

Boundedness results for finite flat group schemes over discrete valuation rings of mixed characteristic

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Abstract. Let p be a prime. Let V be a discrete valuation ring of mixed characteristic $(0, p)$ and index of ramification e . Let $f : G \rightarrow H$ be a homomorphism of finite flat commutative group schemes of p power order over V whose generic fiber is an isomorphism. We provide a new proof of a result of Bondarko and Liu that bounds the kernel and the cokernel of the special fiber of f in terms of e . For $e < p - 1$ this reproves a result of Raynaud. Our bounds are sharper than the ones of Liu, are almost as sharp as the ones of Bondarko, and involve a very simple and short method. As an application we obtain a new proof of an extension theorem for homomorphisms of truncated Barsotti–Tate groups which strengthens Tate’s extension theorem for homomorphisms of p -divisible groups.

Key words: discrete valuation rings, group schemes, truncated Barsotti–Tate groups, p -divisible groups, and Breuil modules.

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1 Introduction

Let p be a rational prime. Let V be a discrete valuation ring of mixed characteristic $(0, p)$. Let K be the field of fractions of V . Let k be the residue field of V . Let e be the index of ramification of V . Let G and H

be two finite flat commutative group schemes of p power order over V . For $n \in \mathbb{N}^*$, let $G[p^n]$ be the schematic closure of $G_K[p^n]$ in G . The goal of the paper is to provide new proofs using *Breuil modules* of the following theorem and of several applications of it.

Theorem 1 *There exists a non-negative integer s that depends only on V and that has the following property.*

For each homomorphism $f : G \rightarrow H$ whose generic fiber $f_K : G_K \rightarrow H_K$ is an isomorphism, there exists a homomorphism $f' : H \rightarrow G$ such that $f' \circ f = p^s \text{id}_G$ and $f \circ f' = p^s \text{id}_H$ and therefore the special fiber homomorphism $f_k : G_k \rightarrow H_k$ has a kernel and a cokernel annihilated by p^s . If moreover H is a truncated Barsotti–Tate group of level $n > s$, then the natural homomorphism $f[p^{n-s}] : G[p^{n-s}] \rightarrow H[p^{n-s}]$ is an isomorphism.

An equivalent form of the first part of Theorem 1 (see the first part of Corollary 2 below) was first obtained in [Bon] using *Cartier–Dieudonné modules*, in [Liu1] using *deformation theories* and *Galois-descent*, and in [Liu2] using *Breuil–Kisin modules*. The number s admits computable upper bounds in terms only of e . For instance, we have $s \leq (\log_p e + \text{ord}_p e + 2)(\text{ord}_p e + 2)$ (cf. Examples 2 and 4). This upper bound of s is weaker than the one obtained by Bondarko $s \leq \lfloor \log_p(\frac{pe}{p-1}) \rfloor$ but it is much stronger than the upper bounds one can get based on the proofs of [Liu1], Theorem 1.0.5 and [Liu2], Theorem 2.4.2. If p does not divide e , then we regain Bondarko’s upper bound (cf. Examples 1 and 2). If $e \leq p - 2$, then $s = 0$ (cf. Example 1) and thus we also regain the following classical theorem of Raynaud.

Corollary 1 *We assume that $e \leq p - 2$ (thus p is odd). Then each finite flat commutative group scheme of p power order over K extends in at most one way to a finite flat commutative group scheme over V .*

Corollary 1 was first proved in [R], Theorem 3.3.3 or Corollary 3.3.6 and more recently in [Bon], [Liu1], [Liu2], and [VZ2], Proposition 15. The first part of the next result is an equivalent form of the first part of Theorem 1.

Corollary 2 *Let $h : G_K \rightarrow H_K$ be a homomorphism over K . Then $p^s h$ extends to a homomorphism $G \rightarrow H$ (i.e., the cokernel of the natural monomorphism $\text{Hom}(G, H) \hookrightarrow \text{Hom}(G_K, H_K)$ is annihilated by p^s). Thus the natural homomorphism $\text{Ext}^1(H, G) \rightarrow \text{Ext}^1(H_K, G_K)$ has a kernel annihilated by p^s .*

The first (resp. the second) part of Corollary 2 was first obtained in [Bon], Theorem A or B, in [Liu1], Theorem 1.0.5, and in [Liu2], Theorem 2.4.2 (resp. in [Bon], Theorem D).

The next two results are a mixed characteristic geometric analogue of the homomorphism form [V], Theorem 5.1.1 of the *crystalline boundedness principle* presented in [V], Theorem 1.2.

Corollary 3 *We assume that G and H are truncated Barsotti–Tate groups of level $n > s$. Let $h : G_K \rightarrow H_K$ be a homomorphism. Then the restriction homomorphism $h[p^{n-s}] : G_K[p^{n-s}] \rightarrow H_K[p^{n-s}]$ extends to a homomorphism $G[p^{n-s}] \rightarrow H[p^{n-s}]$.*

Corollary 4 *We assume that $n > 2s$. Let H be a truncated Barsotti–Tate group of level n over V . Let G be such that we have an isomorphism $h : G_K \rightarrow H_K$. Then the quotient group scheme $G[p^{n-s}]/G[p^s]$ is isomorphic to $H[p^{n-2s}]$ and thus it is a truncated Barsotti–Tate group of level $n - 2s$.*

A weaker form of Corollary 3 was first obtained in [Bon], Theorem E.

Let Y be a normal noetherian integral scheme with field of functions L of characteristic zero. A classical theorem of Tate ([T], Theorem 4) says that for every two p -divisible groups D and F over Y , each homomorphism $D_L \rightarrow F_L$ extends uniquely to a homomorphism $D \rightarrow F$. From Corollary 3 we obtain the following sharper version of this theorem.

Theorem 2 *Let Y and L be as above. Then there exists a non-negative integer s_Y which has the following property.*

Let \mathcal{G} and \mathcal{H} be truncated Barsotti–Tate groups over Y of level $n > s_Y$ and of order a power of the prime p . Let $h : \mathcal{G}_L \rightarrow \mathcal{H}_L$ be a homomorphism. Then there exists a unique homomorphism $g : \mathcal{G}[p^{n-s_Y}] \rightarrow \mathcal{H}[p^{n-s_Y}]$ that induces $h[p^{n-s_Y}]$ over L .

Section 2 recalls the classification of finite flat commutative group schemes of p power order over V in terms of Breuil modules that holds provided k is perfect and V is complete. This classification was conjectured by Breuil (see [Br]), was first proved in [K1] and [K2] for $p > 2$ and for the connected case with $p = 2$, was generalized (using a covariant language) in [VZ1], [Lau1], and [Lau2] for all primes p , and was recently proved for $p = 2$ in [Lau2], [Liu3], and [Kim]. As in [Liu2], our method to prove the above results relies as well on Breuil modules but it performs a refined combinatorial analysis of

Eisenstein polynomials associated to uniformizers of V which allows us to get better upper bounds than the ones of Liu and simpler proofs (see Lemmas 1 to 4 and Proposition 2) that could be generalized to the crystalline context of [Liu2], Theorem 2.4.2. In Section 3 we provide formulas for s as well as explicit upper bounds of it. In Section 4 we prove Theorem 1. Corollaries 1 to 4 and Theorem 2 are proved in Section 5. In Section 6 we present extra applications to different heights associated to G .

2 Breuil modules

Let $V \hookrightarrow V'$ be an extension of discrete valuation rings of mixed characteristic $(0, p)$ such that the index of ramification of V' is also e . Theorem 1 holds for V if it holds for V' . There exists an extension V' which is complete and has a perfect residue field. If we find an upper bound of s only in terms of e which holds for each complete V' with perfect residue field, then this upper bound of s is also good for V .

Thus from now on we assume that V is complete and that k is a perfect field. Let $W(k)$ be the ring of Witt vectors with coefficients in k . We will view V as a $W(k)$ -algebra which is a free $W(k)$ -module of rank e . Let $\text{ord}_p : W(k) \rightarrow \mathbb{N} \cup \{\infty\}$ be the p -adic valuation normalized by the conditions that $\text{ord}_p(p) = 1$ and $\text{ord}_p(0) = \infty$. Let u be a variable and let

$$\mathfrak{S} := W(k)[[u]].$$

We extend the Frobenius endomorphism σ of $W(k)$ to \mathfrak{S} by the rule $\sigma(u) = u^p$. For $n \in \mathbb{N}^*$ let $\mathfrak{S}_n := \mathfrak{S}/p^n\mathfrak{S}$. If M is a \mathfrak{S} -module let

$$M^{(\sigma)} := \mathfrak{S} \otimes_{\sigma, \mathfrak{S}} M.$$

Let π be a uniformizer of V . Let

$$E = E(u) = u^e + a_{e-1}u^{e-1} + \cdots + a_1u + a_0 \in W(k)[u]$$

be the unique Eisenstein polynomial in u which has coefficients in $W(k)$ and which has π as a root. For $i \in \{0, \dots, e-1\}$ we have $a_i \in pW(k)$; moreover a_0 is p times a unit of $W(k)$. We have a $W(k)$ -epimorphism

$$q_\pi : \mathfrak{S} \twoheadrightarrow V$$

that maps u to π .

