Exam 2 Practice Questions Suggested by Quincy Loney

1. Suppose A is a 4×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about the reduced echelon form of A?

Solution: The RREF for A must have a pivot in the first 3 rows followed by a zero row. If there were less than 3 pivots, there would be infinitely many solutions. There is no possibility for more than 3 pivots as there are only 3 columns.

2. Let A be a 2×5 matrix with two pivots.
   (a) Does the homogeneous equation $A\mathbf{x} = \mathbf{0}$ have a non-trivial solution?
   (b) Does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$?

Solution:

(a) Yes. The matrix A has 5 columns and 2 pivots. Hence there are 3 free variables. (Note that one free variable is sufficient for an affirmative answer.)

(b) Yes. The matrix $[A|\mathbf{b}]$ has 2 rows and 2 pivots. Both pivots occur as pivots for A. There is no way to obtain a pivot in the augmented row.

3. For the given matrices, find a nonzero vector in Nul(A) and a nonzero vector in Col(A).

(a) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ -3 & 6 \\ 1 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 & -3 & 1 \\ -4 & 2 & 6 & -2 \end{bmatrix}$

Solution:
The null space of a matrix A is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
The column space of a matrix A is the set of all linear combinations of the columns of A.

(a) Row reduce the augmented matrix to write the basic variables in terms of the free variables. $\left[\begin{array}{cc|c} 2 & -4 & 0 \\ -1 & 2 & 0 \\ -3 & 6 & 0 \\ 1 & -2 & 0 \end{array}\right] \sim$

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$, so x_2 is free and $x_1 = 2x_2$. Choose any non-zero value for x_2, and write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

For example $\mathbf{x} = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{4} \end{bmatrix}$. (We can always check that $A\mathbf{x} = \mathbf{0}$.)

Choose any column of A, or non-zero linear combination of the columns to answer the second part of the question. For example $\begin{bmatrix} -e \\ \frac{e}{2} \\ \frac{3e}{2} \\ -\frac{e}{2} \end{bmatrix}$.

(b) $\left[\begin{array}{rrrr|r} 2 & -1 & -3 & 1 & 0 \\ -4 & 2 & 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 2 & -1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$, so that x_2, x_3, x_4 are free and

$x_1 = 0.5x_2 + 1.5x_3 - 0.5x_4$. One example is $x_2 = 2,\ x_3 = 0 = x_4$ so that $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

The first column is sufficient, $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

4. Let A be the matrix $\begin{bmatrix} 1 & -2 & -1 \\ 4 & -2 & 2 \\ -7 & 1 & -6 \end{bmatrix}$ and let $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$ be a vector in $\mathbb{R}^3$.

(a) Determine if $\mathbf{w}$ is in Nul(A).

(b) Determine if $\mathbf{w}$ is in Col(A).

Solution:

(a) The vector $\mathbf{w}$ is in Nul(A) because $A\mathbf{w} = \mathbf{0}$.

$$\begin{bmatrix} 1 & -2 & -1 \\ 4 & -2 & 2 \\ -7 & 1 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2-4+2 \\ 8-4-4 \\ -14+2+12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) The vector $\mathbf{w}$ is not in Col(A) because the system $A\mathbf{x} = \mathbf{w}$ is inconsistent.

$$\left[\begin{array}{rrr|r} 1 & -2 & -1 & 2 \\ 4 & -2 & 2 & 2 \\ -7 & 1 & -6 & -2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right].$$

5.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 12 \\ -6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 6 \\ h \end{bmatrix}$$

(a) For what values of h is $\mathbf{v}_3$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

(b) For what values of h is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent?

Solution:

(a) One way to solve this question would be to row reduce the augmented matrix

$$\left[\begin{array}{cc|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{array}\right] = \left[\begin{array}{rr|r} 1 & -3 & 5 \\ -4 & 12 & 6 \\ 2 & -6 & h \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -3 & 5 \\ 0 & 0 & 26 \\ 0 & 0 & h-10 \end{array}\right].$$

Since there is a pivot in the augmented column, the system of equations represented by this matrix is inconsistent. Therefore $\mathbf{v}_3$ can not be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and is not in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

Another valid solution is to notice that $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$, so they lie on the same line. It is impossible for $\mathbf{v}_3$ to be on this line since both of its first two components are positive. Hence $\mathbf{v}_3 \notin \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) This set is dependent because $\mathbf{v}_2 = -3\mathbf{v}_1$. Adding a vector to a dependent set, namely adding $\mathbf{v}_3$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ results in a dependent set. A choice for a dependence relation could be $3\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{0}$. Since this set is dependent regardless of the inclusion of $\mathbf{v}_3$, it will be dependent regardless of the choice of h. Therefore the set is dependent for all $h \in \mathbb{R}$.

