Exam 2 (§ 2.1 – 2.3, 4.1 – 4.6)

Math 313-04

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Student's Name (please print): ____

By signing my name below, I agree that I am following all rules and regulations set forth by the Code of Academic Integrity. Furthermore, I agree that I am following all rules set by my instructor and by the course policy for this exam.

Signature: ____

Date: ____________________________

Use correct notation and terminology, and be sure to **justify your answers** when required. The points for each part will be awarded holistically, with attention paid both to correctness and clear explanations. Unless otherwise noted, each part within a problem is worth an equal number of points.

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1. [12 points] Complete the sentences below to precisely define the terms in bold.

(a) A set of vectors $\{v_1, \ldots, v_n\}$ in a vector space V is a **basis** of V if ...

(b) An $n \times n$ matrix A is called **invertible** if ...

2. [12 points] Let A be a 2×2 invertible matrix and assume that $A^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) Solve the equation $A\vec{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

(b) Find A.

(c) Find $(A^T)^{-1}$.

3. [16 points] Determine if the following are possible or impossible. If possible, give a *specific* example. If impossible, explain why.

(a) An $n \times n$ matrix A such that $\mathrm{Col}(A) \neq \mathbb{R}^n$ and such that the equation $A\vec{x} = \vec{v}$ has a unique solution, for some $\vec{v}$ in $\mathbb{R}^n$.

(b) A basis for $\mathbb{P}_2$ which does not contain *any* of the polynomials $1, x$, or x^2.

4. [20 points] In each part, decide if W is a subspace of V. Justify your answer to part (e) **only**.

(a) (3 points) $W = \mathbb{R}^2$, $V = \mathbb{R}^3$.

(b) (3 points) $W = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} : \begin{array}{r} x + y + z = 0 \\ w + x = y \\ w - y + 2z = x \\ w, x, y, z \text{ are real numbers} \end{array} \right\}$, $V = \mathbb{R}^4$.

(c) (3 points) W is the set of all 3×3 *invertible* matrices, $V = M_{3\times 3}$.

(d) (3 points) W is the set of all polynomials $p(x)$ in $\mathbb{P}_2$ with the property that $p(1) = 0$, $V = \mathbb{P}_2$.

(e) (8 points) $W = \left\{ \begin{pmatrix} a & a \\ b & a+b \end{pmatrix} : a, b \text{ are real numbers} \right\}$, $V = M_{2\times 2}$.

5. [14 points] A square matrix A is called **skew-symmetric** if $A^T = -A$. The set V of all 3×3 skew-symmetric matrices is a subspace of $M_{3\times 3}$ with dimension 3 (you do **not** need to show this).

(a) (8 points) Find a basis for V, and prove that it is a basis.

(b) (6 points) Find the coordinate vector of $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$, relative to the basis you found in part (a).

6. [14 points] If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ then $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ (you do **not** need to show this). Find bases for $\text{Nul}(A)$ and $\text{Col}(A)$. You do not need to prove that they are bases. (Make sure that you indicate which basis is which!)

7. [12 points] In this problem you will critique an answer to a linear algebra problem given by Janey Student (Johnny's sister). **Note:** the reduced echelon form Janey found is correct.

Problem: Let

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \text{ and } \vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Is it possible to find a vector $\vec{v}_4$ so that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis of $\mathbb{R}^4$?

Janey Student's Answer: *Yes, it is possible*. We know that $\dim \mathbb{R}^4 = 4$, so a basis for $\mathbb{R}^4$ must contain 4 vectors. Therefore, if we find a vector $\vec{v}_4$ such that $\vec{v}_4$ is not in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$.

Let $\vec{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. The equation

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{v}_4$$

is inconsistent. We know this because

$$\text{rref}\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & | & \vec{v}_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has the row $\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$. This means that $\vec{v}_4$ is not in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and therefore $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$

Is Janey's answer correct? If it is, then draw a picture of Janey in the space below. If not, then find the error in her argument and indicate how you would correct it.