Definition 1 *By a (contravariant) Breuil window relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$ we mean a pair (Q, ϕ) , where Q is a free \mathfrak{S} -module of finite rank and where $\phi : Q^{(\sigma)} \rightarrow Q$ is a \mathfrak{S} -linear map (Frobenius map) whose cokernel is annihilated by E . By a (contravariant) Breuil module relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$ we mean a pair (M, φ) , where M is a finitely generated \mathfrak{S} -module annihilated by a power of p and of projective dimension at most one and where $\varphi : M^{(\sigma)} \rightarrow M$ is a \mathfrak{S} -linear map (Frobenius map) whose cokernel is annihilated by E .*

Definition 2 *Let \mathcal{B} be the category of Breuil modules relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$. Let \mathcal{B}_1 be the full subcategory of \mathcal{B} whose objects are Breuil modules (M, φ) relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$ with M annihilated by p . Let \mathcal{F} be the category of finite flat commutative group schemes of p power order over V . Let \mathcal{F}_1 be the full subcategory of \mathcal{F} whose objects are finite flat commutative group schemes over V annihilated by p .*

We explain shortly the relation to covariant Breuil windows and modules relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$ as introduced in [VZ1], Definitions 1 and 2. Let L be a free \mathfrak{S} -module of finite rank. We have a canonical \mathfrak{S} -linear isomorphism

$$\mathfrak{S} \otimes_{\sigma, \mathfrak{S}} \mathrm{Hom}_{\mathfrak{S}}(L, \mathfrak{S}) \rightarrow \mathrm{Hom}_{\mathfrak{S}}(L^{(\sigma)}, \mathfrak{S})$$

that maps $s \otimes \alpha$ to the σ -linear map $L \rightarrow \mathfrak{S}$, $l \mapsto s\sigma(\alpha(l))$. The last σ -linear map may be also regarded as an element of $\mathrm{Hom}_{\mathfrak{S}}(L^{(\sigma)}, \mathfrak{S})$. As $\mathfrak{S} \xrightarrow{\sigma} \mathfrak{S}$ is flat, we obtain more generally that for each \mathfrak{S} -module M of finite type and for every integer $i \geq 0$ we have a canonical \mathfrak{S} -linear isomorphism

$$\mathrm{Ext}_{\mathfrak{S}}^i(M, \mathfrak{S})^{(\sigma)} \cong \mathrm{Ext}_{\mathfrak{S}}^i(M^{(\sigma)}, \mathfrak{S}).$$

Let (Q, ϕ) be a contravariant Breuil window relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$. Applying the functor $\mathrm{Hom}_{\mathfrak{S}}(-, \mathfrak{S})$ to the \mathfrak{S} -linear map ϕ we obtain a homomorphism

$$\psi : \mathrm{Hom}_{\mathfrak{S}}(Q, \mathfrak{S}) \rightarrow \mathrm{Hom}_{\mathfrak{S}}(Q^{(\sigma)}, \mathfrak{S}) \cong \mathrm{Hom}_{\mathfrak{S}}(Q, \mathfrak{S})^{(\sigma)}.$$

The pair $(\mathrm{Hom}_{\mathfrak{S}}(Q, \mathfrak{S}), \psi)$ is a covariant Breuil window relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$. The rule on objects $(Q, \phi) \mapsto (\mathrm{Hom}_{\mathfrak{S}}(Q, \mathfrak{S}), \psi)$ induces an antiequivalence of the category of contravariant Breuil windows relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$ with the category of covariant Breuil windows relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$.

In the same way the functor $\mathrm{Ext}_{\mathfrak{S}}^1(-, \mathfrak{S})$ induces naturally an antiequivalence of the category of contravariant Breuil modules relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$ with the category of covariant Breuil modules relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$.

If M is annihilated by p , then it is easy to see that M is a free \mathfrak{S}_1 -module (cf. [VZ2], Section 2, p. 578); its rank is also called the rank of (M, φ) . In this paper we will use the shorter terminology (connected) Breuil module.

2.1 Classification results

We recall the following fundamental classification (see [K1], Theorem 2.3.5, [VZ1], Theorem 1, and [Lau1], Theorem 1.1 for $p > 2$ and see [K2], [Lau2], Corollary 7.7, [Liu3], Theorem 1.0.2, and [Kim], Corollary 4.3 for $p = 2$).

Theorem 3 *There exists a contravariant functor $\mathbb{B} : \mathcal{F} \rightarrow \mathcal{B}$ which is an antiequivalence of categories, which is \mathbb{Z}_p -linear, and which takes short exact sequences (in the category of abelian sheaves in the faithfully flat topology of $\text{Spec } V$) to short exact sequences (in the category of \mathfrak{S} -modules endowed with Frobenius maps).*

It is easy to see that \mathbb{B} induces an antiequivalence of categories $\mathbb{B} : \mathcal{F}_1 \rightarrow \mathcal{B}_1$ which takes short exact sequences to short exact sequences. For an object $G = \text{Spec } R$ of \mathcal{F} , let $o(G) \in \mathbb{N}$ be such that $p^{o(G)}$ is the order of G i.e., is the rank of R over V .

We check that if G is an object of \mathcal{F}_1 , then the object $\mathbb{B}(G)$ of \mathcal{B}_1 has rank $o(G)$. The contravariant Dieudonné module $\mathbb{D}(G_k)$ of G_k is equal to $k \otimes_{\sigma, \mathfrak{S}} \mathbb{B}(G)$, where we denote also by σ its composite with the epimorphism $\mathfrak{S} \rightarrow k = \mathfrak{S}/(p, u)$ (the covariant part of this statement follows from either [Z1], Theorem 6 or [Z2], Theorem 1.6 provided G is connected and from [Lau2], Section 8 in general, once we recall that G is the kernel of an isogeny of p -divisible groups over V). This implies that the rank of $\mathbb{B}(G)$ is $o(G)$.

Let H be an object of \mathcal{F} . If p^n annihilates H , then to the chain of natural epimorphisms

$$H \twoheadrightarrow H/H[p] \twoheadrightarrow H/[p^2] \twoheadrightarrow \cdots \twoheadrightarrow H/H[p^n] = 0$$

corresponds a normal series of the Breuil module $\mathbb{B}(H) = (M, \varphi)$

$$0 = (M_n, \varphi_n) \subseteq (M_{n-1}, \varphi_{n-1}) \subseteq \cdots \subseteq (M_0, \varphi_0) = (M, \varphi)$$

by Breuil submodules whose quotient factors are objects of \mathcal{B}_1 . As each M_{i-1}/M_i is a free \mathfrak{S}_1 -module of finite rank, the multiplication by u map $u : M \rightarrow M$ is injective. One computes the order $p^{o(H)}$ of H via the formulas

$$o(H) = o(M, \varphi) := \sum_{i=1}^n \text{rank}_{\mathfrak{S}_1}(M_{i-1}/M_i) = \text{length}_{\mathfrak{S}_{(p)}}(M_{(p)}).$$

Proposition 1 *Let $f : G \rightarrow H$ be a morphism of \mathcal{F} . We write $g := \mathbb{B}(f) : \mathbb{B}(H) = (M, \varphi) \rightarrow \mathbb{B}(G) = (N, \psi)$. Then we have:*

(a) *The homomorphism $f_K : G_K \rightarrow H_K$ is a closed embedding if and only if the cokernel of $g : M \rightarrow N$ is annihilated by some power of u .*

(b) *The homomorphism $f_K : G_K \rightarrow H_K$ is an epimorphism if and only if the \mathfrak{S} -linear map $g : M \rightarrow N$ is a monomorphism.*

(c) *The homomorphism $f_K : G_K \rightarrow H_K$ is an isomorphism if and only if the \mathfrak{S} -linear map $g : M \rightarrow N$ is injective and its cokernel is annihilated by some power of u .*

Proof. Let $\tilde{N} := \text{Coker}(g)$. We prove (a). We first show that the assumption that f_K is not a closed embedding implies that \tilde{N} is not annihilated by a power of u . This assumption implies that there exists a non-trivial flat closed subgroup scheme G_0 of G which is contained in the kernel of f and which is annihilated by p . Let $(N_0, \psi_0) := \mathbb{B}(G_0)$; the \mathfrak{S}_1 -module N_0 is free of positive rank. From the fact that G_0 is contained in the kernel of f and from Theorem 3, we get that we have an epimorphism $\tilde{N} \twoheadrightarrow N_0$. Thus \tilde{N} is not annihilated by a power of u .

