Practice Exam 2, Oct. 26, 2018, Math 304-03- Linear Algebra

Problem 1: 6 points= 2 times 3 points

(a)= 3 times 1 point Define what is:

(i) the span of three vectors v_1, v_2, and v_3 in $\mathbb{R}^n$;

(ii) an ordered basis of a subspace V of $\mathbb{R}^n$;

(iii) an affine subset of $\mathbb{R}^n$.

(b) List all elementary matrices of size 2×2.

Problem 2: 6 points Find the standard matrix A for the linear transformation (map) $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps a point into its rotation clockwise through an angle of $\frac{10\pi}{3}$ (i.e., $T = L_A$).

Problem 3: 6 points Let $X = \{v_1, v_2, v_3\}$ be the set of three vectors, where $v_1 = [2\ 1\ 0\ 3]^T$, $v_2 = [1\ 3\ 2\ 1]^T$, and $v_3 = [1\ 1\ 1\ 1]^T$. Let $w = [4\ 4\ 1\ 6]^T$. Decide if w does or does not belong to Span(X) (the span of the three vectors v_1, v_2, v_3).

Problem 4: 6 points = 3 times 2 points

(a) Check that $v_1 = [1\ 1\ 1]^T$, $v_2 = [1\ -1\ 1]^T$, and $v_3 = [2\ 0\ 3]^T$ form a basis of $\mathbb{R}^3$.

(b) Write $w = [0\ 1\ 0]^T$ as a linear combination of v_1, v_2, and v_3.

(c) Let $K : \mathbb{R}^3 \to \mathbb{R}^3$ be the coordinate transformation associated to the basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$. Find the unique vector $w \in \mathbb{R}^3$ such that $K(w) = [1\ 1\ 1]^T$.

Problem 5: 6 points Compute the image of the unit square under the linear transformation given by the following matrix

$$A = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}.$$

Warning: the neatness counts.

Problem 6: 6 points Compute the inverse of the following invertible matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & -5 & -4 \end{pmatrix}.$$

Problem 7: 6 points Write the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & -5 & -4 \end{pmatrix}$$

as a product of elementary matrices. Hint: use the work of Problem 5.

Problem 8: 6 points= 2 times 3 points Find bases for Col(A) and Row(A), where

$$A = \begin{pmatrix} 1 & 2 & -5 \\ 2 & 2 & -2 \\ 3 & 4 & -7 \end{pmatrix}.$$

Warning: for Row(A) it is requested to use the RREF of A (and its elements belong to $\mathbb{R}_3$).

Problem 9: 6 points Find all values of t such that the set $Y = \{y_1, y_2, y_3\}$ is linearly dependent in $\mathbb{R}^3$, where $y_1 = [1\ t\ 0]^T$, $y_2 = [0\ 1\ 1]^T$, and $y_3 = [t\ 2\ 1]^T$. For each such value of t find a linear dependence relation between the vectors in Y.

Problem 10: 6 points Find a basis for Ker(A), where

$$A = \begin{pmatrix} -1 & -1 & 1 & 4 \\ 2 & 0 & 3 & 5 \\ 1 & -3 & 9 & 22 \\ -10 & -4 & -5 & 1 \end{pmatrix}.$$

Problem 11: 5 points= 5 times 1 point Test your understanding by marking each one of the following seven sentences as true (T) or false (F). Please write your answers to the left of the sentences.

(a) There exists a linear transformation $T : P_n \to \mathbb{R}^n$ which is an isomorphism.

(b) There exists a linear transformation $A : \mathbb{R}^6 \to \mathbb{R}^5$ whose image (column space) has dimension 4.

(c) There exists a linear transformation $A : \mathbb{R}^7 \to \mathbb{R}^8$ whose kernel (null space) has dimension 8.

(d) If $F : V \to W$ is a linear transformation and if $v_1, v_2, v_3 \in V$ are linearly independent, then $F(v_1), F(v_2), F(v_3)$ are linearly independent.

(e) If a composite function $G \circ F$ is one-to-one, then G is one-to-one.

Problem 12: 6 points We consider the following three polynomials $3+2x+x^2$, $-1-2x^2$, and $3+4x-4x^2$ of P_2. Decide if these three polynomials are or are not linearly independent.

Problem 13: 6 points= 2 times 3 points Compute the dimensions of the following subspaces of P_4:

(a) The kernel of the linear transformation $D : P_4 \to P_4$ that maps a polynomial $p(x)$ into its second derivative $p''(x)$ (thus $D(p(x)) = p''(x)$).

(b) The image of the linear transformation $I : P_3 \to P_4$ that maps a polynomial $p(x)$ into $\int_0^x p(t)dt$ (thus $I(p(x)) = \int_0^x p(t)dt$).

Problem 14: 6 points= 2 times 3 points Compute the dimensions of the following subspaces of P_4:

(a) The kernel of the linear transformation $D : P_4 \to P_4$ that maps a polynomial $p(x)$ into its second derivative $p''(x)$ (thus $D(p(x)) = p''(x)$).

(b) The image of the linear transformation $I : P_3 \to P_4$ that maps a polynomial $p(x)$ into $\int_0^x p(t)dt$ (thus $I(p(x)) = \int_0^x p(t)dt$).

Problem 15: 6 points We consider the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L([0\ 1\ 1]^T) = [2\ 3]^T$, $L([1\ 0\ 1]^T) = [2\ 1]^T$, and $L([1\ 1\ 0]^T) = [-2\ 0]^T$. Compute $L([1\ 0\ 3]^T)$.

Problem 16: 6 points We consider the orderred bases $X = (1+x, 2+x)$ and $Y = (1-x, 2-x)$ for P_1. Find the change of basis matrices ${}_XI_Y$ and ${}_YI_X$.

Problem 17: 6 points We consider the following two ordered bases $X = (p_1(x), p_2(x))$ and $Y = (q_1(x), q_2(x))$ for P_1, where $p_1(x) = 1-x$, $p_2(x) = 1+x$, $q_1(x) = 1+2x$, $q_2(x) = 2+x$. Let $F : P_1 \to P_1$ be the linear transformation such that

$${}_XF_Y = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}.$$

If $S = (1, x)$ is the standard ordered bases for P_1, then find the matrix ${}_SF_S$.