modules. Let $M = \mathbb{E}/\mathbb{E}\Psi$ and $M' = \mathbb{E}/\mathbb{E}\Psi'$ be the associated presentations given by Lemma 5.7 (b). For $m \in \mathbb{N}$ we have canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}(D'[p^m], D[p^m]) &\cong \mathrm{Hom}_{\mathbb{E}}(M'/p^m M', M/p^m M)\\ &\cong \mathrm{Ker}(\Psi' : M/p^m M \to M/p^m M),\end{aligned}$$

where the second isomorphism maps a homomorphism ϕ to $\phi(z')$.

Let $m = f_{D',D}$. By the definition of $f_{D',D}$ there exists an infinite set $\mathcal{L}$ and a subset $\{x_l \mid l \in \mathcal{L}\}$ of M such that $\Psi' x_l \in p^m M$ for all $l \in \mathcal{L}$ and such that the reductions $\bar{x}_l \in M/pM$ of the elements x_l are pairwise distinct.

We use Notation 5.11 with respect to M, so that $N = M_{\mathbb{Q}}$, the slopes of N are $\lambda_1 < \cdots < \lambda_r$, and for $j \in \{1, \ldots, r\}$ we have $\beta_j = \nu_j(c)$. For $j \in \{1, \ldots, r\}$ let $\nu'_j : \mathbb{R} \to \mathbb{R}$ be the maximal linear function of slope λ_j such that we have $\nu'_j(t) \le \nu'(t)$ for all $t \in [0, h']$, and let $\beta'_j = \nu'_j(c')$. This defines $\beta' = (\beta'_1, \ldots, \beta'_r) \in \mathbb{R}^r$. We note that β' is not the analogue of β for M' in place of M. By Lemma 5.8 the polygon ν' coincides with $\nu_{\Psi'}$ shifted to the right by c'. Using (5.1) this implies

$$\beta'_j = \nu'_j(c') = \nu_{\Psi',\lambda_j}(0) = \mathrm{w}_{\lambda_j}(\Psi').$$

As $N^{\beta+} \subseteq pM$ by Proposition 5.12, the images of x_l in $N^{\underline{0}}/N^{\beta+}$ are all distinct. By Lemma 5.5 the operator Ψ' induces a homomorphism $N^{\underline{0}}/N^{\beta+} \to N^{\beta'}/N^{(\beta+\beta')+}$ with finite kernel. Hence $\Psi' x_l \notin N^{(\beta+\beta')+}$ for all but finitely many $l \in \mathcal{L}$. Recall that $\underline{m} = (m, \ldots, m) \in \mathbb{R}^r$. As $\Psi' x_l \in p^m M$ and $p^m M \subseteq N^{\underline{m}}$, it follows that $N^{\underline{m}} \not\subseteq N^{(\beta+\beta')+}$. Hence for some index $j \in \{1, \ldots, r\}$ we have $m \leq \beta_j + \beta'_j$. There exists a unique $j^+ \in \mathcal{J}_D \subseteq \{0, \ldots, r^+\}$ such that $\lambda_j = \lambda^+_{j^+}$. By Lemma 9.6 below we have $\beta_j + \beta'_j = \nu^+_{j^+}(c^+)$ and thus $f_{D',D} = m \leq \nu^+_{j^+}(c^+)$. □

Lemma 9.6. *With the above notations, we have $\nu_j(c) + \nu'_j(c') = \nu^+_{j^+}(c^+)$.*

Proof. Let $x^+ \in [0, h^+]$, $x \in [0, h]$, and $x' \in [0, h']$ be the maximal elements such that ν^+, ν, and ν' have slope $< \lambda_j$ on the intervals $[0, x^+]$, $[0, x]$, and $[0, x']$ (respectively). Using that $x^+ = x + x'$, $c^+ = c + c'$, and $\lambda^+_{j^+} = \lambda_j$, we compute $\nu^+_{j^+}(c^+) - \nu_j(c) - \nu'_j(c') = \nu^+_{j^+}(x^+) - \nu_j(x) - \nu'_j(x') = \nu^+(x^+) - \nu(x) - \nu'(x') = 0$. □

Remark 9.7. Corollary 9.4 can be proved directly by letting $D = D'$ in the proof of Theorem 9.1. Then the last two lines of that proof can be replaced by the conclusion $f_D \leq \beta_j + \beta'_j = 2\beta_j \leq 2\nu(c)$, which avoids Lemma 9.6.