To end the proof of part (a) it suffices to show that the assumption that \tilde{N} is not annihilated by a power of u implies that f_K is not a closed embedding. Our assumption implies that also $N_1 := \tilde{N}/p\tilde{N}$ is not annihilated by a power of u . As $\mathfrak{S}_1 = k[[u]]$ is a principal ideal domain, we have a unique short exact sequence of \mathfrak{S}_1 -modules

$$0 \rightarrow N_2 \rightarrow N_1 \rightarrow N_0 \rightarrow 0,$$

where N_2 is the largest \mathfrak{S}_1 -submodule of N_1 annihilated by a power of u and where N_0 is a free \mathfrak{S}_1 -module of positive rank. The \mathfrak{S} -linear map $N_1^{(\sigma)} \rightarrow N_1$ induced naturally by ψ maps $N_2^{(\sigma)}$ to N_2 and thus it induces via quotients a \mathfrak{S} -linear map $\psi_0 : N_0^{(\sigma)} \rightarrow N_0$. As ψ_0 is a quotient map of ψ , its cokernel is annihilated by E . Therefore the pair (N_0, ψ_0) is a Breuil module. From this and Theorem 3 we get that there exists a non-trivial flat closed subgroup scheme G_0 of G which is contained in the kernel of f and for which we have $\mathbb{B}(G_0) = (N_0, \psi_0)$. Thus f_K is not a closed embedding. Therefore (a) holds.

Part (b) is proved similarly to (a). Part (c) follows from (a) and (b). \square

2.2 Basic lemmas

Let $n \in \mathbb{N}^*$. Let H be a truncated Barsotti–Tate group of level n over V . Let $(M, \varphi) := \mathbb{B}(H)$; thus M is a free \mathfrak{S}_n -module of finite rank. Let d be the dimension of H and let h be the height of H . There exists a direct sum decomposition $M = T \oplus L$ into free \mathfrak{S}_n -modules such that the image of φ is $ET \oplus L$ and T has rank d . The existence of such a direct sum decomposition follows (via reduction modulo p^n) from the existence of the normal decomposition of the Breuil window relative to $q_\pi : \mathfrak{S} \twoheadrightarrow V$ of any p -divisible group \mathcal{H} over V such that $\mathcal{H}[p^n] = H$ (for the covariant context with $p > 2$ see [VZ1], Section 2; the case $p = 2$ is the same). The existence of the direct sum decomposition $M = T \oplus L$ implies the existence of two \mathfrak{S}_n -bases $\{e_1, \dots, e_h\}$ and $\{v_1, \dots, v_h\}$ of M such that for $i \in \{d+1, \dots, h\}$ we have $\varphi(1 \otimes e_i) = v_i$ and for $i \in \{1, \dots, d\}$ the difference $\varphi(1 \otimes e_i) - Ev_i$ belongs to the \mathfrak{S} -submodule of M generated by the elements v_{d+1}, \dots, v_h . Indeed, we consider the composite map

$$M \rightarrow M^{(\sigma)} \rightarrow \varphi(M^{(\sigma)}) \rightarrow M/T = L,$$

where the first map $m \mapsto 1 \otimes m$ is σ -linear, where the second map is φ , and where the third map is induced by the natural projection of M on L along T . We tensor this composite map with the residue field k of \mathfrak{S}_n

$$k \otimes_{\mathfrak{S}_n} M \rightarrow k \otimes_{\mathfrak{S}_n} M^{(\sigma)} \rightarrow k \otimes_{\mathfrak{S}_n} \varphi(M^{(\sigma)}) \rightarrow k \otimes_{\mathfrak{S}_n} M/T = k \otimes_{\mathfrak{S}_n} L.$$

As k is a perfect field, all these three maps are surjective. Therefore we find by the lemma of Nakayama a \mathfrak{S}_n -basis $\{e_1, \dots, e_h\}$ of M such that the images $\varphi(1 \otimes e_{d+1}), \dots, \varphi(1 \otimes e_h)$ form a \mathfrak{S}_n -basis of M/T . We can take L to be the \mathfrak{S}_n -submodule of M generated by $v_{d+1} := \varphi(1 \otimes e_{d+1}), \dots, v_h := \varphi(1 \otimes e_h)$. We choose an arbitrary \mathfrak{S}_n -basis $\{v_1, \dots, v_d\}$ of T . As the image of φ is $ET \oplus L$ we obtain the desired \mathfrak{S}_n -basis $\{v_1, \dots, v_h\}$ by making a suitable change of the \mathfrak{S}_n -basis $\{v_1, \dots, v_d\}$ of T .

Lemma 1 *Let $t \in \mathbb{N}^*$. Let $x \in \frac{1}{u^t}M$ be such that $\varphi(1 \otimes x) \in \frac{1}{u^t}M$. We write $x = \sum_{i=1}^h \frac{\alpha_i}{u^t} e_i$, where $\alpha_i \in \mathfrak{S}_n$. Then for each $i \in \{1, \dots, h\}$ we have $E\sigma(\alpha_i) \in u^{t(p-1)}\mathfrak{S}_n$ (equivalently, $\frac{\sigma(\alpha_i)}{u^{tp}} E \in \frac{1}{u^t}\mathfrak{S}_n$) and thus $\alpha_i \in (p^{n-1}, u)\mathfrak{S}_n$.*

Proof. We compute

$$\varphi(1 \otimes x) = \sum_{i=1}^d \frac{\sigma(\alpha_i)}{u^{tp}} \varphi(1 \otimes e_i) + \sum_{i=d+1}^h \frac{\sigma(\alpha_i)}{u^{tp}} \varphi(1 \otimes e_i)$$

$$= \sum_{i=1}^d \frac{\sigma(\alpha_i)}{u^{tp}} E v_i + \sum_{i=d+1}^h \eta_i v_i \in \frac{1}{u^t} M,$$

for suitable elements $\eta_i \in \frac{1}{u^t} \mathfrak{S}_n$. For $i \in \{1, \dots, d\}$ this implies directly that

$$\frac{\sigma(\alpha_i)}{u^{tp}} E \in \frac{1}{u^t} \mathfrak{S}_n. \quad (1)$$

The η_i for $i \in \{d+1, \dots, h\}$ are of the form

$$\eta_i = \frac{\sigma(\alpha_i)}{u^{tp}} + \sum_{j=1}^d \lambda_j \frac{\sigma(\alpha_j)}{u^{tp}},$$

for some elements $\lambda_j \in \mathfrak{S}_n$. If we multiply the last equation by E we obtain from (1) that its belonging relation also holds for $i \in \{d+1, \dots, h\}$. For $i \in \{1, \dots, h\}$, we get that $\sigma(\alpha_i)E(0) = \sigma(\alpha_i(0))a_0 = 0 \in W_n(k)$ and this implies that $\alpha_i(0) \in p^{n-1}W_n(k)$; therefore we have $\alpha_i \in (p^{n-1}, u)\mathfrak{S}_n$. \square

Lemma 2 *Let $t \in \mathbb{N}$. Let N be a \mathfrak{S}_n -submodule of $\frac{1}{u^t}M$ which contains M . We assume that φ induces a \mathfrak{S} -linear map $N^{(\sigma)} \rightarrow N$. Then we have $p^t N \subseteq M$.*

Proof. We will prove this by induction on $t \in \mathbb{N}$. The case $t = 0$ is trivial. For the passage from $t - 1$ to t we can assume that $t > 0$. Let $x \in N$. From Lemma 1 applied to x we get that $px \in u \frac{1}{u^t} M = \frac{1}{u^{t-1}} M$. This implies that $pN \subseteq \frac{1}{u^{t-1}} M$. Let $\tilde{N} := pN + M \subseteq \frac{1}{u^{t-1}} M$. It is easy to see that φ induces a \mathfrak{S} -linear map $\tilde{N}^{(\sigma)} \rightarrow \tilde{N}$. By induction applied to \tilde{N} we get that $p^{t-1} \tilde{N} \subseteq M$. This implies that $p^t N \subseteq M$. This ends the induction. \square

The technical part of the method we use in this paper can be summarized as follows. With t and N as in Lemma 2, we will vary the uniformizer π of V to obtain a universal sharp upper bound t_0 of t (i.e., to show that in fact we have the first universal inclusion $N \subseteq \frac{1}{u^{t_0}} M$) and to refine Lemma 2 so that we get a smaller number $s \in \{0, \dots, t_0\}$ for which we have the second universal inclusion $p^s N \subseteq M$.

3 Motivation, formulas, and bounds for s

In this section we present formulas for s and upper bounds of it in terms of e . The main technical result that lies behind these formulas is also presented in this section (see Proposition 2 below).