6. Let $A = \begin{bmatrix} 1 & -2 & -1 \\ 4 & -2 & 2 \\ -7 & 1 & -6 \end{bmatrix}$.

(a) Find a basis for Nul(A)

(b) Find a basis for Col(A)

Solution:

(a) Find the general solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 4 & -2 & 2 & 0 \\ -7 & 1 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. The parametric form of the general solution is

$\mathbf{x} = s\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, so a basis for Nul($A$) is $\mathcal{B} = \left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

(b) The pivots for A occur in the first two columns, therefore the first two column vectors form a basis for Col(A), $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}\right\}$.

7. Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$. Explain why S cannot be a basis for $\mathbb{R}^n$

(a) When $k < n$.

(b) When $k > n$.

Solution:

(a) When $k < n$, the matrix $A = [\mathbf{v}_1 \cdots \mathbf{v}_k]$ has fewer columns than rows. Thus there can not be a pivot in every row. This means that there is at least one zero row in RREF(A). Augmenting A to solve the system $A\mathbf{x} = \mathbf{b}$, we can now find elements in $\mathbb{R}^n$ which are not in the span of S. So, S is not a basis of $\mathbb{R}^n$.

(b) When $k > n$ the matrix $A = [\mathbf{v}_1 \cdots \mathbf{v}_k]$ has fewer rows than columns. Thus there can not be a pivot in every column. This means that there is at least one column which is dependent on the columns preceding it. Since S is not a linearly independent set, it is not a basis of $\mathbb{R}^n$.

8. Find the vector $\mathbf{x}$ determined by the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ and the basis $\mathcal{B} = \left\{\begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}\right\}$.

Solution: Make sure that the weight c_i corresponds to ordered basis vector $\mathbf{b}_i$ for each $1 \le i \le 3$.

$$\mathbf{x} = -2\begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} - 1\begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ -6 \\ 7 \end{bmatrix}.$$

Note that the calculation above is merely the product of the change-of-basis matrix $P_{\mathcal{B}}$ and the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$. Symbolically we just found $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.

9. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$ relative to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}$.

Solution: Using the notation from the previous question, $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.

To solve for $[\mathbf{x}]_{\mathcal{B}}$, treat this matrix equation exactly like any other matrix equation $A\mathbf{x} = \mathbf{b}$. Augment $P_{\mathcal{B}}$ with $\mathbf{x}$ and row reduce. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -3 & -1 & 3 \\ 3 & 8 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$. Therefore $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

10. Show that the polynomials $1, \quad 1-t, \quad 2-4t+t^2, \quad 6-18t+9t^2-t^3$ form a basis of $\mathbb{P}_3$ and find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $17 - 27t + 6t^2$ relative to this basis.

Solution: Note that if the four polynomials are linearly independent then they form a basis since $\dim \mathbb{P}_3 = 4$. Even though it may be easy to see that this particular set of polynomials is independent because each is of a different degree, we should practice using the techniques of this chapter.

We can express each of the polynomials as coordinate vectors relative to the standard basis of $\mathbb{P}_3$; $\{1, \quad t, \quad t^2, \quad t^3\}$.

Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -18 \\ 9 \\ -1 \end{bmatrix} \right\}$ which is an ordered list of the coordinate vectors. The change-of-basis matrix $P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ with columns from $\mathcal{B}$, has 4 pivots, so the columns are independent.

Write $17 - 27t + 6t^2$ as the column vector $\begin{bmatrix} 17 \\ -27 \\ 6 \\ 0 \end{bmatrix}$ with respect to the standard basis, and then solve the appropriate matrix equation (of the form $A\mathbf{x} = \mathbf{b}$).

$[P_{\mathcal{B}}|\mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 6 & 17 \\ 0 & -1 & -4 & -18 & -27 \\ 0 & 0 & 1 & 9 & 6 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ which means that $\quad 17 - 27t + 6t^2 = 2(1) + 3(1 - t) + 6(2 - 4t + t^2) + 0(6 - 18t + 9t^2 - t^3)$.

11. Let V be a finite dimensional vector space. True-False-Why

(a) If there is a spanning set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ of vectors in V, then $\dim V \le p$.

(b) If there is a linearly independent set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ of vectors in V, then $\dim V \ge p$.

(c) If there is a linearly dependent set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ of vectors in V, then $\dim V \le p$.

(d) If there is a linearly dependent set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ of vectors in V, then $\dim V \ge p$.

Solution:

(a) True because a spanning set can always be pared down to a basis. If it is already an independent set, then $\dim V = p$.

(b) True because an independent set can be expanded to a basis. If it is already a spanning set, then $\dim V = p$.

(c) False. Consider $\mathbb{R}^2$ and $\{\mathbf{0}\}$. The set is dependent and has one element, yet $\dim \mathbb{R}^2 = 2$.