Homomorphism numbers have bounded variation under isogenies:

Proposition 9.8. *Let E denote a p-divisible group, and let $g : D \to D'$ be an isogeny of p-divisible groups over k such that p annihilates the kernel of g. Then we have $|f_{D,E} - f_{D',E}| \leq 1$ and $|f_D - f_{D'}| \leq 2$, thus $|n_D - n_{D'}| \leq 2$ by Theorem 7.3.*

Proof. Let $g' : D' \to D$ be the isogeny such that $g'g = p \cdot 1_D$. We write $H_m = \mathrm{Hom}(D[p^m], E[p^m])$ and $H'_m = \mathrm{Hom}(D'[p^m], E[p^m])$. For each $m \in \mathbb{N}$ we have a commutative diagram

$$\begin{tikzcd}
H_{m+2} \arrow[rr, "\tau_{m+2,1}"] \arrow[d, "u \mapsto ug'"'] & & H_1 \arrow[d, "\rho"] \\
H'_{m+2} \arrow[r, "\tau_{m+2,2}"] & H'_2 \arrow[r, "u \mapsto ug"] & H_2
\end{tikzcd}$$

where ρ maps a homomorphism $u : D[p] \to E[p]$ to the obvious composition $D[p^2] \to D[p] \xrightarrow{u} E[p] \to E[p^2]$. Let $m = f_{D',E}$. Then the image of $\tau_{m+2,2}$ is finite by the definition of $f_{D',E}$. As ρ is injective, the image of $\tau_{m+2,1}$ is finite too, thus $f_{D,E} \leq m + 1$. By interchanging the roles of D and D' we get that $|f_{D,E} - f_{D',E}| \leq 1$. A similar argument (or Corollary 8.12) gives also $|f_{E,D} - f_{E,D'}| \leq 1$, thus $|f_D - f_{D'}| \leq |f_D - f_{D,D'}| + |f_{D,D'} - f_{D'}| \leq 2$. □

Remark 9.9. The preceding proposition offers another approach to bound f_D from above; see Remark 6.4. For a non-ordinary minimal p-divisible group D_0 we have $n_{D_0} = f_{D_0} = e_{D_0} = \ell_{D_0} = 1$; see [Oo3, Thm. 1.2] and [Va3, Thm. 1.6]. Here $\ell_{D_0} = 1$ is proved in Proposition 8.14, and the rest follows by Theorems 8.11 and 7.3. Thus Proposition 9.8 (or [Va3, Prop. 1.4.4]) gives $f_D \leq 1 + 2q_D$. Together with Theorem 1.5 we get $f_D \leq 1 + 2\lfloor \nu(c) \rfloor$. As $\lfloor 2\nu(c) \rfloor \leq 1 + 2\lfloor \nu(c) \rfloor$ with equality if and only if $\lfloor 2\nu(c) \rfloor$ is odd, this is slightly weaker than Corollary 9.4, but we get as close as possible taking into account that this approach necessarily gives an odd upper bound.

9.2 Lower bounds in the isoclinic case

We assume now that the p-divisible group D over k is isoclinic and continue to study the possible values of $f_D = \ell_D$ in this case.

Let us recall from [Va3] how to compute ℓ_D for isoclinic groups. Let M be the Dieudonné module of D and let $O \subseteq \mathrm{End}_{W(k)}(M)$ be its level module. We have $O = O_0$, cf. Definition 8.1. For each integer t let $\alpha_t(M)$ be the maximal integer and let $\beta_t(M)$ be the minimal integer such that

$$p^{\beta_t(M)} M \subseteq F^t M \subseteq p^{\alpha_t(M)} M.$$

Let $\delta_t(M) = \beta_t(M) - \alpha_t(M)$.

Lemma 9.10. *The natural number $\delta_t(M)$ is equal to the minimal integer δ such that $p^\delta \mathrm{End}_{W(k)}(M) \subseteq \mathrm{End}_{W(k)}(F^t M)$ as subgroups of $\mathrm{End}_{W(k)}(M)_{\mathbb{Q}}$.*

Proof. The optimality of $\alpha_t(M)$ and $\beta_t(M)$ means that the inclusion maps $p^{\beta_t(M)} M \to F^t M$ and $p^{-\alpha_t(M)} M^\vee \to (F^t M)^\vee$ have non-zero reductions modulo p. This property carries over to the tensor product of the two maps, which is an inclusion $p^{\beta_t(M)-\alpha_t(M)} \mathrm{End}_{W(k)}(M) \to \mathrm{End}_{W(k)}(F^t M)$. $\square$