For a real number x , let $\lfloor x \rfloor$ be the integral part (floor) of x (i.e., the greatest integer smaller or equal to x). We define

$$m := \text{ord}_p(e).$$

Let $a_e := 1$ and

$$E_0 := \sum_{i \in p\mathbb{N} \cap [0, e]} a_i u^i = a_{p\lfloor \frac{e}{p} \rfloor} u^{p\lfloor \frac{e}{p} \rfloor} + \cdots + a_{2p} u^{2p} + a_p u^p + a_0 \in W(k)[u^p].$$

Let $E_1 := E - E_0 \in W(k)[u]$. We define the numbers τ and ι as follows.

If $m = 0$, then let $\tau(\pi) := 1$ and $\iota(\pi) := 0$.

If $m \geq 1$, let $\tau(\pi) \in \mathbb{N}^* \cup \{\infty\}$ be the content $\text{ord}_p(E_1)$ of E_1 . Thus

$$\tau(\pi) := \text{ord}_p(E_1) = \min\{\text{ord}_p(a_i) \mid i \in \{1, 2, \dots, e-1\} \setminus p\mathbb{N}^*\}.$$

If $m \geq 1$ and $\tau(\pi) < \infty$, let $\iota(\pi) \in \{1, 2, \dots, e-1\} \setminus p\mathbb{N}^*$ be the smallest number such that we have

$$\tau(\pi) = \text{ord}_p(a_{\iota(\pi)}).$$

For all $m \geq 0$ we define

$$\tau = \tau_V := \min\{\tau(\pi) \mid \pi \text{ a uniformizer of } V\}.$$

If $\tau < \infty$, then we also define

$$\iota = \iota_V := \min\{\iota(\pi) \mid \pi \text{ a uniformizer of } V \text{ with } \tau(\pi) = \tau\}.$$

Lemma 3 *We have $\tau \leq m + 1 < \infty$.*

Proof. If $m = 0$, then this holds as $\tau = 1$. Thus we can assume that $m \geq 1$. We consider a new uniformizer $\tilde{\pi} := \pi + p$ of V . The unique Eisenstein polynomial $\tilde{E}(u) = u^e + \tilde{a}_{e-1}u^{e-1} + \cdots + \tilde{a}_1u + \tilde{a}_0$ in u with coefficients in $W(k)$ that has $\tilde{\pi}$ as a root is $\tilde{E}(u) = E(u - p)$. Thus $\tilde{a}_{e-1} = -pe + a_{e-1} = -p^{m+1}e' + a_{e-1}$, where $e' := p^{-m}e \in \mathbb{N}^* \setminus p\mathbb{N}^*$. Therefore p^{m+2} does not divide either a_{e-1} or \tilde{a}_{e-1} . This implies that either $\tau(\pi) \leq m + 1$ or $\tau(\tilde{\pi}) \leq m + 1$. Thus $\tau \leq m + 1 < \infty$. \square

Proposition 2 *Let n and t be positive integers. Let $C = C(u) \in \mathfrak{S}$ be a power series whose constant term is not divisible by p^n . We assume that*

$$E\sigma(C) \in (u^t, p^n)\mathfrak{S}. \quad (2)$$

If $\tau(\pi) = \infty$, then we have $t \leq ne$. If $\tau(\pi) \neq \infty$, then we have

$$t \leq \min\{\tau(\pi)e + \iota(\pi), ne\}.$$

Moreover, if $m = 0$, then we have $p\sigma(C) \in (u^t, p^n)\mathfrak{S}$ and if $m \geq 1$, then we have $p^{\tau(\pi)+1}\sigma(C) \in (u^t, p^n)\mathfrak{S}$ (if $m \geq 1$ and the content of C is 1, then in fact we have $\tau(\pi) + 1 \geq n$).

Proof. Clearly we can remove from C all monomials of some degree i that satisfies the inequality $pi \geq t$. Therefore we can assume that C is a polynomial of degree d such that $pd < t$. As $E_0\sigma(C)$ and $E_1\sigma(C)$ do not contain monomials of the same degree, the relation $E\sigma(C) \in (u^t, p^n)\mathfrak{S}$ implies that

$$E_0\sigma(C) \in (u^t, p^n)\mathfrak{S} \quad \text{and} \quad E_1\sigma(C) \in (u^t, p^n)\mathfrak{S}.$$

We first consider the case when p does not divide e (i.e., $m = 0$). Thus E_0 divided by p is a unit of \mathfrak{S} . Therefore we have $p\sigma(C) \in (u^t, p^n)\mathfrak{S}$. As $p \deg(C) = pd < t$, we get that $p\sigma(C) \equiv 0 \pmod{p^n}$. As moreover $E_1 \equiv u^e \pmod{p}$, we get $u^e\sigma(C) \in (u^t, p^n)\mathfrak{S}$. As the constant term of C is not divisible by p^n this implies that $t \leq e = \min\{\tau(\pi)e + \iota(\pi), ne\}$.

From now on we will assume that $p|e$. By the Weierstraß preparation theorem ([Bou], Chapter 7, Section 3, number 8) we can assume that C is a Weierstraß polynomial of degree d (i.e., a monic polynomial of degree d such that $C - u^d$ is divisible by p) and thus of content 1. Indeed, to check this let $c \in \{0, \dots, n-1\}$ be such that p^c is the content of C . We set $\bar{C} := (1/p^c)C$. The constant term of \bar{C} is not divisible by p^{n-c} . As $E\sigma(\bar{C}) \in (u^t, p^{n-c})\mathfrak{S}$, it suffices to prove the proposition for $(\bar{C}, n-c)$ instead of (C, n) . But \bar{C} is a unit times a Weierstraß polynomial.

Before we continue, we first prove the following lemma.

Lemma 4 *Let n and t be positive integers. We assume that p divides e . Let*

$$C = C(u) = u^d + c_{d-1}u^{d-1} + \dots + c_1u + c_0 \in W(k)[u]$$

be a Weierstraß polynomial such that $pd < t$ and $c_0 \notin p^nW(k)$. We also assume that

$$E_0\sigma(C) \in (u^t, p^n)\mathfrak{S}.$$

Then $d = (n - 1)\frac{e}{p}$ and for each $i \in \{0, 1, \dots, n - 1\}$ we have:

$$\text{ord}_p(c_{i\frac{e}{p}}) = n - i - 1, \quad \text{and} \quad \text{ord}_p(c_j) \geq n - i, \quad \text{for} \quad 0 \leq j < i\frac{e}{p}. \quad (3)$$

Moreover we also have $t \leq ne$.

Proof: For $l \in \{0, \dots, d - 1\}$, let $\gamma_l = \sigma(c_l) \in W(k)$. Thus

$$\sigma(C) = u^{dp} + \gamma_{d-1}u^{(d-1)p} + \dots + \gamma_1u^p + \gamma_0.$$

We define c_d and γ_d to be 1. For $l < 0$ or $l > d$, we define $c_l = \gamma_l = 0$. We have $\text{ord}_p(\gamma_l) = \text{ord}_p(c_l)$ for all $l \in \mathbb{Z}$. Moreover we set

$$E_0\sigma(C) = \sum_{j=0}^{d+\frac{e}{p}} \beta_{jp}u^{jp}, \quad \beta_{jp} \in W(k).$$

By our assumption β_{jp} is divisible by p^n if $jp < t$ and in particular if $j \leq d$. For $j \in \{0, \dots, d + \frac{e}{p}\}$ we have the identity

$$\beta_{jp} = a_0\gamma_j + a_p\gamma_{j-1} + \dots + a_e\gamma_{j-\frac{e}{p}}. \quad (4)$$

By induction on $j \in \{0, \dots, d\}$ we show that (3) holds. This includes the equality part of (3) if $j = i\frac{e}{p}$. Our induction does not require that $i < n$. But of course, the assumption that $i \geq n$ in the equality part of (3) (resp. the assumptions that $j \leq d$ and $i \geq n + 1$ in the inequality part of (3)), leads (resp. lead) to a contradiction as the order of any c_j can not be negative.

The case $j = 0$ follows by looking at the constant term of $E_0\sigma(C)$. The passage from $j - 1$ to j goes as follows. Let us first assume that $(i - 1)\frac{e}{p} < j < i\frac{e}{p}$ for some integer $i \geq 1$ (we do not require $i\frac{e}{p} \leq d$).

We show that the assumption $\text{ord}_p(c_j) = \text{ord}_p(\gamma_j) < n - i$ leads to a contradiction. Indeed, in this case the first term of the right hand side of (4) would have order $\leq n - i$ but all the other terms of the right hand side of (4) would have order strictly bigger than $n - i$. Note for example that $j - \frac{e}{p} < (i - 1)\frac{e}{p}$ and therefore $\text{ord}_p(\gamma_{j-\frac{e}{p}}) \geq n - i + 1$ by induction assumption. As $\text{ord}_p(\beta_{jp}) \geq n$ we obtain a contradiction to (4).

