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On two theorems for flat, affine group schemes over a discrete valuation ring

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Abstract: We include short and elementary proofs of two theorems that characterize reductive group schemes over a discrete valuation ring, in a slightly more general context.

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1 Introduction

Let k be a field. Let $p \in \{0\} \cup \{n \in \mathbb{N} | n \text{ is a prime}\}$ be the characteristic of k. Let V be a discrete valuation ring of residue field k. Let π be a uniformizer of V and let $K := V[\frac{1}{\pi}]$ be the field of fractions of V. Let $F = \mathrm{Spec}(P)$ and $G = \mathrm{Spec}(R)$ be flat, affine group schemes over V. We will assume that F is a reductive group scheme over V; so F is smooth over V and its fibres are connected and reductive groups over fields. In this paper we present elementary and short proofs of the following two basic theorems on reductive group schemes over V.

Theorem 1.1. *Let $f : F \to G$ be a homomorphism such that its generic fibre $f_K : F_K \to G_K$ is a closed embedding. Then the following four properties hold:*

(a) Suppose f_K is an isomorphism and G is of finite type. Then F is the smoothening of G in the sense of [1, Thm. 5, p. 175] (the definition is recalled in 2.3.2).

(b) The kernel Ker$(f_k : F_k \to G_k)$ is a unipotent, connected group of dimension 0.

(c) The homomorphism f is finite.

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(d) If $p = 2$, then we assume that $F_{\bar{K}}$ has no normal subgroup that is an SO_{2n+1} group for some $n \in \mathbb{N}$. Then f itself is a closed embedding.

Theorem 1.2. *Suppose G_K is a connected, smooth group over K and the identity component $(G_k)^0_{red}$ of the reduced group $(G_k)_{red}$ is a reductive group over k of the same dimension as G_K. Then the following three properties hold:*

(a) The group scheme G is of finite type over V, has a connected special fibre G_k, and its generic fibre G_K is a reductive group.

(b) If $p = 2$, we assume that the group $G_{\bar{K}}$ has no normal subgroup that is isomorphic to SO_{2n+1} for some $n \in \mathbb{N}$. Then G is a reductive group scheme over V.

(c) The normalization G^n of G is a finite G-scheme. Moreover, there is a faithfully flat V-algebra $\tilde{V}$ that is a discrete valuation ring and such that the normalization $(G_{\tilde{V}})^n$ of $G_{\tilde{V}}$ is a reductive group scheme over $\tilde{V}$.

If V is of mixed characteristic $(0, p)$, then 1.1 (a) and (b) were first proved in [13, proof of 3.1.2.1 c)]. For the sake of completeness, in §3 we recall the proofs of 1.1 (a) and (b) in a slightly enlarged manner that recalls elementary material. The passage from 1.1 (a) and (b) to 1.1 (d) (resp. 1.1 (c)) is a direct consequence of the classification of the ideals of the Lie algebras of adjoint and simply connected semisimple groups over $\bar{k}$ (resp. is only a variant of Zariski Main Theorem). In [13, 3.1.2.1 c)] we overlooked the phenomenon of exceptional nilpotent such ideals and so the extra hypothesis of 1.1 (d) for $p = 2$ does not show up in [13, 3.1.2.1 c)]. Theorem 1.1 (resp. Theorem 1.2) is also proved in [12], under the extra assumption that G_K is of finite type over K (resp. that G_k is of finite type over k). The methods of [12] are essentially the same as of [13] except that the proofs of [12] contain many unneeded parts. Here we follow [13] to get short and efficient proofs of 1.1 and 1.2. The importance of 1.1 (resp. 1.2) stems from its fundamental applications to integral canonical models of Shimura varieties of Hodge (resp. to dual groups), cf. [13] (resp. cf. [12]). We also mention that 1.1 (c) and (d) are powerful tools in extending results on semisimple groups over a field, from characteristic 0 to arbitrary positive characteristic. As an exemplification, in §5 we include such an application that pertains to adjoint groups.

The proofs of 1.1 and 1.2 are carried on in §3 and §4 (respectively). Few notations and preliminaries needed in §3 and §4 are gathered in §2. We would like to thank U of Arizona for good conditions for the writing of this work. We would like to thank G. Prasad for pointing out that [13, 3.1.2.1 c)] omits to add the extra hypothesis of 1.1 (d) for $p = 2$.

2 Preliminaries

In 2.1 we list notations. Elementary properties of Lie algebras, dilatations, representation theory, and quasi-sections are recalled in 2.2 to 2.5. In 2.6 we present in an accessible way a result of the classical Bruhat–Tits theory for reductive groups over K.

2.1 Notations

Let k, V, K, π, $F = \mathrm{Spec}(P)$, and $G = \mathrm{Spec}(R)$ be as in §1. Let V^{sh} be the strict henselization of V. Let $\mathcal{W}$ be the set of finite, discrete valuation ring extensions of the completion of V^{sh}. If H is a reductive group scheme over an affine scheme $\mathrm{Spec}(A)$, let $Z(H)$, H^{der}, H^{ad}, and H^{ab} denote the center, the derived group, the adjoint group, and respectively the abelianization of H. So $Z^{\mathrm{ab}}(H) = H/H^{\mathrm{der}}$ and $H^{\mathrm{ad}} = H/Z(H)$. Let H^{sc} be the simply connected semisimple group cover of H^{der}. For a free A-module N of finite rank, let $GL(N)$ be the group scheme over $\mathrm{Spec}(A)$ of linear automorphisms of N.

2.2 Lie algebras

Let $\mathrm{Lie}(H)$ be the Lie algebra of an affine group H of finite type over k or over V. We view $\mathrm{Lie}(H)$ as a (left) H-module via the adjoint representation. Until 2.3 we assume that $k = \bar{k}$ and that H is over k. For $x \in \mathrm{Lie}(H)$ we have a unique and functorial Jordan decomposition $x = x_s + x_n$ such that for any monomorphism $H \hookrightarrow GL_m$ ($m \in \mathbb{N}$), $x_s \in \mathrm{Lie}(GL_m)$ is semisimple, $x_n \in \mathrm{Lie}(GL_m)$ is nilpotent, and x_s and x_n are polynomials in $x \in \mathrm{Lie}(GL_m)$ (cf. [2, Ch. 1, §4]). We say x is nilpotent (resp. semisimple) if $x = x_n$ (resp. $x = x_s$). So if H is $\mathbb{G}_a$ (resp. $\mathbb{G}_m$), then x is nilpotent (resp. semisimple).