Lemma 9.11 ([Va3, Prop. 4.3 (a)])**.** *We have $\ell_D = \max\{\delta_t(M) \mid t \in \mathbb{N}\}$.*

Proof. The level module $O = O_0$ is the intersection in $\mathrm{End}_{W(k)}(M)_{\mathbb{Q}}$ of all $\mathrm{End}_{W(k)}(F^t M)$ for $t \in \mathbb{N}$. Thus ℓ_D is minimal with $p^{\ell_D} \mathrm{End}_{W(k)}(M) \subseteq \mathrm{End}_{W(k)}(F^t M)$ for all $t \in \mathbb{N}$. The lemma follows from Lemma 9.10. $\square$

Lemma 9.12. *Assume that $a_D = 1$.*

(i) For $0 \leq t \leq c$ we have $\alpha_t(M) = 0$.

(ii) For $0 \leq t \leq d$ we have $\beta_t(M) = t$.

Proof. We write $M = \mathbb{E}/\mathbb{E}\Psi$ as in Lemma 5.7 (c). Let $z = 1 + \mathbb{E}\Psi \in M$. For $0 \leq t \leq c$ we have $F^t M \subseteq M$, and $F^t M \not\subseteq pM$ because $F^t z$ is part of the $W(k)$-basis Υ of M defined in Lemma 5.7 (a). This proves (i). Similarly, for $0 \leq t \leq d$ we have $V^t M \subseteq M$ and $V^t M \not\subseteq pM$, which is equivalent to $p^t M \subseteq F^t M$ and $p^t M \not\subseteq pF^t M$. This proves (ii). $\square$

We have the following lower bound of ℓ_D.

Proposition 9.13. *If D is isoclinic with $a_D = 1$ then $\ell_D \geq \min\{c, d\}$.*

Proof. Let $t = \min\{c, d\}$. We calculate $\ell_D \geq \delta_t(M) = \beta_t(M) - \alpha_t(M) = t$ using Lemmas 9.11 and 9.12. $\square$

Corollary 9.14. *Assume that D is isoclinic with $a_D = 1$ and $|c - d| \leq 2$. Then we have $\ell_D = \min\{c, d\}$.*

Proof. We have $\min\{c, d\} \leq \ell_D = f_D \leq \lfloor 2\nu(c) \rfloor \leq \lfloor 2cd/(c+d) \rfloor = \min\{c, d\}$, cf. Theorem 8.11 and Corollary 9.4. $\square$

The lower bound in Proposition 9.13 is optimal:

Example 9.15. Let D be the isoclinic p-divisible group with Dieudonné module $M = \mathbb{E}/\mathbb{E}\Psi$ for $\Psi = F^c + V^d$ where $cd > 0$. Then $\ell_D = \min\{c, d\}$.

Proof. This is a particular case of [Va3, Thm. 1.5.2]. We give here a direct proof. As the $W(k)$-basis Υ of M defined in Lemma 5.7 (a) is annihilated by $F^{c+d} + p^d$ we have $F^{c+d}M = p^d M$. It follows that $\alpha_{t+c+d}(M) = \alpha_t(M) + d$ and $\beta_{t+c+d}(M) = \beta_t(M) + d$ for $t \in \mathbb{Z}$. We also have $\alpha_{-t}(M) = -\beta_t(M)$. Thus Lemma 9.12 gives a complete description of $\delta_t(M)$ for all t, which implies that $\delta_t(M) \leq \min\{c, d\}$ with equality when $t = \min\{c, d\}$. Therefore $\ell_D = \min\{c, d\}$ by Lemma 9.11. $\square$

We recall that Corollary 1.4 is a direct consequence of Theorem 1.3 and of the following optimality result.

Proposition 9.16. *Assume that ν is linear and $cd > 0$. Then there exists a p-divisible group D over k with Newton polygon ν such that $\ell_D = \lfloor 2\nu(c) \rfloor$. (Note that $n_D = e_D = f_D = \ell_D$ by Theorems 7.3 and 8.11.)*

Proof. By passing to the dual if necessary we can assume that $c \geq d$. We have $\nu(c) = cd/(c + d)$. Let $m = \lfloor 2\nu(c) \rfloor$. We will construct explicitly the Dieudonné module of D to be $M = \mathbb{E}/\mathbb{E}\Psi$ for a suitable $\Psi \in \mathbb{E}$ as in Lemma 5.7 (b). Let $z = 1 + E\Psi \in M$. By Corollary 9.4 and Theorem 8.11, in order that $\ell_D = m$ it suffices to show that $\ell_D \geq m$.