Finally we consider the passage from $j - 1$ to j in the case when $j = i\frac{e}{p}$. Then we use the equation

$$\beta_{ie} = a_0 \gamma_{i \frac{e}{p}} + a_p \gamma_{i \frac{e}{p} - 1} + \cdots + a_e \gamma_{(i-1) \frac{e}{p}}. \quad (5)$$

By the induction assumption we have $\text{ord}_p(a_e \gamma_{(i-1) \frac{e}{p}}) = n - i$. From the inequality part of (3) we get the inequalities

$$\text{ord}_p(a_p \gamma_{i \frac{e}{p} - 1}) \geq n - i + 1, \dots, \text{ord}_p(a_{e-p} \gamma_{1+(i-1) \frac{e}{p}}) \geq n - i + 1.$$

As $\beta_{ie} = \beta_{jp} \in p^n W(k)$, it follows from (5) that

$$\text{ord}_p(a_0 \gamma_{i \frac{e}{p}}) = \text{ord}_p(a_e \gamma_{(i-1) \frac{e}{p}}) = n - i.$$

Thus $\text{ord}_p(\gamma_{i \frac{e}{p}}) = n - i - \text{ord}_p(a_0) = n - i - 1$. This ends our induction.

We check that $d = (n - 1) \frac{e}{p}$. If d is of the form $i \frac{e}{p}$, then $0 = \text{ord}_p(c_d) = n - i - 1$ gives $i = n - 1$ and thus d is as required. We are left to show that the assumption that d is not of this form, leads to a contradiction. Let $i \in \mathbb{N}^*$ be such that $(i - 1) \frac{e}{p} < d < i \frac{e}{p}$. Then the inequality $0 = \text{ord}_p(c_d) \geq n - i$ gives $i \geq n$. But then $\text{ord}_p(c_{(i-1) \frac{e}{p}}) = n - i \leq 0$ contradicts the assumption that C is a Weierstraß polynomial and therefore that $c_{(i-1) \frac{e}{p}} \in pW(k)$.

As p divides e and $d = (n - 1) \frac{e}{p}$, $E_0 \sigma(C) \in (u^t, p^n) \mathfrak{S}$ is a monic polynomial of degree $e + pd = ne$ and thus we have $t \leq ne$. \square

Corollary 5 *With the notations of Lemma 4, let $l \in \{0, 1, \dots, e - 1\}$. Let $E_2 = E_2(u) = u^l + b_{l-1} u^{l-1} + \cdots + b_1 u + b_0 \in W(k)[u]$ be a Weierstraß polynomial of degree l . If we have $E_2 \sigma(C) \in (u^t, p^n) \mathfrak{S}$, then $l \geq t$.*

Proof: If $n = 1$, then $d = 0$ and $C = c_0$ is a unit of $W(k)$; thus $E_2 \in (u^t, p^n) \mathfrak{S}$ and therefore $l \geq t$. Thus we can assume that $n \geq 2$. We write:

$$E_2 \sigma(C) = \sum_{i=0}^{l+pd} \delta_i u^i.$$

Let $q := \lfloor \frac{l}{p} \rfloor < \frac{e}{p}$. We have the equation:

$$\delta_l = \gamma_0 + b_{l-p} \gamma_1 + \cdots + b_{l-qp} \gamma_q. \quad (6)$$

For $i \in \{1, \dots, q\}$ we have $\text{ord}_p(\gamma_i) = \text{ord}_p(c_i) \geq n - 1$ (cf. Lemma 4 and $n \geq 2$); therefore $\text{ord}_p(b_{l-ip} \gamma_i) \geq n$. As $\text{ord}_p(\gamma_0) = \text{ord}_p(c_0) = n - 1$, from (6)

we get $\text{ord}_p(\delta_l) = n - 1$. From this and the assumption $E_2\sigma(C) \in (u^t, p^n)\mathfrak{S}$, we get $l \geq t$.

We note that the inequality $l \geq t$ implies that actually the case $n \geq 2$ does not occur (cf. the sequence of relations $e > l \geq t > pd = (n - 1)e$). \square

We are now ready to prove Proposition 2 in the case when $p|e$ (i.e., $m \geq 1$) and the content of C is 1. It suffices to show that $\tau(\pi) + 1 \geq n$. We can assume that $\tau(\pi) + 1 \leq n$ and it suffices to show that $\tau(\pi) + 1 = n$.

As $\tau(\pi) = \text{ord}_p(E_1)$, by Weierstraß preparation theorem we can write

$$E_1 = p^{\tau(\pi)} E_2 \theta,$$

where $\theta \in \mathfrak{S}$ is a unit and where $E_2 \in W(k)[u]$ is a Weierstraß polynomial of degree $\iota(\pi) < e$. As $E_1\sigma(C) \in (u^t, p^n)\mathfrak{S}$ and as $n > \tau(\pi)$, we get that

$$E_2\sigma(C) \in (u^t, p^{n-\tau(\pi)})\mathfrak{S}.$$

As $d = (n - 1)\frac{e}{p} \geq \tau(\pi)\frac{e}{p}$ and $c_d = 1$, we can consider the monic polynomial

$$C_1 = C_1(u) = u^d + c_{d-1}u^{d-1} + \cdots + c_{\tau(\pi)\frac{e}{p}}u^{\tau(\pi)\frac{e}{p}} \in W(k)[u].$$

It follows from Lemma 4 that $\text{ord}_p(c_j) \geq n - \tau(\pi)$ for $j < \tau(\pi)\frac{e}{p}$. Thus $C_1 - C \in p^{n-\tau(\pi)}\mathfrak{S}$ and therefore we obtain

$$E_2\sigma(C_1) = E_2\sigma(C_1 - C) + E_2\sigma(C) \in (u^t, p^{n-\tau(\pi)})\mathfrak{S}. \quad (7)$$

We write $C_1 = u^{\tau(\pi)\frac{e}{p}}C_2$. Then the constant term of C_2 is $c_{\tau(\pi)\frac{e}{p}}$ and thus it is not divisible by $p^{n-\tau(\pi)}$, cf. (3). As $n > \tau(\pi)$, the relation $(n - 1)e = pd < t$ implies that $t - \tau(\pi)e > 0$. Thus from (7) we get that

$$E_2\sigma(C_2) \in (u^{t-\tau(\pi)e}, p^{n-\tau(\pi)})\mathfrak{S}. \quad (8)$$

A similar argument shows that

$$E_0\sigma(C_2) \in (u^{t-\tau(\pi)e}, p^{n-\tau(\pi)})\mathfrak{S}. \quad (9)$$

From Corollary 5 applied to the quintuple $(t - \tau(\pi)e, C_2, E_0, E_2, n - \tau(\pi))$ instead of (t, C, E_0, E_2, n) , we get that $\iota(\pi) = \deg(E_2) \geq t - \tau(\pi)e$. As $\iota(\pi) \leq e - 1$ and as $n \geq \tau(\pi) + 1$, we conclude that $t \leq \tau(\pi)e + \iota(\pi) = \min\{\tau(\pi)e + \iota(\pi), ne\}$. As $\iota(\pi) \leq e - 1$, the relations $(n - 1)e = pd < t \leq \min\{\tau(\pi)e + \iota(\pi), ne\}$ imply that $n \leq \tau(\pi) + 1$. Thus $n = \tau(\pi) + 1$ and this ends the proof of the Proposition 2 in the second case $p|e$. \square

3.1 Formulas for s

For a uniformizer π of V , Lemma 1 and Proposition 2 motivate the introduction of the following invariant

$$t(\pi) := \lfloor \frac{\tau(\pi)e + \iota(\pi)}{p-1} \rfloor \in \mathbb{N} \cup \{\infty\}.$$

Always there exists a π such that $t(\pi)$ is finite, cf. Lemma 3.

Based on the last sentence of Proposition 2, we define $\epsilon \in \{0, 1\}$ as follows. If $m = 0$, then $\epsilon := 0$ and if $m \geq 1$, then $\epsilon := 1$. Next we will define a number $s = s_V \in \mathbb{N}$ as follows. For all non-negative integers i let

$$s_i := i(\tau + \epsilon) \quad \text{and} \quad t_i := \lfloor \frac{\tau e + \iota}{(p-1)p^i} \rfloor.$$

Thus $t_0 = \lfloor \frac{\tau e + \iota}{p-1} \rfloor$ is the minimum of $t(\pi)$ for all possible uniformizers π of V and we have $t_{i+1} = \lfloor \frac{t_i}{p} \rfloor$ for all $i \geq 0$. Let $z \in \mathbb{N}$ be the smallest number such that we have $t_z - t_{z+1} \leq \tau + \epsilon$. Equivalently, let $z \in \mathbb{N}$ be the smallest number such that we have

$$s_z + t_z = \min\{s_i + t_i | i \in \mathbb{N}\} =: s = s_V.$$

We note that

$$0 = s_0 < s_1 < \cdots < s_z \quad \text{and} \quad 0 \leq t_z < t_{z-1} < \cdots < t_0.$$

Moreover the following relations hold:

$$s = s_z + t_z < s_{z-1} + t_{z-1} < \cdots < s_1 + t_1 < s_0 + t_0 = t_0 = \lfloor \frac{\tau e + \iota}{p-1} \rfloor. \quad (10)$$

By Lemma 3 there exists a π such that $\tau(\pi) \leq m+1$ (with $m = \text{ord}_p(e)$). We have $s \leq \lfloor \frac{\tau e + \iota}{p-1} \rfloor$ and thus, as $\iota \leq e-1$, we get

$$s \leq \frac{2e-1 + e \text{ord}_p(e)}{p-1}. \quad (11)$$

If $t_0 \geq 1$, let $v := \lfloor \log_p t_0 \rfloor$ and we consider the p -adic expansion

$$t_0 = \sum_{j=0}^v a_j p^j, \quad 0 \leq a_j < p, \quad a_v \neq 0. \quad (12)$$

Example 1 We assume that $e \leq p-2$. Thus p is odd, $m = \iota = 0$, and $\tau = 1$. We have $s \leq \frac{e}{p-1}$, cf. (10). Therefore $s = 0$.

