Exam 2 – Solutions (§ 2.1 – 2.3, 4.1 – 4.6)

Math 313-04

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Student's Name (please print): ____

By signing my name below, I agree that I am following all rules and regulations set forth by the Code of Academic Integrity. Furthermore, I agree that I am following all rules set by my instructor and by the course policy for this exam.

Signature: ____

Date: ____________________________

Use correct notation and terminology, and be sure to **justify your answers** when required. The points for each part will be awarded holistically, with attention paid both to correctness and clear explanations. Unless otherwise noted, each part within a problem is worth an equal number of points.

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1. [12 points] Complete the sentences below to precisely define the terms in bold.

(a) A set of vectors $\{v_1, \ldots, v_n\}$ in a vector space V is a **basis** of V if ... (i) the set $\{v_1, \ldots, v_n\}$ is linearly independent; and (ii) $\text{span}\{v_1, \ldots, v_n\} = V$.

(b) An $n \times n$ matrix A is called **invertible** if ... A has an inverse – i.e. if there is a matrix B such that $BA = I_n = AB$, where I_n is the $n \times n$ identity matrix.

2. [12 points] Let A be a 2×2 invertible matrix and assume that $A^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) Solve the equation $A\vec{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

Solution: $x = A^{-1} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

(b) Find A.

Solution: $A = (A^{-1})^{-1}$, and we can find this by row reducing $(A^{-1}|I_2)$:

$$\begin{pmatrix} 1 & 0 & | & 1 & 0 \\ 0 & 1 & | & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & | & 1 & -1 \\ 0 & 1 & | & 0 & 1 \end{pmatrix}.$$

So $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

(c) Find $(A^T)^{-1}$.

Solution: The hard way to do this is to find A^T (from part (b)), and then find the inverse. The easy way is to remember that $(A^T)^{-1} = (A^{-1})^T$. So $(A^T)^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

3. [16 points] Determine if the following are possible or impossible. If possible, give a *specific* example. If impossible, explain why.

(a) An $n \times n$ matrix A such that $\mathrm{Col}(A) \neq \mathbb{R}^n$ and such that the equation $A\vec{x} = \vec{v}$ has a unique solution, for some $\vec{v}$ in $\mathbb{R}^n$.

Solution: Impossible – $\mathrm{Col}(A) \neq \mathbb{R}^n$ means that columns of A do *not* span $\mathbb{R}^n$. By the Invertible Matrix Theorem, this implies that A has fewer than n pivot positions. Therefore, the equation $Ax = v$ has free variables, and so the equation either has no solution or it has infinitely many solutions. In particular, it never has a unique solution.

(b) A basis for $\mathbb{P}_2$ which does not contain *any* of the polynomials $1, x$, or x^2.

Solution: Possible – $\{2, 2x, 2x^2\}$.

4. [20 points] In each part, decide if W is a subspace of V. Justify your answer to part (e) **only**.

(a) (3 points) $W = \mathbb{R}^2$, $V = \mathbb{R}^3$.

Solution: No: $\mathbb{R}^2$ is not even a sub*set* of $\mathbb{R}^3$

(b) (3 points) $W = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} : \begin{array}{r} x+y+z=0 \\ w+x=y \\ w-y+2z=x \\ w,x,y,z \text{ are real numbers} \end{array} \right\}$, $V = \mathbb{R}^4$.

Solution: Yes: $W = \mathrm{Nul}(A)$, where

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 2 \end{pmatrix}.$$

(c) (3 points) W is the set of all 3×3 *invertible* matrices, $V = M_{3\times 3}$.

Solution: No: The zero vector (i.e. the 3×3 zero matrix) is not in W.

(d) (3 points) W is the set of all polynomials $p(x)$ in $\mathbb{P}_2$ with the property that $p(1) = 0$, $V = \mathbb{P}_2$.

Solution: Yes: this was one of the review day problems (actually, the review day problem considered polynomials in $\mathbb{P}_2$, but the idea is exactly the same).