We will show that Ψ can be chosen such that M is isoclinic and there exists $x \in M \setminus pM$ with $F^c x = p^m z$. This implies that $\beta_c(M) \geq m$. By Lemmas 9.11 and 9.12 we get $\ell_D \geq \delta_c(M) = \beta_c(M) \geq m$ as required.

It remains to construct Ψ and x. If $c = d$, then $m = c = d$ and we can take $\Psi = F^c + V^c$ and $x = V^c z$ (or use Corollary 9.14). If $c > d$, then we can take $\Psi = F^c + \Phi + V^d$ for

$$\Phi = p^{2d-m} F^{c-2d} = p^{c-m} V^{2d-c}.$$

A priori Φ is an element of $\mathbb{D} = \mathbb{E} \otimes \mathbb{Q}$, but actually Φ lies in $\mathbb{E}$ (as $m < 2d$ and $m < c$). We take $x = -(p^{m-d}F^d + V^d)z$. It is easy to see that as $\Psi z = 0$ we have $F^c x = p^m z$. As $m \geq d$ and $c > d$, we have $x \in M \setminus pM$. The Newton polygon of M is linear of slope $\lambda = d/(c+d)$. Indeed, as the exponents of F in Ψ satisfy $c > c - 2d > -d$, this is equivalent to $2d - m \geq 2d\lambda$ (cf. Lemma 5.8) and thus $m \leq 2cd/(c+d)$ which obviously holds. $\square$

Example 9.17. More generally, for each integer m with $\min\{c, d\} \leq m \leq \lfloor 2cd/(c+d) \rfloor$ there exists an isoclinic p-divisible group D with $\ell_D = m$. Indeed, we can assume that $c \geq d$ and that $m < \lfloor 2cd/(c+d) \rfloor$. Define D such that its Dieudonné module is $\mathbb{E}/\mathbb{E}\Psi$ with $\Psi = F^c + p^{2d-m}F^{c-2d} + V^d$. Then the proof of Proposition 9.16 gives $\ell_D \geq m$.

We sketch a proof of the other inequality $\ell_D = f_D \leq m$ using the method of the proof of Theorem 9.1. Let w be the minimal F-valuation of slope $\lambda = d/(c+d)$ on $N = M_{\mathbb{Q}}$ such that $w(M) \geq 0$. Let $\beta = cd/(c+d)$. Let $\psi : M \to M/p^{m+1}M$ be induced by Ψ. We say that a subgroup of M has infinite reduction if its image in M/pM is infinite.

Assume that $f_D > m$. This means that $\operatorname{Ker} \psi$ has infinite reduction. As $p^{m+1}M \subseteq N^{w \geq m+1}$ and as $\mathbb{w}_\lambda(\Psi) = \beta$, using Lemma 5.5 it follows that the kernel of the restriction $\psi' : M \cap N^{w \geq m+1-\beta} \to M/p^{m+1}M$ has infinite reduction as well. Thus the kernel of the induced map $\psi'' : M \cap N^{w \geq m+1-\beta} \to M/(p^{m+1}M + N^{w > 2\beta})$ has infinite reduction. If we identify M with $W(k)^{c+d}$ using the basis Υ defined in Lemma 5.7 (a), then the map ψ'' defined for Ψ is equal to the analogous map ψ''_0 defined for $\Psi_0 = F^c + V^d$. This is true because $\mathbb{w}_\lambda(\Phi) + m + 1 - \beta > 2\beta$, which holds as $\mathbb{w}_\lambda(\Phi) = 3\beta - m$.

We continue with $M_0 = \mathbb{E}/\mathbb{E}\Psi_0$. Let $z = 1 + \mathbb{E}\Psi_0 \in M_0$. As before let w be the minimal F-valuation of slope λ on $N_0 = (M_0)_{\mathbb{Q}}$ such that $w(M_0) \geq 0$. Let $M_1 = \Psi_0(M_0)$ and $M_2 = p^{d+1}M_0 + N_0^{w > 2\beta}$. It is easy to see that $M_2 \subseteq pM_1$, using that the elements $(p^i F^{c-i} z)_{0 \leq i \leq d}$, $(p^d F^j z)_{0 < j < c-d}$, and $(p^i V^{d-i} z)_{0 < i \leq d}$ form a basis of M_1. By the previous paragraph, the kernel of the map $\psi_0 : M_0 \to M_0/M_2$ induced by Ψ_0 has infinite reduction. Thus the kernel of $\psi_0 : M_0 \to M_0/pM_1$ has infinite reduction, and the kernel of $\psi_0 : M_0/pM_0 \to M_1/pM_1$ is infinite. This is impossible.