Example 2 We assume that $m = 0$ and $e \geq p-1$. As $m = 0$ we have $\tau = \tau + \epsilon = 1$ and thus (s_z, t_z) is either $(v, 1)$ or $(v+1, 0)$. Therefore for $m = 0$ and $e \geq p-1$ we get $s = v+1 = 1 + \lfloor \log_p(\frac{e}{p-1}) \rfloor = \lfloor \log_p(\frac{pe}{p-1}) \rfloor$ and this is the same expression as in [Bon]. We have $s = 1$ if and only if $p-1 \leq e \leq p^2 - p - 1$. In general, for $m = 0$ we have the following practical upper bound $s \leq 1 + \log_p e$.

Example 3 We assume that $E = u^p - p$. Then $\tau(\pi) = \infty$. For $n \in \mathbb{N}^*$ we have $E\sigma(\sum_{i=1}^n p^{n-i}u^{i-1}) = u^{pn} - p^n$. Thus if $M = \mathfrak{S}_n$ and $\varphi : M^{(\sigma)} \rightarrow M$ takes 1 to the image of E in $M = \mathfrak{S}_n$, then for $N := M + \frac{1}{u^n}(\sum_{i=1}^n p^{n-i}u^{i-1})M$ we have $\varphi(N^{(\sigma)}) = M \subset N$, $p^n N = 0 \subset M$, $p^{n-1}N \not\subset M$ and therefore Lemma 2 is optimal in general. The smallest t for which one has $N \subset \frac{1}{u^t}M$ is n itself and thus it does not have an upper bound independently of n .

Let $\tilde{\pi} = c_0p + c_1\pi + \dots + c_{p-1}\pi^{p-1}$ be another uniformizer of V . We have $c_0, c_1, c_1^{-1}, c_2, \dots, c_{p-1} \in W(k)$. It is easy to see that $\tilde{\pi}^p$ is congruent modulo p^2 to the element pc_1^p of $pW(k)$. Thus, if $\tilde{E}(u) = u^p + \tilde{a}_{p-1}u^{p-1} + \dots + \tilde{a}_0$ is the Eisenstein polynomial that has $\tilde{\pi}$ as a root, then for $i \in \{1, \dots, p-1\}$ we have $\tilde{a}_i \in p^2W(k)$. Therefore $\tau \geq 2$. But $\tau \leq m+1 = 2$, cf. Lemma 3. Thus $\tau = 2$. From inequalities (10) we get that we have $s \leq \frac{2p+\iota}{p-1}$. Thus for $p = 3$ we have $s \leq 4$ and for $p \geq 5$ we have $s \leq 3$.

Example 4 We assume that $m \geq 1$. We have $t_{v+1} \leq \frac{t_0}{p^{v+1}} < 1$ and thus $t_{v+1} = 0$. Therefore $z \leq v+1$ and $s \leq s_{v+1} + t_{v+1} = s_{v+1} = (v+1)(\tau+1)$. We have the inequalities $(\tau+1)e/(p-1) \geq t_0$ and $\log_p t_0 \geq v$. Moreover $\tau \leq m+1$ by Lemma 3. We find the inequality

$$s \leq (\log_p(m+2) + \log_p e - \log_p(p-1) + 1)(m+2).$$

As $m = \text{ord}_p e$, $\log_p(m+2) < m+1$, and $\log_p(p-1) \in [0, 1)$, we find that

$$s < (\log_p e + \text{ord}_p e + 2)(\text{ord}_p e + 2).$$

If $(p, m) \neq (2, 1)$, then $\log_p(m+2) \leq m$ and we get that

$$s \leq (\log_p e + \text{ord}_p e + 1)(\text{ord}_p e + 2).$$

4 Proof of Theorem 1

We prove that Theorem 1 holds for the number s of Subsection 3.1 if V is complete and has perfect residue field k . We already noted in Section 2 that it is enough to treat this case.

We choose the uniformizer π of V in such a way that $\tau = \tau(\pi)$ is minimal and $\iota = \iota(\pi)$. Let $z \in \mathbb{N}$, the sequence of pairs $(s_0, t_0), \dots, (s_z, t_z)$, and $s \in \mathbb{N}$ be as in Subsection 3.1. Thus $t_0 = \lfloor (\tau e + \iota)/(p-1) \rfloor = t_0(\pi)$ and $s = s_z + t_z$.

Let $f : G \rightarrow H$ be a homomorphism of finite flat commutative group schemes of p power order over V , which induces an isomorphism $f_K : G_K \rightarrow H_K$ in generic fibers. We first remark that if there exists a homomorphism $f' : H \rightarrow G$ such that $f \circ f' = p^s \text{id}_H$, then we have $f'_K = p^s f_K^{-1}$ and thus the equality $f' \circ f = p^s \text{id}_G$ also holds as it holds generically; moreover, as the equalities continue to hold in the special fibre we conclude that the kernel and cokernel of f_k are annihilated by p^s .

We choose an epimorphism $\xi_H : \tilde{H} \rightarrow H$ from a truncated Barsotti–Tate group \tilde{H} . We consider the following fiber product

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{f}} & \tilde{H} \\ \xi_G \downarrow & & \downarrow \xi_H \\ G & \xrightarrow{f} & H \end{array}$$

in the category \mathcal{F} . Then \tilde{f}_K is an isomorphism. Assume that there exists a homomorphism $\tilde{f}' : \tilde{H} \rightarrow \tilde{G}$ such that $\tilde{f} \circ \tilde{f}' = p^s \text{id}_{\tilde{H}}$. Then $\xi_G \circ \tilde{f}'$ is zero on the finite flat group scheme $\text{Ker}(\xi_H)$ because this is true for the generic fibers. Thus there exists $f' : H \rightarrow G$ such that $f' \circ \xi_H = \xi_G \circ \tilde{f}'$. One easily verifies that $f \circ f' = p^s \text{id}_H$.

Thus to prove the existence of f' we can assume that $f = \tilde{f}$ and that $H = \tilde{H}$ is a truncated Barsotti–Tate group of level $n > s$.

Next we translate the existence of f' in terms of Breuil modules. Let (M, φ) and (N, ψ) be the Breuil modules of H and G (respectively). We know by Proposition 1 (c) that to f corresponds a \mathfrak{S} -linear monomorphism $M \hookrightarrow N$ whose cokernel is annihilated by some power u^t . We will assume that t is the smallest natural number with this property. We put aside the case $t = 0$ (i.e., the case when $f : G \rightarrow H$ is an isomorphism) which is trivial. The existence of $f' : H \rightarrow G$ is equivalent to the inclusion

$$p^s N \subseteq M. \tag{13}$$

Before we prove this inclusion we show that Theorem 1 follows from it. It remains to prove the last sentence of the Theorem 1. Hence again let H be a truncated Barsotti–Tate group of level $n > s$.

The identity $f \circ f' = p^s \text{id}_H$ means that we have a commutative diagram:

$$\begin{array}{ccc} H & \xrightarrow{f'} & G \\ & \searrow p^s & \downarrow f \\ & & H. \end{array}$$

We note that a homomorphism of finite flat group schemes over V is zero if it induces the zero homomorphism at the level of generic fibers. Therefore we obtain a commutative diagram:

$$\begin{array}{ccc} H/(H[p^s]) & \xrightarrow{\check{f}'} & G \\ & \searrow & \downarrow f \\ & & H. \end{array}$$

We see that \check{f}' is a closed immersion because the diagonal arrow is. Now we apply the functor $[p^{n-s}]$ (see Section 1) to the last diagram:

$$\begin{array}{ccc} H/(H[p^s]) & \xrightarrow{\check{f}'[p^{n-s}]} & G[p^{n-s}] \\ & \searrow & \downarrow f[p^{n-s}] \\ & & H[p^{n-s}]. \end{array}$$

The horizontal homomorphism is again a closed immersion. As it is a homomorphism between finite flat group schemes of the same order, it has to be an isomorphism. The diagonal arrow is trivially an isomorphism. Thus $f[p^{n-s}] : G[p^{n-s}] \rightarrow H[p^{n-s}]$ is an isomorphism and therefore $G[p^{n-s}]$ is a truncated Barsotti–Tate group of level $n - s$. This shows the last sentence of Theorem 1. It remains to prove the inclusion (13).

