August 26, 2011

Dear Kisin,

I liked your argument (prop. 1.3.2) in JAMS **23** 4, p.976 to get, over a valuation ring, a closed flat $G \subset \mathrm{GL}(M)$ to be the scheme theoretic fixer of a collection of "tensors". In your definition of "tensor", you allow symmetric and exterior powers. This is not necessary.

a) $\overset{d}{\wedge} \subset \overset{d}{\otimes}$ (over any ring, and locally a direct factor): $\overset{d}{\wedge}$ is the image of the antisymmetrization map from $\overset{d}{\otimes}$ to $\overset{d}{\otimes}$. You can hence dispense with $\overset{d}{\wedge}$.

b) You need the stabilizer some N direct factor in a Sym^d (of a sum of copies of M) [I will address your $\det(M)^{-n}$ later]. This is the same as stabilizing $N^\perp$ in $(\mathrm{Sym}^d)^\vee$. Now, $(\mathrm{Sym}^d(X))^\vee = \Gamma^d(X^\vee)$, and in the same way that over any ring Sym^d is a quotient of $\overset{d}{\otimes}$, Γ^d is a submodule of $\overset{d}{\otimes}$, locally direct factor (it is the symmetric tensors). So

$$\det(N^\perp) \text{ is a line in } \overset{\dim}{\wedge}\, \Gamma^d(\cdots) \subset \overset{\dim}{\otimes}\, \overset{d}{\otimes}(\cdots).$$

Note that if $\bar{G}$ is the monoid schematic closure of $G \subset \mathrm{GL}(M)$ in $\mathrm{End}(M)$, G is the stabilizer of the ideal of $\bar{G}$ in $\mathrm{End}(M)$. Yo can just use $\mathrm{Sym}^*(M \otimes M_0^\vee)$ instead of $\mathcal{O}_{\mathrm{GL}}$.

Your assumption "reductive general fiber" is rather strong. Here is a trick, I learned while reading Vasiu, which can replace it in useful cases.

1) If $L \subset W$ is a line (over a base S; W locally free and L locally direct factor), the stabilizer of L in $\mathrm{GL}(W)$ equals the stabilizer of $L^{\otimes r} \subset W^{\otimes r}$, for any $r > 0$.

I was at first afraid of $p \mid r$, but the fear disappears for the more general statement:

"for $L_i \subset W_i$, the stabilizer, in $\Pi\, \mathrm{GL}(W_i)$, of the L_i, equals the stabilizer of $\otimes L_i$ in $\otimes W_i$".

2) If we are in a tannakian situation where all 1-dim objects are tensor powers of a fixed L_0, and if we already have tensor powers of L_0 as well as L_0^{-1} inside some $\overset{a}{\otimes} M \overset{b}{\otimes} M^\vee$ (such as $\det(M)$, $\det(M^\vee)$), one can apply 1).

Best,

P. Deligne

P.S.: Can you make the result explicit in the following example, over $\mathbb{Z}_p$: $G \subset \mathrm{GL}(4, \mathbb{Z}_p)$ the schematic closure of the symplectic group, $\subset \mathrm{GL}(4, \mathbb{Q}_p)$, fixing $e_1{\wedge}e_2 + pe_3{\wedge}e_4$?

CC: A. Vasiu