Therefore we have $\ell_D = f_D \leq m$.

9.3 Possible values of isomorphism numbers

For a non-ordinary Newton polygon ν let $\mathscr{N}_\nu$ be the set of all possible values of $n_D = e_D = f_D = \ell_D$ for p-divisible groups D with Newton polygon ν. We make two fragmentary remarks on the structure of $\mathscr{N}_\nu$. First, as two isogenous groups can be linked by a chain of isogenies with kernels annihilated by p, in view of Proposition 9.8 the difference between two consecutive numbers in $\mathscr{N}_\nu$ is at most 2. In certain cases we can say more (cf. Proposition 6.2):

Proposition 9.18. *Assume that $\lfloor 2\nu(c) \rfloor$ is odd and lies in $\mathscr{N}_\nu$. Then $\mathscr{N}_\nu$ contains all odd integers between 1 and $\lfloor 2\nu(c) \rfloor$.*

Proof. Let $m = \lfloor \nu(c) \rfloor$. We choose D such that $f_D = \lfloor 2\nu(c) \rfloor = 2m+1$. By Theorem 1.5 there exists a chain of isogenies $D = D_m \to \cdots \to D_0$ such that all consecutive kernels are annihilated by p and D_0 is minimal. As $f_{D_0} = 1$ (resp. $f_D = 2m+1$), for each $i \in \{0, 1, \ldots, m\}$ we get from Proposition 9.8 that $f_{D_i} \leq 1 + 2i$ (resp. $f_{D_i} \geq 1 + 2i$). Thus $f_{D_i} = 1 + 2i$. $\square$

9.4 The principally quasi-polarised case

Let D be a p-divisible group over k of dimension d equipped with a principal quasi-polarisation $\lambda : D \to D^\vee$; thus $c = d > 0$. The isomorphism number $n_{D,\lambda}$ of (D, λ) is the least level m such that $(D[p^m], \lambda[p^m])$ determines (D, λ) up to isomorphism. It is proved in [GV, Subsect. 6.3] that $n_{D,\lambda} \leq n_D$ if $p > 2$ and $n_{D,\lambda} \leq n_D + 1$ if $p = 2$. If D is not supersingular, then Theorem 1.3 implies that $n_D \leq d - 1$. If D is supersingular and $d > 0$, then we have $n_{D,\lambda} \leq d$ by [NV1, Thm. 1.3]. Together we get in all cases:

Corollary 9.19. *We have $n_{D,\lambda} \leq d$.*

This bound is optimal by [NV1, Example 3.3].

9.5 The number N_h

Recall that N_h is defined in Section 2 as the minimal number such that for all p-divisible groups D and E over k of height at most h, we have $e_{D,E} \leq N_h$.

Proposition 9.20. *We have $N_h = \lfloor h/2 \rfloor$.*

Proof. By Theorem 8.11 and Corollary 9.2 we have $N_h \leq h/2$. By Example 9.15 there exists an isoclinic p-divisible group D over k of slope $1/2$ and dimension $\lfloor h/2 \rfloor$ with $f_D = \lfloor h/2 \rfloor$; its height is $2\lfloor h/2 \rfloor \leq h$. As we have $e_{D,D} = f_D$ by Theorem 8.11, we get that $N_h \geq \lfloor h/2 \rfloor$. We conclude that $N_h = \lfloor h/2 \rfloor$. □

Remark 9.21. In the last proof the reference to Example 9.15 can be replaced by [NV1, Example 3.3] if we use that $f_D = n_D$.