We will prove by induction on $j \in \{0, \dots, z\}$ that we have $p^{s_j} N \subseteq \frac{1}{u^{t_j}} M$.

We remark that by Lemma 2 this implies that $p^{s_j+t_j} N \subset M$. As $s = s_z + t_z$ the induction gives the desired inclusion (13) and ends the proof of Theorem 1. We also remark that already the base of the induction $j = 0$ implies the Theorem 1 but with the much weaker bound $s = t_0$.

For the base of the induction it suffices to show that $t \leq t_0$. Let $x \in N$ be such that $u^{t-1}x \notin M$. With $\{e_1, \dots, e_h\}$ a \mathfrak{S}_n -basis of M as before Lemma 1, we write

$$x = \sum_{i=1}^h \frac{\alpha_i}{u^t} e_i, \quad \alpha_i \in \mathfrak{S}_n.$$

From Lemma 1 we get that for each $i \in \{1, \dots, h\}$ we have $E\sigma(\alpha_i) \in u^{t(p-1)}\mathfrak{S}_n$. By the minimality of t , there exists $i_0 \in \{1, \dots, h\}$ such that α_{i_0} is not divisible by u . Let $C = C(u) \in \mathfrak{S}$ be such that its reduction modulo p^n is α_{i_0} . The constant term of C is not divisible by p^n and we have $E\sigma(C) \in (u^{t(p-1)}, p^n)\mathfrak{S}$. From this and Proposition 2 we get that $t(p-1) \leq \min\{\tau e + \iota, ne\}$. Thus $t \leq t_0 = \lfloor (\tau e + \iota)/(p-1) \rfloor$.

If $0 < j < z$, then the inductive step from $j-1$ to j goes as follows. We assume that $p^{s_{j-1}}N \subseteq \frac{1}{u^{t_{j-1}}}M$. Let $l_{j-1} \in \{0, \dots, t_{j-1}\}$ be the smallest number such that we have $p^{s_{j-1}}N \subseteq \frac{1}{u^{l_{j-1}}}M$. If $l_{j-1} = 0$, then $p^{s_{j-1}}N \subseteq M$ and thus, as $s_{j-1} < s_j$, we also have $p^{s_j}N \subseteq M \subseteq \frac{1}{u^{t_j}}M$. Therefore we can assume that $1 \leq l_{j-1} \leq t_{j-1}$. Let $y \in p^{s_{j-1}}N$. We write

$$y = \sum_{i=1}^h \frac{\eta_i}{u^{n_i}} e_i,$$

where $\eta_i \in \mathfrak{S}_n \setminus u\mathfrak{S}_n$ and where $n_i \in \{0, \dots, l_{j-1}\}$. Let $C_i = C_i(u) \in \mathfrak{S}$ be such that its reduction modulo p^n is η_i . We want to show that $p^{\tau+\epsilon}y \in \frac{1}{u^{t_j}}M$. For this it suffices to show that for each $i \in \{1, \dots, h\}$ we have $p^{\tau+\epsilon}\frac{\eta_i}{u^{n_i}} \in \frac{1}{u^{t_j}}\mathfrak{S}_n$. To check this we can assume that $n_i \geq t_j + 1$. As

$$t_j + 1 = \lfloor \frac{t_{j-1}}{p} \rfloor + 1 \geq \frac{t_{j-1} + 1}{p} \geq \frac{l_{j-1} + 1}{p},$$

we get that $pn_i - l_{j-1} \geq 1$. As $\frac{\eta_i}{u^{n_i}} = \frac{\eta_i u^{l_{j-1} - n_i}}{u^{l_{j-1}}}$, from Lemma 1 (applied with $y, \varphi(1 \otimes y) \in p^{s_{j-1}}N \subseteq \frac{1}{u^{l_{j-1}}}M$) we get that $E\sigma(C_i u^{l_{j-1} - n_i}) \in (u^{(p-1)l_{j-1}}, p^n)\mathfrak{S}$. This implies that

$$E\sigma(C_i) \in (u^{pn_i - l_{j-1}}, p^n)\mathfrak{S} \subseteq (u, p^n)\mathfrak{S}.$$

As $\eta_i \in \mathfrak{S}_n \setminus u\mathfrak{S}_n$, the constant term of C_i is not divisible by p^n . Thus from Proposition 2 applied to $(C_i, pn_i - l_{j-1})$ instead of (C, t) and from the definition of ϵ in Subsection 3.1, we get $\sigma(p^{\tau+\epsilon}C_i) = p^{\tau+\epsilon}\sigma(C_i) \in (u^{pn_i - l_{j-1}}, p^n)\mathfrak{S}$.

This implies that we can write $p^{\tau+\epsilon}C_i = A_i + B_i$, where $A_i \in p^n\mathfrak{S}$ and where $B_i \in u^{n_i - \lfloor \frac{t_{j-1}}{p} \rfloor} \mathfrak{S}$. Thus

$$p^{\tau+\epsilon} \frac{\eta_i}{u^{n_i}} \in \frac{1}{u^{\lfloor \frac{t_{j-1}}{p} \rfloor}} \mathfrak{S}_n \subseteq \frac{1}{u^{\lfloor \frac{t_{j-1}}{p} \rfloor}} \mathfrak{S}_n = \frac{1}{u^{t_j}} \mathfrak{S}_n.$$

Therefore $p^{\tau+\epsilon}y \in \frac{1}{u^{t_j}}M$. As $s_j = s_{j-1} + \tau + \epsilon$ and as $y \in p^{s_{j-1}}N$ is arbitrary, we conclude that $p^{s_j}N = p^{\tau+\epsilon}p^{s_{j-1}}N \subseteq \frac{1}{u^{t_j}}M$. This ends the induction. \square

5 Proofs of Corollaries 1 to 4 and Theorem 2

5.1 Proofs of the Corollaries 1 to 4

Corollary 1 follows from Theorem 1 and Example 1.

In connection to the Corollaries 2 to 4, let \tilde{G} be the schematic closure in $G \times_V H$ of the graph of the homomorphism $h : G_K \rightarrow H_K$ and let $i : \tilde{G} \hookrightarrow G \times_V H$ be the resulting closed embedding homomorphism. Via the two projections $q_1 : G \times_V H \rightarrow G$ and $q_2 : G \times_V H \rightarrow H$ we get homomorphisms $\rho_1 : \tilde{G} \rightarrow G$ and $\rho_2 : \tilde{G} \rightarrow H$. The generic fiber $\rho_{1,K}$ of ρ_1 is an isomorphism.

We prove the first part of Corollary 2. We consider the commutative diagram:

$$\begin{array}{ccccc} \tilde{G} & \xrightarrow{i} & G \times_V H & \xrightarrow{q_2} & H \\ & \searrow \rho_1 & \downarrow q_1 & & \\ & & G & & \end{array}$$

By Theorem 1 there exists a homomorphism $\rho'_1 : G \rightarrow \tilde{G}$ such that $\rho'_1 \circ \rho_1 = p^s \text{id}_{\tilde{G}}$. Then $q_2 \circ i \circ \rho'_1$ is the desired extension of $p^s h$.

To check the last part of Corollary 2, let $0 \rightarrow G \rightarrow J \rightarrow H \rightarrow 0$ be a short exact sequence whose generic fibre splits. It defines an arbitrary element $\nu \in \text{Ker}(\text{Ext}^1(H, G) \rightarrow \text{Ext}^1(H_K, G_K))$. Let $h : H_K \rightarrow J_K$ be a homomorphism that is a splitting of $0 \rightarrow G_K \rightarrow J_K \rightarrow H_K \rightarrow 0$. Let $g : H \rightarrow J$ be such that its generic fibre is $p^s h$, cf. first part of Corollary 2. Let $0 \rightarrow G \rightarrow J_s \rightarrow H \rightarrow 0$ be the pull back of $0 \rightarrow G \rightarrow J \rightarrow H \rightarrow 0$ via $p^s \text{id}_H$. Then there exists a unique section $g_s : H \rightarrow J_s$ of $0 \rightarrow G \rightarrow J_s \rightarrow H \rightarrow 0$ whose composite with $J_s \rightarrow J$ is g . Thus $p^s \nu = 0$. This proves Corollary 2.

We prove Corollary 3. From Corollary 2 we get that $p^s h$ extends to a homomorphism $G \rightarrow H$ which induces a homomorphism $G[p^{n-s}] = G/G[p^s] \rightarrow H[p^{n-s}]$ whose generic fibre is $h[p^{n-s}]$. This proves Corollary 3.