2.2.1 Lemma 2.1

Lemma 2.1. *Let H be a reductive group over $k = \bar{k}$. Let $\mathfrak{n}$ be a non-zero ideal of Lie(H) that is formed by nilpotent elements and is a simple H-module. Then $p = 2$ and H has a normal subgroup H_0 isomorphic to SO_{2n+1} ($n \in \mathbb{N}$) and such that $\mathfrak{n} \subset$ Lie(H_0).*

Proof. The image of $\mathfrak{n}$ in $\mathrm{Lie}(H^{\mathrm{ab}})$ is trivial. So $\mathfrak{n} \subset \mathrm{Lie}(H^{\mathrm{der}})$. So we can assume that $H = H^{\mathrm{der}}$ is semisimple. Let $\mathfrak{n}^{\mathrm{ad}} := \mathrm{Im}(\mathfrak{n} \to \mathrm{Lie}(H^{\mathrm{ad}}))$. As $\mathrm{Lie}(Z(H))$ is formed by semisimple elements, the Lie homomorphism $\mathfrak{n} \to \mathfrak{n}^{\mathrm{ad}}$ is an isomorphism. Thus $\mathfrak{n}^{\mathrm{ad}}$ is a non-zero ideal of $\mathrm{Lie}(H^{\mathrm{ad}})$ that is a simple H^{ad}-module. Let $H^{\mathrm{ad}} = \prod_{i \in I} H_i$ be the product decomposition into simple groups. As H^{ad} is adjoint, there is no element of $\mathrm{Lie}(H^{\mathrm{ad}})$ fixed by H^{ad}. So as $\mathfrak{n}$ is a simple H^{ad}-module, we get that there is $i_0 \in I$ such that $\mathfrak{n}^{\mathrm{ad}} \subset \mathrm{Lie}(H_{i_0})$. Thus we can assume H^{ad} is a simple group. Let T be a maximal torus of H^{ad} and let $\mathcal{L}$ be the Lie type of H^{ad}. As $\mathrm{Lie}(H^{\mathrm{ad}})$ has non-zero semisimple elements, we have $\mathfrak{n}^{\mathrm{ad}} \neq \mathrm{Lie}(H^{\mathrm{ad}})$. So as $\mathfrak{n}^{\mathrm{ad}}$ is a simple H^{ad}-module, from [11, Prop. 1.11] we get that either $\mathfrak{n}^{\mathrm{ad}} = \mathrm{Im}(\mathrm{Lie}(H^{\mathrm{sc}}) \to \mathrm{Lie}(H^{\mathrm{ad}}))$ or $(p, \mathcal{L}) \in \{(2, F_4), (3, G_2), (2, B_n), (2, C_n) | n \in \mathbb{N}\}$.

If $\mathfrak{n}^{\mathrm{ad}} = \mathrm{Im}(\mathrm{Lie}(H^{\mathrm{sc}}) \to \mathrm{Lie}(H^{\mathrm{ad}}))$, then $\mathfrak{n}^{\mathrm{ad}}$ has non-zero semisimple elements except when $\dim_k(\mathrm{Lie}(Z(H^{\mathrm{sc}})))$ is the rank of $\mathcal{L}$, i.e. except when $(p, \mathcal{L}) = (2, A_1)$ (cf. loc. cit.).

If $(p, \mathcal{L})$ is $(2, F_4)$ (resp. $(3, G_2)$), then $\mathfrak{n}^{\mathrm{ad}}$ is the unique proper ideal of $\mathrm{Lie}(H^{\mathrm{ad}})$ and so it is generated by the direct sum $\mathfrak{s}$ of the eigenspaces of the adjoint action of T on $\mathrm{Lie}(H^{\mathrm{ad}})$ that correspond to short roots. Thus $\dim_k(\mathfrak{n}^{\mathrm{ad}})$ is 26 (resp. 7) and $\mathfrak{n}^{\mathrm{ad}} \subset \mathfrak{s} \oplus \mathrm{Lie}(T)$, cf. [7, pp. 408–409] applied to a semisimple group over $\mathbb{F}_p$ whose extension to k is H^{sc}. But $\mathcal{L}$ has 24 (resp. 6) short roots, cf. [3, PLATES VIII and IX]. So $\dim_k(\mathrm{Lie}(T) \cap \mathfrak{n}^{\mathrm{ad}})$

is $2 = 26 - 24$ (resp. $1 = 7 - 6$) and so $\mathfrak{n}^{\mathrm{ad}}$ has non-zero semisimple elements. If $(p, \mathcal{L}) = (2, C_n)$ with $n \geq 3$, then we similarly argue that $\mathfrak{n}^{\mathrm{ad}}$ is the ideal associated to short roots and that $\dim_k(\mathrm{Lie}(T) \cap \mathfrak{n}^{\mathrm{ad}}) = 2n^2 - n - 1 - \varepsilon - 2(n^2 - n) = n - 1 - \varepsilon > 1$, where ε is 1 if n is even and is 0 if n is odd (cf. [7, p. 409] and [3, PLATE III]).

Thus we have $(p, \mathcal{L}) = (2, B_n)$, with $n \in \mathbb{N}$. As $\mathfrak{n}^{\mathrm{ad}}$ is a simple H^{ad}-submodule of $\mathrm{Lie}(H^{\mathrm{ad}})$, from [11, Prop. 1.11] we get that that $\mathfrak{n}^{\mathrm{ad}}$ is the direct sum of the eigenspaces of the adjoint action of T on $\mathrm{Lie}(H^{\mathrm{ad}})$ that correspond to short roots, that $\dim_k(\mathfrak{n}^{\mathrm{ad}}) = 2n$, and that $\mathfrak{n}^{\mathrm{ad}} \subset \mathrm{Im}(\mathrm{Lie}(H^{\mathrm{sc}}) \to \mathrm{Lie}(H^{\mathrm{ad}}))$. Let $\mathfrak{n}^{\mathrm{sc}}$ be the inverse image of $\mathfrak{n}$ in $\mathrm{Lie}(H^{\mathrm{sc}})$. We have $\dim_k(\mathfrak{n}^{\mathrm{sc}}) = \dim_k(\mathfrak{n}) + \dim_k(\mathrm{Lie}(\mathrm{Ker}(H^{\mathrm{sc}} \to H))) = 2n + \dim_k(\mathrm{Lie}(\mathrm{Ker}(H^{\mathrm{sc}} \to H)))$. But as $\mathfrak{n}^{\mathrm{sc}}$ is an H^{sc}-module, we have $\mathrm{Lie}(Z(H^{\mathrm{sc}})) \subset \mathfrak{n}^{\mathrm{sc}}$ (cf. [11, Prop. 1.11]). Thus $\dim_k(\mathfrak{n}^{\mathrm{sc}}) = 2n + 1$. Thus $\dim_k(\mathrm{Lie}(\mathrm{Ker}(H^{\mathrm{sc}} \to H))) = 2n + 1 - 2n = 1$ and so $\mathrm{Ker}(H^{\mathrm{sc}} \to H) = Z(H^{\mathrm{sc}}) \tilde{\to} \mu_2$. Thus H is an adjoint group and so isomorphic to SO_{2n+1}. □