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References

[Be] P. Berthelot, Théorie de Dieudonné sur un anneau de valuation parfait, *Ann. Sci. École Norm. Sup.* (4) **13** (1980), 225–268

[dJO] J. de Jong and F. Oort, Purity of the stratification by Newton polygons, *J. Amer. Math. Soc.* **13** (2000), no. 1, 209–241

[De] M. Demazure, *Lectures on p-divisible groups*, Lecture Notes in Math., Vol. **302**, Springer-Verlag, Berlin-New York, 1972

[GV] O. Gabber and A. Vasiu, Dimensions of group schemes of automorphisms of truncated Barsotti–Tate groups, to appear in *Int. Math. Res. Notices*, 49 pages published online, http://imrn.oxfordjournals.org/content/early/2012/07/26/imrn.rns165.abstract?keytype=ref&ijkey=gVkQc5Otk95hU2C

[Ill] L. Illusie, Déformations de groupes de Barsotti–Tate (d'après A. Grothendieck), in *Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell* (Paris, 1983/84), *Astérisque*, No. **127** (1985), 151–198

[Ka] N. Katz, Slope filtration of F-crystals, *Journées de Géométrie Algébrique de Rennes* (Rennes, 1978), Vol. I, *Astérisque*, No. **63** (1979), 113–163

[La] E. Lau, Displays and formal p-divisible groups, *Invent. Math.* **171** (2008), no. 3, 617–628

[Ma] Y. I. Manin, The theory of formal commutative groups in finite characteristic, *Russian Math. Surv.* **18** (1963), no. 6, 1–83

[NV1] M.-H. Nicole and A. Vasiu, Minimal truncations of supersingular p-divisible groups, *Indiana Univ. Math. J.* **56** (2007), no. 6, 2887–2897

[NV2] M.-H. Nicole and A. Vasiu, Traverso's isogeny conjecture for p-divisible groups, *Rend. Semin. Mat. Univ. Padova* **118** (2007), 73–83

[NVW] M.-H. Nicole, A. Vasiu, and T. Wedhorn, Purity of level m stratifications, *Ann. Sci. École Norm. Sup.* (4) **43** (2010), no. 6, 925–955

[Oo1] F. Oort, Newton polygon strata in the moduli of abelian varieties, *Moduli of Abelian Varieties* (Texel Island, 1999), 417–440, Progr. Math., **195**, Birkhäuser, Basel, 2001

[Oo2] F. Oort, Foliations in moduli spaces of abelian varieties, *J. Amer. Math. Soc.* **17** (2004), no. 2, 267–296

[Oo3] F. Oort, Minimal p-divisible groups, *Ann. of Math.* (2) **161** (2005), no. 2, 1021–1036

[Oo4] F. Oort, Simple p-kernels of p-divisible groups, *Adv. Math.* **198** (2005), no. 1, 275–310

[OZ] F. Oort and Th. Zink, Families of p-divisible groups with constant Newton polygon, *Doc. Math.* **7** (2002), 183–201

[RZ] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, Vol. **141**, Princeton University Press, Princeton, NJ, 1996

[Tr1] C. Traverso, Sulla classificazione dei gruppi analitici commutativi di caratteristica positiva, *Ann. Scuola Norm. Sup. Pisa* (3) **23** (1969), 481–507

[Tr2] C. Traverso, p-divisible groups over fields, *Symposia Mathematica*, Vol. **XI** (Convegno di Algebra Commutativa, INDAM, Rome, 1971), 45–65, Academic Press, London, 1973

[Tr3] C. Traverso, Specializations of Barsotti–Tate groups, *Symposia Mathematica*, Vol. **XXIV** (Sympos., INDAM, Rome, 1979), 1–21, Academic Press, London-New York, 1981

[Va1] A. Vasiu, Crystalline boundedness principle, *Ann. Sci. École Norm. Sup.* **39** (2006), no. 2, 245–300

[Va2] A. Vasiu, Level m stratifications of versal deformations of p-divisible groups, *J. Algebraic Geom.* **17** (2008), no. 4, 599–641

[Va3] A. Vasiu, Reconstructing p-divisible groups from their truncations of small level, *Comment. Math. Helv.* **85** (2010), no. 1, 165–202

[Yu] C.-F. Yu, On finiteness of endomorphisms of abelian varieties, *Math. Res. Lett.* **17** (2010), no. 2, 357–370

[Zi1] Th. Zink, The display of a formal p-divisible group, *Cohomologies p-adiques et applications arithmétiques*, Vol. I, *Astérisque*, No. **278** (2002), 127–248

[Zi2] Th. Zink, de Jong-Oort purity for p-divisible groups, *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin*, Vol. II, 693–701, Progr. Math., **270**, Birkhäuser Boston, Inc., Boston, MA, 2009

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