We prove Corollary 4 for $n > 2s$. As $h : G_K \rightarrow H_K$ is an isomorphism, $\rho_{2,K}$ is also an isomorphism. Thus $\rho_2[p^{n-s}] : \tilde{G}[p^{n-s}] \rightarrow H[p^{n-s}]$ is an isomorphism, cf. Theorem 1. By applying Theorem 1 to the Cartier dual $(\rho_1[p^{n-s}])^t : (G[p^{n-s}])^t \rightarrow (\tilde{G}[p^{n-s}])^t$ of $\rho_1[p^{n-s}] : \tilde{G}[p^{n-s}] \rightarrow G[p^{n-s}]$, we get that $(G[p^{n-s}])^t[p^{n-2s}]$ is isomorphic to $(\tilde{G}[p^{n-s}])^t[p^{n-2s}]$ and thus with $H^t[p^{n-2s}]$. Therefore $\{(G[p^{n-s}])^t[p^{n-2s}]\}^t$ is isomorphic to $H[p^{n-2s}]$. From this and the fact that we have a short exact sequence $0 \rightarrow G[p^s] \rightarrow G[p^{n-s}] \rightarrow \{(G[p^{n-s}])^t[p^{n-2s}]\}^t \rightarrow 0$, we get that the Corollary 4 holds. \square

5.2 Proof of Theorem 2

Clearly the homomorphism $g : \mathcal{G}[p^{n-s_Y}] \rightarrow \mathcal{H}[p^{n-s_Y}]$ is unique if it exists. We proved the case $Y = \text{Spec } V$ in Corollary 3. Let $y \in Y$ be a point; let $\kappa(y)$ be its residue field and let $\text{Spec } \mathcal{R}_y$ be its local ring. If $h[p^{n-s_Y}]$ extends to a homomorphism $g_y : \mathcal{G}_{\mathcal{R}_y}[p^{n-s_Y}] \rightarrow \mathcal{H}_{\mathcal{R}_y}[p^{n-s_Y}]$, then $h[p^{n-s_Y}]$ extends also to a homomorphism $g_{U_y} : \mathcal{G}_{U_y}[p^{n-s_Y}] \rightarrow \mathcal{H}_{U_y}[p^{n-s_Y}]$ over an open neighborhood U_y of y in Y . It follows from our assumptions that the extension g_y of $h[p^{n-s_Y}]$ exists for each point $y \in Y$ of codimension 1. Indeed, if $\text{char } \kappa(y) \neq p$ the group schemes $\mathcal{G}_{\mathcal{R}_y}$ and $\mathcal{H}_{\mathcal{R}_y}$ are étale and therefore an extension g_y trivially exists. If $\text{char } \kappa(y) = p$, then Corollary 3 implies that g_y exists provided we have $n > s_y$ for a suitable non-negative integer s_y that depends only on the ramification index of the discrete valuation ring \mathcal{R}_y . As the set $\Omega_p(Y)$ of points of Y of codimension 1 and of characteristic p is finite, we can define

$$s_Y := \max\{s_y | y \in \Omega_p(Y)\} \in \mathbb{N}.$$

With this s_Y , there exists an extension $g_U : \mathcal{G}_U[p^{n-s_Y}] \rightarrow \mathcal{H}_U[p^{n-s_Y}]$ of $h[p^{n-s_Y}]$ over an open subscheme $U \subseteq Y$ such that $\text{codim}_Y(Y \setminus U) \geq 2$. As Y is a normal noetherian integral scheme, the existence of an extension $g : \mathcal{G}[p^{n-s_Y}] \rightarrow \mathcal{H}[p^{n-s_Y}]$ of g_U is a general fact which holds for every locally free coherent \mathcal{O}_Y -modules \mathcal{M} and \mathcal{N} and for each homomorphism $\alpha_U : \mathcal{M}_U \rightarrow \mathcal{N}_U$ of \mathcal{O}_U -modules. \square

6 Upper bounds on heights

In this section we assume that V is complete and k is perfect. Let $\pi \in V$ and $q_\pi : \mathfrak{S} \rightarrow V$ be as in Section 2. We will study three different heights of a finite flat commutative group scheme G of p power order over V .

Definition 3 (a) *By the Barsotti–Tate height of G we mean the smallest non-negative integer $h_1(G)$ such that G is a closed subgroup scheme of a truncated Barsotti–Tate group over V of height $h_1(G)$.*

(b) *By the Barsotti–Tate co-height of G we mean the smallest non-negative integer $h_2(G)$ such that G is the quotient of a truncated Barsotti–Tate group over V of height $h_2(G)$.*

(c) *By the Brueil (or generator) height of G we mean the smallest number $h_3(G)$ of generators of the \mathfrak{S} -module N , where $(N, \psi) := \mathbb{B}(G)$.*

6.1 Simple inequalities

We have $h_2(G) = h_1(G^t)$ and $p^{h_3(G)}$ is the order of $G_k[p]$. If G is a truncated Barsotti–Tate group, then $h_1(G) = h_2(G) = h_3(G)$ are equal to the height of G . Based on these properties, it is easy to check that in general we have

$$h_3(G) = h_3(G^t) \leq \min\{h_1(G), h_2(G)\}.$$

Lemma 5 *We have $h_1(G) \leq 2h_3(G)$.*

Proof. The proof of this is similar to the proof of [VZ1], Proposition 2 (ii) but worked out in a contravariant way. If $h_4(G) \in \{0, \dots, h_3(G)\}$ is the smallest number of generators of $\text{Im}(\psi)/EN$, then as in loc. cit. we argue that there exists a Breuil window (Q, ϕ) relative to $q_\pi : \mathfrak{S} \rightarrow V$ which has rank $h_3(G) + h_4(G)$ and which is equipped naturally with a surjection $(Q, \phi) \rightarrow (N, \psi)$. More precisely, starting with \mathfrak{S} -linear maps $\chi_T : T := \mathfrak{S}^{h_3(G)} \rightarrow N$ and $\chi_L : L := \mathfrak{S}^{h_4(G)} \rightarrow \text{Im}(\psi)$ such that χ_T is onto and $\text{Im}(\psi) = \text{Im}(\chi_L) + EN$, one can take $Q := T \oplus L$ and the surjection $\chi_T \oplus \chi_L : Q \rightarrow N$.

The existence of the surjection $(Q, \phi) \rightarrow (N, \psi)$ implies that G is a closed subgroup scheme of the p -divisible group of height $h_3(G) + h_4(G)$ over V associated to (Q, ϕ) . Thus $h_1(G) \leq h_3(G) + h_4(G) \leq 2h_3(G)$. \square

If $o(G)$ is as in Section 2, then we obviously have

$$h_3(G) \leq o(G). \tag{14}$$

If we have a short exact sequence

$$0 \rightarrow G_1 \rightarrow G \rightarrow G_2 \rightarrow 0$$

of finite flat group schemes over V , then as the functor \mathbb{B} takes short exact sequences to short exact sequences (in the category of \mathfrak{S} -modules endowed with Frobenius maps), we have the subadditive inequality

$$h_3(G) \leq h_3(G_1) + h_3(G_2). \quad (15)$$

Proposition 3 *For every truncated Barsotti–Tate group H over V of height r and for each G whose generic fiber is isomorphic to H_K , we have*

$$h_3(G) \leq (2s + 1)r.$$

Therefore we have $\max\{h_1(G), h_2(G), h_3(G)\} \leq (4s + 2)r$.

Proof. Let n be the level of H . If $n \leq 2s$, then from (14) we get that $h_3(G) \leq o(G) \leq 2sr$. We now assume that $n > 2s$. Then from Corollary 4 we get that $G[p^{n-s}]/G[p^s]$ is isomorphic to $H[p^{n-2s}]$. Therefore we have $h_3(G[p^{n-s}]/G[p^s]) = r$.

As the orders of $G[p^s]$ and $G/G[p^{n-s}]$ are equal to sr , from inequalities (14) and (15) we get first that $h_3(G[p^{n-s}]) \leq h_3(G[p^{n-s}]/G[p^s]) + h_3(G[p^s]) \leq r + sr = (s + 1)r$ and second that $h_3(G) \leq h_3(G[p^{n-s}]) + h_3(G/G[p^{n-s}]) \leq (s + 1)r + sr = (2s + 1)r$.

The group scheme G^t satisfies the same property as G i.e., G_K^t is isomorphic to H_K^t . Thus we also have $h_3(G^t) \leq (2s + 1)r$. From this and Lemma 5 we get that $h_2(G) = h_1(G^t) \leq 2h_3(G) \leq (4s + 2)r$. Similarly, $h_1(G) \leq 2h_3(G) \leq (4s + 2)r$. Thus $\max\{h_1(G), h_2(G), h_3(G)\} \leq (4s + 2)r$. \square

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