2.3 Dilatations

In this section we assume G_K is reduced. Let S be a reduced subgroup of G_k. Let J be the ideal of R that defines S and let I_R be the ideal of R that defines the identity section of G. Let R_1 be the R-subalgebra of $R[\frac{1}{\pi}]$ generated by $\frac{x}{\pi}$, where $x \in J$. By the dilatation of G centered on S one means the affine scheme $G_1 := \mathrm{Spec}(R_1)$; it has a canonical structure of a flat, affine group scheme over V and the morphism $G_1 \to G$ is a homomorphism whose special fibre factors through the closed embedding $S \hookrightarrow G_k$ (cf. [1, Prop. 1 and 2, pp. 63–64]). In §3 we will need the following elementary Lemma.

2.3.1 Lemma 2.2

Lemma 2.2. *Suppose G is a closed subgroup of a smooth group scheme H over V of relative dimension l. Then Ker$(G_{1k} \to G_k)$ is isomorphic to a subgroup of $\mathbb{G}_a^{l-\dim(S)}$. Moreover, if $G = H$, then Ker$(G_{1k} \to G_k) \tilde{\to} \mathbb{G}_a^{l-\dim(S)}$.*

Proof. We can assume V is complete. As G_1 is a closed subgroup of the dilatation of H centered on S (cf. [1, Prop. 2 (c) and (d), p. 64]), we can also assume that $G = H$. Let $\hat{R}$ and $\hat{I}_R$ be the completions of R and I_R with respect to the I_R-topology. Let $s := \dim(S)$. As G is smooth over V, we can write $\hat{R} = V \oplus \hat{I}_R = V[[x_1, ..., x_{l-s}, y_1, ..., y_s]]$, where x_1, ..., x_{l-s}, y_1, ..., $y_s \in \hat{I}_R$ are such that the ideal $(x_1, ..., x_{l-s}, \pi)$ of $\hat{R}$ defines the completion of S along its identity section. We have $R_1 \otimes_R \hat{R} = V[[x_1, ..., x_{l-s}, y_1, ..., y_s]][\frac{x_1}{\pi}, ..., \frac{x_{l-s}}{\pi}]$. Let $\delta : \hat{R} \to \widehat{\hat{R} \otimes_V \hat{R}}$ be the comultiplication map of the formal Lie group of G. As S is a subgroup of G_k, for $i \in \{1, ..., l-s\}$ we have $\delta(x_i) = x_i \otimes 1 + 1 \otimes x_i + \sum_{j \in I_i^a}(a_{ij} \otimes b_{ij}) + \sum_{j \in I_i^b}(a_{ij} \otimes b_{ij})$, where I_i^a and I_i^b are finite sets, where each a_{ij} and b_{ij} belong to $\hat{I}_R$, and where for each $j \in I_i^a$ (resp. $j \in I_i^b$) the element $a_{ij} \in \hat{I}_R$ (resp. $b_{ij} \in \hat{I}_R$) is divisible by either some x_u or by some πy_v; here $u \in \{1, ..., l-s\}$ and $v \in \{1, ..., s\}$.

We have $\mathrm{Ker}(G_{1k} \to G_k) = \mathrm{Spec}(A_1)$, where $A_1 := R_1 \otimes_R \hat{R}/(x_1, ..., x_{l-s}, y_1, ..., y_s)$.

Let $\bar{x}_i$ be the image of $\frac{x_i}{\pi}$ in A_1. So A_1 is a k-algebra generated by $\bar{x}_1$, ..., $\bar{x}_{l-s}$. Taking the identity $\delta(\frac{x_i}{\pi}) = \frac{x_i}{\pi} \otimes 1 + 1 \otimes \frac{x_i}{\pi} + \sum_{j \in I_i^a} \frac{a_{ij}}{\pi} \otimes b_{ij} + \sum_{j \in I_i^b} a_{ij} \otimes \frac{b_{ij}}{\pi}$ modulo the ideal $(x_1, ..., x_{l-s}, y_1, ..., y_s)$ of $R_1 \otimes_R \hat{R}$, we get that the comultiplication map $\delta_1 : A_1 \to A_1 \otimes_k A_1$ of the group $\mathrm{Ker}(G_{1k} \to G_k)$ is such that $\delta_1(\bar{x}_i) = \bar{x}_i \otimes 1 + 1 \otimes \bar{x}_i$ for any $i \in \{1, ..., l-s\}$. As G_1 is smooth over V (cf. [1, Prop. 3, p. 64]) of relative dimension l, $\mathrm{Ker}(G_{1k} \to G_k) = \mathrm{Ker}(G_{1k} \to S)$ has dimension at least $l - s$. So as A_1 is k-generated by $\bar{x}_1$, ..., $\bar{x}_{l-s}$ and its dimension is at least $l - s$, we get that $A_1 = k[\bar{x}_1, ..., \bar{x}_{l-s}]$ is a polynomial k-algebra. From the description of δ_1 we get that $\mathrm{Ker}(G_{1k} \to G_k)$ is isomorphic to $\mathbb{G}_a^{l-s}$. □

2.3.2 Smoothening

We assume that G_K is smooth over K and that G is of finite type over V. We take S to be the Zariski closure in G_k of the special fibres of all morphisms $\mathrm{Spec}(V^{\mathrm{sh}}) \to G$ of V-schemes. We refer to $G_1 \to G$ as the canonical dilatation of G; it is a morphism of finite type. There is $m \in \mathbb{N}$ and a finite sequence of canonical dilatations $G' := G_m \to G_{m-1} \to ... \to G_1 \to G_0 := G$ such that G' is uniquely determined by the following two properties (cf. [1, pp. 174–175]):

(i) the affine group scheme G' is smooth and of finite type over V;

(ii) if Y is a smooth V-scheme and if $Y \to G$ is a morphism of V-schemes, then $Y \to G$ factors uniquely through the homomorphism $G' \to G$.

From very definitions, we get that $G'(V) = G(V)$ and that the smoothening $(G_{V^{\mathrm{sh}}})'$ of $G_{V^{\mathrm{sh}}}$ is $G'_{V^{\mathrm{sh}}}$. We also point out that the V-schemes G_1, ..., $G_m = G'$ are of finite type.