(e) (8 points) $W = \left\{ \begin{pmatrix} a & a \\ b & a+b \end{pmatrix} : a, b \text{ are real numbers} \right\}$, $V = M_{2\times 2}$.

Solution: Yes: the zero vector (i.e. the 2×2 zero matrix) is in W, by choosing $a = b = 0$. If $A = \begin{pmatrix} a & a \\ b & a+b \end{pmatrix}$ and $B = \begin{pmatrix} c & c \\ d & c+d \end{pmatrix}$ are arbitrary elements of W, and if r is a scalar, then

$$A + B = \begin{pmatrix} (a+c) & (a+c) \\ (b+d) & (a+c)+(b+d) \end{pmatrix} \text{ and } rA = \begin{pmatrix} ra & ra \\ rb & ra+rb \end{pmatrix}.$$

So both $A + B$ and rA are also in W.

5. [14 points] A square matrix A is called **skew-symmetric** if $A^T = -A$. The set V of all 3×3 skew-symmetric matrices is a subspace of $M_{3\times 3}$ with dimension 3 (you do **not** need to show this).

(a) (8 points) Find a basis for V, and prove that it is a basis.

Solution: If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ and if $A^T = -A$, then this means that $a = e = j = 0$, $d = -b, g = -c$, and $h = -f$. So we can write

$$A = \begin{pmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{pmatrix} = b \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + f \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

This shows that the set

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

spans V. Since $\dim(V) = 3$, Theorem 4.12 implies that $\mathcal{B}$ is automatically a basis for V.

Note: There are many other possible bases you might have found. The basis you found in this part will also impact your answer to part (b).

(b) (6 points) Find the coordinate vector of $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$, relative to the basis you found in part (a).

Since

$$A = 1 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

the coordinate vector is

$$[A]_{\mathcal{B}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

6. [14 points] If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ then $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ (you do **not** need to show this). Find bases for $\text{Nul}(A)$ and $\text{Col}(A)$. You do not need to prove that they are bases. (Make sure that you indicate which basis is which!)

Solution: A basis for the column space is

$$\mathcal{B}_{col} \left\{ \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \right\}.$$

Writing the solutions to $Ax = 0$ in parametric vector form gives

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Thus, a basis for $\text{Nul}(A)$ is

$$\mathcal{B}_{nul} = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Note that the rank-nullity theorem says that $\dim(\text{Col}(A)) + \dim(\text{Nul}(A)) = 4$. This is consistent with the sizes of the bases we found.

7. [12 points] In this problem you will critique an answer to a linear algebra problem given by Janey Student (Johnny's sister). **Note:** the reduced echelon form Janey found is correct.

Problem: Let

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \text{ and } \vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Is it possible to find a vector $\vec{v}_4$ so that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis of $\mathbb{R}^4$?

Janey Student's Answer: *Yes, it is possible*. We know that $\dim \mathbb{R}^4 = 4$, so a basis for $\mathbb{R}^4$ must contain 4 vectors. Therefore, if we find a vector $\vec{v}_4$ such that $\vec{v}_4$ is not in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$.

Let $\vec{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. The equation

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{v}_4$$

is inconsistent. We know this because

$$\text{rref}\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & | & \vec{v}_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has the row $\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$. This means that $\vec{v}_4$ is not in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and therefore $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$

Is Janey's answer correct? If it is, then draw a picture of Janey in the space below. If not, then find the error in her argument and indicate how you would correct it.

Solution: The answer is not correct. It is only possible to extend a *linearly independent* set of vectors to a basis. The set $\{v_1, v_2, v_3\}$ is *not* linearly independent, since $v_3 = 2v_1 - v_2$. Janey's mistake is that the theorem she wants to use does not actually apply in this case.

In fact, since $\{v_1, v_2, v_3\}$ is linearly independent, so is *any* set that contains v_1, v_2, v_3; in particular, it is impossible for a basis of $\mathbb{R}^4$ to contain all of v_1, v_2, and v_3.