(d) False. Consider $\mathbb{R}^2$ and $\{e_1, e_2, \mathbf{0}\}$. The set is dependent, and has 3 elements. yet $\dim \mathbb{R}^2 = 2$.

12. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices. Define $T : M_{2\times 2} \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2a + 3b - 5c + 2d \\ a + b - c - d \\ 3a + 2b - 7d \end{bmatrix}$.

(a) Describe the kernel of T and a find basis for Ker(T).

(b) Is T one-to-one?

(c) Find a basis for the range of T.

(d) Is T onto?

Solution:

(a) Ker(T) is the set of all 2×2 matrices which are mapped to $\mathbf{0} \in \mathbb{R}^3$, i.e. $\begin{bmatrix} 2a + 3b - 5c + 2d \\ a + b - c - d \\ 3a + 2b - 7d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Solve the system of equations given by the coordinates by row reduction.

$$\left[\begin{array}{cccc|c} 2 & 3 & -5 & 2 & 0 \\ 1 & 1 & -1 & -1 & 0 \\ 3 & 2 & 0 & -7 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -5 & 0 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{aligned} a &= -2r + 5s \\ b &= 3r - 4s \\ c &= r \\ d &= s \end{aligned}$$ for free variables r and s.

Thus $\text{Ker}(T) = \left\{\begin{bmatrix} -2r + 5s & 3r - 4s \\ r & s \end{bmatrix} \in M_{2\times 2} \middle| r, s \in \mathbb{R}\right\}$.

A possible basis for Ker(T) can be $\left\{\begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 5 & -4 \\ 0 & 1 \end{bmatrix}\right\}$.

(b) This map is not one-to-one because it has a non-trivial kernel. In particular both basis vectors found above are mapped to $\mathbf{0} \in \mathbb{R}^3$.

(c) The range of T is the set of all vectors in $\mathbb{R}^3$ that are outputs of the function T, i.e. vectors of the form $\begin{bmatrix} 2a + 3b - 5c + 2d \\ a + b - c - d \\ 3a + 2b - 7d \end{bmatrix}$, which are spanned by $\left\{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -7 \end{bmatrix}\right\}$. These are the columns of the coefficient matrix that was row reduced in part (a).

Using the pivot columns we can write the basis $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$.

(d) Since the dimension of the range is 2, the range of T can not be all of $\mathbb{R}^3$, therefore this map is not onto.

13. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices. Define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$.
    (a) Show that T is a linear transformation.
    (b) Let B be in $M_{2\times 2}$ such that $B = B^T$. Find an A such that $T(A) = B$.
    (c) Show that the range of T is the set of B such that $B = B^T$.
    (d) Describe the kernel of T.

Solution:

(a) $T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) = T\left(\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \right) =$

$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} + \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix} = \ldots = T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) + T\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right)$.

The scalar multiple portion works as expected as well.

(b) If $B = B^T$, then $T(B) = B+B^T = B+B = 2B$. Choose $A = 0.5B$ so that $T(A) = 0.5T(B) = (0.5)(2)B = B$.

(c) Let B be in the range of T. Then $B = T(A)$ for some $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in $M_{2\times 2}$. We can calculate $B = T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} \\ a_{12}+a_{21} & a_{22}+a_{22} \end{bmatrix}$. Now it is easy to see that B^T is equal to B.

(d) $T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} \\ a_{12}+a_{21} & a_{22}+a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ means, $a_{11} = 0 = a_{22}$ and $a_{12} = -a_{21}$. So the kernel is the set of matrices of the from $\begin{bmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{bmatrix}$.

14. Let A be an $n \times n$ matrix whose columns are linearly independent. Explain why the columns of A^2 span $\mathbb{R}^n$.

Solution: Since A is a square matrix with linearly independent columns, A is row equivalent to I_n and therefore invertible. Since $A^2 = AA$ is a product of invertible matrices, it too is invertible and is row equivalent to I_n. Therefore the columns of A^2 span R^n.

15. Suppose a 6×8 matrix has rank 5.
    (a) Find dimNul(A)
    (b) Find dimRow(A)
    (c) Find rank A^T

Solution:

(a) The matrix has 8 columns and 5 of them are pivot columns, so 3 of them are non-pivot columns.

(b) The rank is also the dimension of the row space.

(c) The column space of a matrix is the same as the row space of its transpose.

16. Suppose a 6×8 matrix has two pivot columns.
    (a) Find $\dim \text{Nul}(A)$
    (b) Is $\text{Col}(A) = \mathbb{R}^2$?

Solution:

(a) There are 8 columns so 6 of them are non-pivot columns.

(b) The column space is a subspace of $\mathbb{R}^6$ which is <u>isomorphic</u> to $\mathbb{R}^2$... <u>not equal to</u>.