2.4 Representations

We denote also by $\delta : R \to R \otimes_V R$ the comultiplication map of G. Let L_0 be a finite subset of I_R. For $l_0 \in L_0$ we write $\delta(l_0) = \sum_{j \in I(l_0)} a_{0j} \otimes l_{0j}$, where $I(l_0)$ is a finite set and a_{0j}, $l_{0j} \in R$. Let L be the V-submodule of R generated by 1, by l_0's, and by l_{0j}'s. It is known that we have $\delta(L) \subset R \otimes_V L$, cf. [9, 2.13]. So L is a G-module and thus we have a homomorphism $\rho(L) : G \to GL(L)$ between flat, affine group schemes over V. Let $\mathcal{B}$ be a V-basis of L contained in $\{1\} \cup I_R$. For $l \in \mathcal{B} \cap I_R$, we write $\delta(l) = \sum_{l' \in \mathcal{B}} a_{ll'} \otimes l'$, where each $a_{ll'} \in R$. As $\delta(l) - l \otimes 1 + 1 \otimes l \in I_R \otimes_V I_R$, we have $a_{1l} = l$. But if $GL(L) = \mathrm{Spec}(A(L))$ and if $q(L) : A(L) \to R$ is the V-homomorphism that defines $\rho(L)$, then $R(L) := \mathrm{Im}(q(L))$ is the V-algebra generated by the $a_{ll'}$'s. As a conclusion we have:

(i) The V-subalgebra of R generated by L is contained in $R(L)$. So if G (resp. if G_K) is of finite type over V (resp. over K) and if L_0 generates the V-algebra R (resp. the K-algebra $R[\frac{1}{\pi}]$), then $\rho(L)$ (resp. $\rho(L)_K$) is a closed embedding homomorphism.

2.5 Quasi-sections

Let X be a reduced, flat V-scheme of finite type. Let $y \in X(k)$. From [6, Cor. (17.16.2)] we get the existence of a finite field extension $\tilde{K}$ of K such that there is a faithfully flat, local V-subalgebra of $\tilde{K}$ that is of finite type and we have a morphism $z : \mathrm{Spec}(\tilde{V}) \to X$ whose image contains y.

2.5.1 Lemma 2.3

Lemma 2.3. *(a) If V is complete, then we can assume $\tilde{V}$ is a discrete valuation ring.*

(b) Let $a : Y \to X$ be a morphism between reduced, flat V-schemes of finite type. Suppose that V is complete, that $k = \bar{k}$, and that for any $W \in \mathcal{W}$ the map $a(W) : Y(W) \to X(W)$ is onto. Then the map $a(k) : Y(k) \to X(k)$ is surjective.

Proof. As V is complete, it is also a Nagata ring (cf. [10, (31.C), Cor. 2]) and thus the normalization $V^{\mathrm{n}}_{\tilde{K}}$ of V in $\tilde{K}$ is a finite V-algebra. So as $\tilde{V}$ we can take any local ring of the normalization of $\tilde{V}$ in $\tilde{K}$ (it is a local ring of $V^{\mathrm{n}}_{\tilde{K}}$) that is a discrete valuation ring. From this (a) follows. By taking W to be $\tilde{V}$, we get that $y \in \mathrm{Im}(a(k))$. So as y was arbitrary, we get that (b) holds. □

2.6 Lemma 2.4

Lemma 2.4. *Suppose V is complete and $k = \bar{k}$. Let $f : F \to G$ be a homomorphism such that f_K is an isomorphism. Then $f(V) : F(V) \to G(V)$ is an isomorphism. If moreover G is smooth, then f is an isomorphism.*

Proof. Let T be a maximal split torus of F, cf. [5, Vol. III, Exp. XIX, 6.1]. We show that the assumption that $F(V) \lneqq G(V)$ leads to a contradiction. We have $F(K) = F(V)T(K)F(V)$, cf. Cartan decomposition of [4, 4.4.3]. So as $F(V) \lneqq G(V)$, there is $g \in G(V) \cap (T(K) \setminus T(V))$. We write $T = \mathbb{G}_m^s = \mathrm{Spec}(V[w_1, ..., w_s][\frac{1}{w_1 \dots w_s}])$. Let $w \in P$ be such that under the V-homomorphisms $P \twoheadrightarrow V[w_1, ..., w_s][\frac{1}{w_1 \dots w_s}] \to K$ that define $g \in T(K) \leqslant F(K)$, it is mapped into an element of the set $\{w_i, w_i^{-1} | i \in \{1, ..., s\}\}$ that maps into $K \setminus V$. As f_K is an isomorphism we can identify $P[\frac{1}{\pi}] = R[\frac{1}{\pi}]$. Let $n \in \mathbb{N}$ be such that $\pi^n w \in R$. Under the V-homomorphism $R \to K$ that defines g^{n+1}, the element $\pi^n w$ maps into an element of $K \setminus V$. Thus $g^{n+1} \notin G(V)$. Contradiction. So $F(V) = G(V)$.

Let now G be smooth. We show that the assumption that f is not an isomorphism leads to a contradiction. We can assume $k = \bar{k}$. As $R \neq P$, there is $w_0 \in P \setminus R$ such that $x := \pi w_0 \in R \setminus \pi R$. Let $\bar{g} : R \to k$ be a k-homomorphism such that $\bar{g}(x) \neq 0$. Let $g : R \to V$ be a V-homomorphism that lifts $\bar{g}$ (as V is henselian and G is smooth). The K-homomorphism $g[\frac{1}{\pi}] : R[\frac{1}{\pi}] \to K$ maps w_0 into an element of $K \setminus V$. So g defines an element of $G(V) \setminus F(V)$. So $F(V) \lneqq G(V)$. Contradiction. Thus f is an isomorphism.□

3 Proof of Theorem 1.1

In this chapter we prove 1.1. To prove 1.1 we can assume that V is complete, that $k = \bar{k}$, and that f_K is an isomorphism. So $f : F \to G$ is defined by a V-monomorphism

$P \hookrightarrow R$ that induces a K-isomorphism $P[\frac{1}{\pi}] \xrightarrow{\sim} R[\frac{1}{\pi}]$ to be viewed as an identity. Let $\rho(L) : G \to GL(L)$ be as in 2.4, with $L_0 \in I_R \subset R$ such that it generates the K-algebra $P[\frac{1}{\pi}] = R[\frac{1}{\pi}]$. The generic fibre of $\rho(L)$ is a closed embedding, cf. 2.4 (i) and the choice of L_0. To prove 1.1 for f, it suffices to prove 1.1 for $\rho(L) \circ f$. So we can assume $\rho(L)$ is a closed embedding; so G is a reduced, flat, closed subgroup of $GL(L)$ and so of finite type over V.

3.1 Proofs of 1.1 (a) and (b)

Let $G' = G_m \to ... \to G_1 \to G_0 = G$ be as in 2.3.2. As F is smooth over V and due to 2.3.2 (ii), $f : F \to G$ factors through a homomorphism $f' : F \to G'$. As G' is smooth, f' is an isomorphism (cf. 2.6). So 1.1 (a) holds.

For $i \in \{0, ..., m-1\}$, G_i is a reduced, flat group scheme of finite type (cf. end of 2.3.2) and so a closed subgroup of some general linear group H_i (cf. 2.4 (i)). So each group $\mathrm{Ker}(G_{i+1k} \to G_{ik})$ is a subgroup of a product of a finite number of copies of $\mathbb{G}_a$, cf. 2.3.1. As f' is an isomorphism, $\mathrm{Ker}(f_k) = \mathrm{Ker}(F_k \to G_k)$ has a composition series whose factors are subgroups of $\mathrm{Ker}(G_{i+1k} \to G_{ik})$ ($i \in \{0, ..., m-1\}$). Thus $\mathrm{Ker}(f_k)$ is a unipotent group in the sense of [5, Vol. II, Exp. XVII, 1.1]. As $\mathrm{Ker}(f_k) \lhd F_k$ and as F_k has a trivial unipotent radical (being reductive), $\mathrm{Ker}(f_k)$ has dimension 0. But F_k is connected and so its action on $(\mathrm{Ker}(f_k))_{\mathrm{red}}$ via inner conjugation is trivial. So $(\mathrm{Ker}(f_k))_{\mathrm{red}} \leqslant Z(F_k)$.

Let $\bar{g} \in (\mathrm{Ker}(f_k))(k)$. By induction on $i \in \{0, ..., m\}$ we show that $\bar{g} \in \mathrm{Ker}(F(k) \to G_i(k))$. The case $i = 0$ is obvious as $G_0 = G$. For $i \in \{0, ..., m-1\}$ the passage from i to $i+1$ goes as follows. Let S_i be the reduced subgroup of G_{ik} such that G_{i+1} is the dilatation of G_i centered on S_i. Let $\tilde{H}_{i+1}$ be the dilatation of H_i centered on S_i; we have a closed embedding homomorphism $G_{i+1} \hookrightarrow \tilde{H}_{i+1}$ (cf. [1, Prop. 2 (c) and (d), p. 64]). We consider a closed embedding homomorphism $\tilde{H}_{i+1} \hookrightarrow GL_{n_i}$ (with $n_i \in \mathbb{N}$, cf. 2.4 (i)). We have homomorphisms $F \to G_{i+1} \hookrightarrow \tilde{H}_{i+1} \hookrightarrow GL_{n_i}$. Let $\bar{h} \in GL_{n_i}(k)$ be the image of $\bar{g}$ via $F \to GL_{n_i}$. As $\bar{g} \in Z(F_k)(k)$, $\bar{h}$ is a semisimple element. As $\mathrm{Ker}(H_{i+1k} \to H_{ik})$ is a product of $\mathbb{G}_a$ groups (cf. 2.3.1) and as $\bar{g} \in \mathrm{Ker}(F(k) \to H_i(k))$, $\bar{h}$ is a unipotent element. So $\bar{h}$ is the identity element. So $\bar{g} \in \mathrm{Ker}(F(k) \to G_{i+1}(k))$. This ends the induction.

As $f' : F \to G' = G_m$ is an isomorphism, we get that $\bar{g}$ is the identity element of $F(k)$. Thus $(\mathrm{Ker}(f_k))_{\mathrm{red}}$ is a trivial group. So $\mathrm{Ker}(f_k)$ is also connected. So 1.1 (b) holds.

3.2 Proof of 1.1 (c)

As V is a Nagata ring, R is also a Nagata ring (cf. [10, (31.H)]). So the normalization $G^{\mathrm{n}} = \mathrm{Spec}(R^{\mathrm{n}})$ of G is a finite G-scheme. The homomorphism $F \to G$ factors through a morphism $F \to G^{\mathrm{n}}$. We have $F(V) = G(V) = G^{\mathrm{n}}(V)$, cf. 2.6; this also holds if V is replaced by a $W \in \mathcal{W}$. Thus $f(k) : F(k) \to G^{\mathrm{n}}(k)$ is an epimorphism, cf. 2.5.1 (b) applied to $F \to G^{\mathrm{n}}$. So G^{n}_k is connected and the morphism $F_k \to (G^{\mathrm{n}}_k)_{\mathrm{red}}$ is dominant. So both F and G^{n} have unique local rings O_1 and O_2 (respectively) that are faithfully flat V-algebras and discrete valuation rings. The natural V-homomorphism $O_2 \to O_1$ is dominant and becomes an isomorphism after inverting π. So we can identify $O_1 = O_2$. So as $F_K = G^{\mathrm{n}}_K$, the normal, noetherian, affine schemes F and G^{n} have the same set of local rings that are discrete valuation rings. Thus we have $R^{\mathrm{n}} = P$ (cf. [10, (17.H), Thm. 38]) and so $F \to G^{\mathrm{n}}$ is an isomorphism. So f is a finite morphism. So 1.1 (c) holds.

3.3 Proof of 1.1 (d)

As $\mathrm{Ker}(f_k)$ is unipotent, it is a subgroup of the unipotent radical of a Borel subgroup of some general linear group over k (cf. [5, Vol. II, Exp. XVII, 3.5]). So $\mathrm{Lie}(\mathrm{Ker}(f_k))$ is formed by nilpotent elements (cf. 2.2) and is normalized by F_k. The root data of F_k and $F_{\bar{K}}$ are isomorphic (cf. [5, Vol. III, Exp. XXII, 2.8]) and determine the isomorphism classes of F_k and $F_{\bar{K}}$ (cf. [5, Vol. III, Exp. XXIII, 5.1]). So the hypothesis of 1.1 (d) implies that either $p > 2$ or $p = 2$ and F_k has no normal subgroup isomorphic to SO_{2n+1} ($n \in \mathbb{N}$). So $\mathrm{Lie}(\mathrm{Ker}(f_k))$ has no non-trivial simple F_k-submodule, cf. 2.2.1. Thus $\mathrm{Lie}(\mathrm{Ker}(f_k)) = 0$. From this and the connectedness part of 1.1 (b), we get that $\mathrm{Ker}(f_k)$ is trivial. So f_k is a closed embedding. As f_k and f_K are closed embeddings, from Nakayama's lemma we get that the finite morphism $F \times_G \mathrm{Spec}(\mathcal{O}) \to \mathrm{Spec}(\mathcal{O})$ is a closed embedding for any local ring $\mathcal{O}$ of G. Thus f is a closed embedding and so an isomorphism (as f_K is so). So 1.1 (d) holds. This ends the proof of 1.1.

3.4 Remarks

(a) The reference in [14, proof of 4.1.2] to [13, 3.1.2.1 c)] does not always work for $p = 2$ (cf. 1.1 (d)). However, in [14, proof of 4.1.2] one can always choose $\rho_{B(k)}$ and L such that ρ is a closed embedding (cf. 2.4 (i)).

(b) We continue to assume that f_K is an isomorphism and that $k = \bar{k}$. As $\mathrm{Lie}(Z(F_k))$ is formed by semisimple elements, the connected, unipotent group $Z(F_k) \cap \mathrm{Ker}(f_k)$ is trivial. Suppose now that $p = 2$ and f is not an isomorphism. So f_k is not a closed embedding (see 3.3) and so $(f_k)_{\mathrm{red}} : F_k \to (G_k)_{\mathrm{red}}$ is not a closed embedding. We get that $F_k^{\mathrm{ad}} \to (G_k)_{\mathrm{red}}/f_k(Z(F_k))$ is an isogeny (cf. 1.1 (c)) that is a finite product $\prod_{j\in J} f_j : F_j \to G_j$ of isogenies f_j with F_j as a simple, adjoint group. Any f_j is a purely inseparable isogeny (cf. 1.1 (b)) and there is $j_0 \in J$ such that f_{j_0} is not an isomorphism. We have $\mathrm{Ker}(f_k) := \prod_{j\in J} \mathrm{Ker}(f_j)$ and so $\mathrm{Ker}(f_{j_0})$ is a non-trivial, unipotent subgroup of F_j whose Lie algebra is formed by nilpotent elements (see 3.3). From 2.2.1 and its proof we get that F_{j_0} is an SO_{2n+1} group and that $\mathrm{Lie}(\mathrm{Ker}(f_{j_0}))$ contains the unique simple F_{j_0}-submodule $\mathfrak{n}_{j_0}$ of $\mathrm{Lie}(F_{j_0})$ of dimension $2n$. The quotient of F_{j_0} by $\mathfrak{n}_{j_0}$ (see [2, §17]) is an Sp_{2n} group (cf. [2, 23.6, p. 261]). So f_{j_0} factors through a purely inseparable isogeny $g_{j_0} : Sp_{2n} \to G_{j_0}$ whose kernel is a unipotent group. So $\mathrm{Lie}(\mathrm{Ker}(g_{j_0}))$ is formed by nilpotent elements and so it is trivial (cf. 2.2.1). Thus g_{j_0} is an isomorphism. As a conclusion, the root data of F_k and $(G_k)_{\mathrm{red}}$ are not isomorphic and so F_k and $(G_k)_{\mathrm{red}}$ are not isomorphic.

4 Proof of Theorem 1.2

In this chapter we prove 1.2. To prove 1.2 we can assume that V is complete, that $k = \bar{k}$, and that tr.deg.$(k) \geq 1$.

4.1 The group Γ

As $k = \bar{k}$, $(G_k)^0_{\text{red}}$ is a split reductive group. Let T_1, ..., T_s be a finite number of $\mathbb{G}_m$ subgroups of $(G_k)^0_{\text{red}}$ that generate $(G_k)^0_{\text{red}}$. For $i \in \{1, ..., s\}$ let $y_i \in T_i(k)$ be an element of infinite order (here is the place where we need, in the case when $p \in \mathbb{N}$, that tr.deg.$(k) \geq 1$). So the Zariski closure in T_i of the subgroup of $T_i(k)$ generated by y_i, is T_i itself. Let Γ be the subgroup of $G(k)$ generated by y_1, ..., y_s. We conclude:

(i) The Zariski closure of Γ in G_k is $(G_k)^0_{\text{red}}$.

4.2 The finite type case

In this section we prove 1.2 under the assumption that G is of finite type over V. For $i \in \{1, ..., s\}$ let V_i be a finite V-algebra that is a discrete valuation ring and such that there is $w_i \in G(V_i)$ that lifts y_i (cf. 2.5.1 (a)). By replacing V with its normalization in the composite field of the fields $V_i[\frac{1}{\pi}]$ ($i \in \{1, ..., s\}$), we can assume that for $i \in \{1, ..., s\}$ there is $w_i \in G(V)$ that lifts y_i. Let G' be as in 2.3.2. As $w_1, ..., w_s \in G'(V) = G(V)$, from 4.1 (i) we get that the Zariski closure of $\mathrm{Im}(G'_k(k) \to G_k(k))$ in $(G_k)_{\text{red}}$ contains $(G_k)^0_{\text{red}}$. So $(G_k)^0_{\text{red}} \leqslant \mathrm{Im}(G'_k \to G_k)$. So if G'^0_k is the identity component of G'_k, then we have an isogeny $G'^0_k \to (G_k)^0_{\text{red}}$. From this and [2, 14.11] we get that the unipotent radical of G'^0_k is trivial. So G'^0_k is a reductive group, cf. [2, 11.21].

Let G'^0 be the open subgroup of G' formed by G'^0_k and by G_K. Let $(G'_j)_{j \in J}$ be a covering of G' by open affine subschemes such that each G'_{jk} is an open subscheme of either G'^0_k or $G'_k \setminus G'^0_k$. The product scheme $G'^0 \times_{G'} G'_j$ is: (i) G'_j if $G'_{jk} \hookrightarrow G'^0_k$, and (ii) G'_{jK} if $G'_{jk} \hookrightarrow G'_k \setminus G'^0_k$. So $G'^0 \times_{G'} G'_j$ is affine for any $j \in J$. Thus the morphism $G'^0 \to G'$ is affine and so G'^0 is affine. So G'^0 is a smooth, affine group scheme over V whose special fibre is a reductive group and whose generic fibre is connected. This implies that G'^0 is a reductive group scheme over V, cf. [5, Vol. III, Exp. XIX, 2.6 and 2.7]. So G_K is a reductive group. So 1.2 (a) holds.

Let $y \in G(k)$. From 2.6 applied to $G'^0 \to G'$, we get that $G'^0 = G'$ and that $G'^0(V) = G'(V) = G(V)$. So by replacing V with a $W \in \mathcal{W}$, we can assume there is $w \in G(V)$ that lifts y (cf. 2.5.1 (a)). So $y \in \mathrm{Im}(G'^0(k) \to G(k))$. So we have an epimorphism $G'^0_k \twoheadrightarrow (G_k)_{\text{red}}$ and so G_k is connected. So 1.2 (a) holds. We check 1.2 (b). From 1.1 (d) we get that the homomorphism $G'^0 \to G$ is a closed embedding and so an isomorphism (as its generic fibre is so). So 1.2 (b) holds. We check that 1.2 (c) holds. The homomorphism $G'^0 \to G^{\mathrm{n}}$ is an isomorphism (cf. 1.1 (c)) and so G^{n} is a reductive group scheme. So 1.2 (c) also holds. This ends the proof of 1.2 for the case when G is of finite type.

4.3 The general case

To end the proof of 1.2 we are left to show that G is of finite type. Let I_0 be the ideal of R that defines $(G_k)^0_{\mathrm{red}}$. Let $\mathcal{L}$ be the set of all G–modules L obtained as in 2.4. For $L \in \mathcal{L}$, let L_0, L, $R(L)$, and $\rho(L) : G \to GL(L)$ be as in 2.4. We can identify $G(L) := \mathrm{Spec}(R(L))$ with the Zariski closure in $GL(L)$ of the image of $\rho(L)$. As G_K is smooth over K and connected, it is also of finite type. We now choose L_0 such that $\rho(L)_K$ is an isomorphism (cf. 2.4 (i)) and L_0 modulo I_0 generates the finite type k-algebra R/I_0. So $R(L)$ surjects onto R/I_0 (cf. 2.4 (i)) and thus $\rho(L)_k$ induces a closed embedding homomorphism $(G_k)^0_{\mathrm{red}} \hookrightarrow (G(L)_k)^0_{\mathrm{red}}$ between smooth, connected groups of dimension $\dim(G_K)$. So by reasons of dimensions, we get that $(G(L)_k)^0_{\mathrm{red}}$ is a reductive group isomorphic to $(G_k)^0_{\mathrm{red}}$. As in 4.2, by replacing V with some $W \in \mathcal{W}$ we can assume that for any $i \in \{1, ..., s\}$ there is $w_{iL} \in G(L)(V)$ that lifts $y_i \in (G(L)_k)^0_{\mathrm{red}}(k) = (G_k)^0_{\mathrm{red}}(k)$. So from 4.2 applied to $G(L)$ (instead of G), we get that the normalization $G(L)^{\mathrm{n}} = \mathrm{Spec}(R(L)^{\mathrm{n}})$ of $G(L)$ is a reductive group scheme over V.

Let $\tilde{L} \in \mathcal{L}$ be such that $L \subset \tilde{L}$. So we have a homomorphism $\rho(\tilde{L}, L) : G(\tilde{L}) \to G(L)$ such that $\rho(L) = \rho(\tilde{L}, L) \circ \rho(L)$. So $R(L) \hookrightarrow R(\tilde{L}) \hookrightarrow R \hookrightarrow R(L)[\frac{1}{\pi}] = R(\tilde{L})[\frac{1}{\pi}] = R[\frac{1}{\pi}]$. Let $\rho(\tilde{L}, L)^{\mathrm{n}} : G(\tilde{L})^{\mathrm{n}} \to G(L)^{\mathrm{n}}$ be the morphism defined by $\rho(\tilde{L}, L)$. Let $W \in \mathcal{W}$ be such that there is $w_{i\tilde{L}} \in G(\tilde{L})(W)$ that lifts $y_i \in (G(\tilde{L})_k)^0_{\mathrm{red}}(k) = (G_k)^0_{\mathrm{red}}(k)$. The normalization $(G(\tilde{L})_W)^{\mathrm{n}}$ of $G(\tilde{L})_W$ is a reductive group scheme over W, cf. 4.2 applied to $G(\tilde{L})_W$. The morphisms $(G(\tilde{L})_W)^{\mathrm{n}} \to G(\tilde{L})^{\mathrm{n}}_W \to G(L)^{\mathrm{n}}_W \to G(L)_W$ define a homomorphism $\rho(\tilde{L}, L, W)^{\mathrm{n}} : (G(\tilde{L})_W)^{\mathrm{n}} \to G(L)^{\mathrm{n}}_W$ between reductive group schemes over W whose generic fibre is an isomorphism. So $\rho(\tilde{L}, L, W)^{\mathrm{n}}$ is an isomorphism, cf. 2.6. This implies that $\rho(\tilde{L}, L)^{\mathrm{n}}$ is an isomorphism. So $R(L) \hookrightarrow R(\tilde{L}) \hookrightarrow R(L)^{\mathrm{n}} = R(\tilde{L})^{\mathrm{n}} \hookrightarrow R[\frac{1}{\pi}]$. So as $R = \cup_{\tilde{L} \in \mathcal{L}} R(\tilde{L})$, we have V-monomorphisms $R(L) \hookrightarrow R \hookrightarrow R(L)^{\mathrm{n}}$. But $R(L^{\mathrm{n}})$ is a finite $R(L)$-algebra, cf. 4.2. So as $R(L)$ is a noetherian V-algebra, we get that R is a finite $R(L)$-algebra and so a finitely generated V-algebra. This ends the proof of 1.2.

5 An application

We assume that $p \in \mathbb{N}$ is a prime and that $k = \bar{k}$. We take V to be the Witt ring $W(k)$ of k. The goal of this section is to exemplify how one can use 1.1 (d) to extend results on semisimple groups over a field of characteristic 0 to results on semisimple groups over k. Let H_k be an absolutely simple, adjoint group over k. Let Q_k be a parabolic subgroup of H_k different from H_k. Let U_k be the unipotent radical of Q_k and let L_k be a Levi subgroup of Q_k.

5.1 Proposition

Proposition 5.1. *Let $\rho_k : L_k \to GL(Lie(U_k))$ be the representation of the inner conjugation action of L_k on $Lie(U_k)$. Then ρ_k is a closed embedding.*

Proof. Let H be the adjoint group scheme over $V = W(k)$ that lifts H_k. Let Q be a parabolic subgroup of H that lifts Q_k. Let U be the unipotent radical of Q and let L be a Levi subgroup of Q that lifts L_k. Let $\rho : L \to GL(\mathrm{Lie}(U))$ be the representation of the inner conjugation action of L on $\mathrm{Lie}(U)$. Let T be a maximal torus of L. Let B be a Borel subgroup of G such that $T \leqslant B \leqslant Q$. Let $\mathrm{Lie}(H) = \mathrm{Lie}(T) \oplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ be the Weyl decomposition of $\mathrm{Lie}(H)$ with respect to T. Let $\Delta = \{\alpha_1, ..., \alpha_r\}$ be the basis of the root system Φ that corresponds to B; here $r \in \mathbb{N}$ is the rank of H_k. Let Φ_U be the subset of Φ such that $\mathrm{Lie}(U) = \oplus_{\alpha\in\Phi_U} \mathfrak{g}_\alpha$. For each $i \in \{1, ..., r\}$, there is $\alpha \in \Phi_U$ that is the sum of α_i with an element of $\Phi_U \cup \{0\}$ which is a linear combination with coefficients in $\mathbb{N}\cup\{0\}$ of elements of $\Delta \setminus \{\alpha_i\}$. As H is adjoint, this implies that the inner conjugation action of T on $\mathrm{Lie}(U)$ is via characters of T that generate the group of characters of T. Thus the restriction of ρ to T is a closed embedding. So the identity component of $\mathrm{Ker}(\rho_K)$ is a semisimple group over K that has rank 0. Thus $\mathrm{Ker}(\rho_K)$ is a finite, étale subgroup of $Z(H_K)$. As $Z(H_K) \leqslant T_K$ and as the intersection $T_K \cap \mathrm{Ker}(\rho_K)$ is trivial, we get that $\mathrm{Ker}(\rho_K)$ is trivial. So ρ_K is a closed embedding. From 1.1 (d) we get that ρ is a closed embedding, except perhaps when $p = 2$ and L_k has a normal subgroup that is an SO_{2n+1} group for some $n \in \mathbb{N}$. So for the rest of the proof we can assume that $p = 2$ and that L_k has a normal subgroup S_k that is an SO_{2n+1} group for some $n \in \mathbb{N}$, $n \le r$.

This implies that $r \ge 2$ and that H_k has a subgroup normalized by T_k and which is a $PGL_2 = SO_3$ group. So H_k is an SO_{2r+1} group, cf. [14, 3.8]. If $\mathrm{Lie}(\mathrm{Ker}(\rho_k)) = \{0\}$, then as in the end of 3.3 we argue that ρ is a closed embedding. So to end the proof, we only need to show that the assumption that $\mathrm{Lie}(\mathrm{Ker}(\rho_k)) \neq \{0\}$ leads to a contradiction. As in 3.3 we argue that 1.1 (a) implies that $\mathrm{Lie}(\mathrm{Ker}(\rho_k))$ is formed by nilpotent elements. Based on 2.2.1 and its proof, we can assume that S_k is such that $\mathrm{Lie}(S_k) \cap \mathrm{Lie}(\mathrm{Ker}(\rho_k))$ contains the ideal $\mathfrak{n}$ of $\mathrm{Lie}(S_k)$ generated by eigenspaces of the adjoint action of T on $\mathrm{Lie}(H^{\mathrm{ad}})$ that correspond to short roots; we have $\dim_k(\mathfrak{n}) = 2n$ (cf. proof of 2.2.1). Let U_k^- be the unipotent subgroup of H_k that is the opposite of U_k with respect to T_k; so $\mathrm{Lie}(U_k^-) = \oplus_{\alpha\in\Phi_U} \mathfrak{g}_{-\alpha} \otimes_V k$. Let $w_0 \in H_k(k)$ be such that it normalizes both T_k and L_k and we have $w_0 U_k w_0^{-1} = U_k^-$, cf. [3, (XI), PLATE II]. As $\mathfrak{n} \subset \mathrm{Lie}(\mathrm{Ker}(\rho_k))$, $\mathfrak{n}$ centralizes $\mathrm{Lie}(U_k)$. As $\mathfrak{n}$ is a characteristic ideal of S_k, $\mathfrak{n}$ is normalized by w_0 and so it also centralizes $\mathrm{Lie}(U_k^-)$. Moreover, $\mathfrak{n}$ is normalized by $\mathrm{Lie}(L_k)$. Thus $\mathfrak{n}$ is normalized by $\mathrm{Lie}(H_k) = \mathrm{Lie}(U_k) \oplus \mathrm{Lie}(L_k) \oplus \mathrm{Lie}(U_k^-)$ and so $\mathfrak{n}$ is an ideal of $\mathrm{Lie}(H_k)$. But $\mathrm{Lie}(H_k)$ has a unique minimal ideal that has dimension $2r$, cf. [8, (B_r) of 0.13]. Thus $2n = \dim_k(\mathfrak{n}) \ge 2r \ge 2n$. Thus $n = r$ and so $S_k = H_k$. This implies that $Q_k = H_k$ and so we reached a contradiction to the relation $Q_k \neq H_k$. □

References

[1] S. Bosch, W. Lütkebohmert and M. Raynaud: *Néron models*, Springer-Verlag, 1990.

[2] A. Borel: "Linear algebraic groups", *Grad. Texts in Math.*, Vol. 126, Springer-Verlag, 1991.

[3] N. Bourbaki: *Lie groups and Lie algebras*, Springer-Verlag, 2002, Chapters 4–6.

[4] F. Bruhat and J. Tits: "Groupes réductifs sur un corps local: I. Données radicielles valuées", *Inst. Hautes Études Sci. Publ. Math.*, Vol. 41, (1972), pp. 5–251.

[5] M. Demazure, A. Grothendieck and ét al.: *Schémas en groupes. Vol. I-III*, Lecture Notes in Math., Vol. 151–153, Springer-Verlag, 1970.

[6] A. Grothendieck: "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schéma (Quatrième Partie)", *Inst. Hautes Études Sci. Publ. Math.*, Vol. 32, (1967).

[7] G. Hiss: "Die adjungierten Darstellungen der Chevalley-Gruppen", *Arch. Math.*, Vol. 42, (1982), pp. 408–416.

[8] J.E. Humphreys: *Conjugacy classes in semisimple algebraic groups*, In: Math. Surv. and Monog., Vol. 43, Amer. Math. Soc., Providence, 1995.

[9] J.C. Jantzen: *Representations of algebraic groups. Second edition.*, In: Math. Surveys and Monog., Vol. 107, Amer. Math. Soc., Providence, 2000.

[10] H. Matsumura: *Commutative algebra. Second edition.*, The Benjamin/Cummings Publ. Co., Inc., Reading, Massachusetts, 1980.

[11] R. Pink: "Compact subgroups of linear algebraic groups", *J. of Algebra*, Vol. 206, (1998), pp. 438–504.

[12] G. Prasad and J.-K. Yu: *On quasi-reductive group schemes*, math.NT/0405381, 34 pages revision, June 2004.

[13] A. Vasiu: "Integral canonical models of Shimura varieties of preabelian type", *Asian J. Math.*, Vol. 3(2), (1999), pp. 401–518.

[14] A. Vasiu: "Surjectivity criteria for p-adic representations, Part I", *Manuscripta Math.*, Vol. 112(3), (2003), pp. 325–355.