Math 304 Linear Algebra Fall 2018 Exam 1V1 Solutions

(1) (15 Points) Let $L_A : \mathbf{R}^5 \to \mathbf{R}^4$ be the linear function $L_A(X) = AX$ associated with the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$.

(a) Find all vectors in $\mathrm{Ker}(L_A) = \{X \in \mathbf{R}^5 \mid L_A(X) = 0\}$ in terms of free variables.

Solution: To find $\mathrm{Ker}(L_A)$ we must solve a linear system by row reducing

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 \\ 3 & 4 & 5 & 6 & 7 & 0 \\ 4 & 5 & 6 & 7 & 8 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r + 2s + 3t \\ x_2 = -2r - 3s - 4t \\ x_3 = r \in \mathbf{R} \\ x_4 = s \in \mathbf{R} \\ x_5 = t \in \mathbf{R} \end{array}.$$

$$\mathrm{Ker}(L) = \left\{ \begin{bmatrix} r + 2s + 3t \\ -2r - 3s - 4t \\ r \\ s \\ t \end{bmatrix} \in \mathbf{R}^5 \;\middle|\; r, s, t \in \mathbf{R} \right\}.$$

(b) Find all vectors in $\mathrm{Range}(L_A) = \{Y = L_A(X) \in \mathbf{R}^4 \mid X \in \mathbf{R}^5\}$ in terms of consistency conditions on the entries of $Y = [y_i]$.

Solution: $Y = [y_i] \in \mathrm{Range}(L)$ iff the following system is consistent:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & y_1 \\ 2 & 3 & 4 & 5 & 6 & y_2 \\ 3 & 4 & 5 & 6 & 7 & y_3 \\ 4 & 5 & 6 & 7 & 8 & y_4 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & -3y_1 + 2y_2 \\ 0 & 1 & 2 & 3 & 4 & 2y_1 - y_2 \\ 0 & 0 & 0 & 0 & 0 & y_1 - 2y_2 + y_3 \\ 0 & 0 & 0 & 0 & 0 & y_1 - y_2 - y_3 + y_4 \end{array}\right] \begin{array}{c} \text{is consistent iff} \\ 0 = y_1 - 2y_2 + y_3 \\ \text{and} \\ 0 = y_1 - y_2 - y_3 + y_4 \end{array}$$

(c) Determine whether L is injective and whether L is surjective.

Solution: L is not injective since by (a) more than one vector in $\mathbf{R}^5$ is sent to the zero vector, and L is not surjective since by (b) not all vectors of $\mathbf{R}^4$ are in $\mathrm{Range}(L)$.

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(2) (15 Points) Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ and $S : \mathbf{R}^2 \to \mathbf{R}^3$ be the functions

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ -x_1 - x_2 + 5x_3 \end{bmatrix} \text{ and } S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \\ 2y_1 + 3y_2 \end{bmatrix}.$$

(a) Find the matrices A and B such that $S(Y) = AY$ and $T(X) = BX$.

Solution: $S(Y) = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \\ 2y_1 + 3y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = AY$ and

$T(X) = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ -x_1 - x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = BX.$

(b) Use composition of functions to find the formula for the composition $(S \circ T)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Solution: $(S \circ T)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S\left(T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = S\begin{bmatrix} x_1 + 2x_2 - x_3 \\ -x_1 - x_2 + 5x_3 \end{bmatrix}$

$$= \begin{bmatrix} (x_1 + 2x_2 - x_3) + (-x_1 - x_2 + 5x_3) \\ (x_1 + 2x_2 - x_3) - (-x_1 - x_2 + 5x_3) \\ 2(x_1 + 2x_2 - x_3) + 3(-x_1 - x_2 + 5x_3) \end{bmatrix} = \begin{bmatrix} x_2 + 4x_3 \\ 2x_1 + 3x_2 - 6x_3 \\ -x_1 + x_2 + 13x_3 \end{bmatrix}.$$

(c) Use your answer to part (b) to find the matrix C such that $(S \circ T)(X) = CX$.

Solution: $(S \circ T)(X) = \begin{bmatrix} x_2 + 4x_3 \\ 2x_1 + 3x_2 - 6x_3 \\ -x_1 + x_2 + 13x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 \\ 2 & 3 & -6 \\ -1 & 1 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = CX.$

(d) What should be the relationship between the matrices A, B and C? Do your matrices satisfy that relation?

Solution: The relationship should be that $AB = C$ (matrix multiplication). Check that $AB = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} (1-1) & (2-1) & (-1+5) \\ (1+1) & (2+1) & (-1-5) \\ (2-3) & (4-3) & (-2+15) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 \\ 2 & 3 & -6 \\ -1 & 1 & 13 \end{bmatrix} =$ C.

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(3) (15 Points) Answer each question separately. No justifications are needed.

(a) If $A \in \mathbf{R}_n^m$ and $B \in \mathbf{R}_p^n$ are matrices such that $AB = 0_p^m$, then what condition on $rank(A)$ would *guarantee* that $B = 0_p^n$?

Solution: $AB = 0_p^m$ means for each column $1 \le k \le p$ of B we have $A(Col_k(B)) = 0_1^m$. The condition that $rank(A) = n$ would guarantee that the only solution to $[A|0_1^m]$ is trivial, so $Col_k(B) = 0_1^n$ so $B = 0_p^m$.

(b) If $A \in \mathbf{R}_n^m$ and $AX = 0$ has only the trivial solution, what is the most you can say about the relation between m and n?

Solution: If $n > m$ then $AX = 0$ would have at least one free variable, giving nontrivial solutions. So we must have $n \le m$.

(c) For a nonzero matrix $A \in \mathbf{R}_8^5$, what are the possible values of $rank(A)$?

Solution: If $A \in \mathbf{R}_8^5$ is not the zero matrix, $rank(A)$ is the number of leading ones in its RREF, so $1 \le rank(A) \le 5 = Min(5, 8)$ since each leading one occupies a row, there is at least one, and no more than the number of rows.

(d) What condition on the rank of $A \in \mathbf{R}_n^m$ is equivalent to the homogeneous linear system $AX = 0$ having nontrivial solutions?

Solution: When $rank(A) < n$ the homogeneous linear system $AX = 0$ has nontrivial solutions since there are free variables corresponding to columns without leading ones in the RREF.

(e) What condition on the rank of $A \in \mathbf{R}_n^m$ is equivalent to the non-homogeneous linear system $AX = B$ being inconsistent for some B?

Solution: If $rank(A) < m$ then $AX = B$ is inconsistent for some B because $[A|B]$ row reduces to $[C|D]$ with C in RREF, and C has at least one row of zeros, giving consistency conditions.

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(4) (15 Points) For $A \in \mathbf{R}^m_n$, let $L_A : \mathbf{R}^n \to \mathbf{R}^m$ be the linear function $L_A(X) = AX$. For $1 \le j \le n$ let $\mathbf{e}_j \in \mathbf{R}^n$ be the matrix with 1 in row j and 0 in all other rows. No justifications are needed for these questions.

(a) What is the relationship between A and $L_A(\mathbf{e}_j)$ for $1 \le j \le n$?

Solution: For $1 \le j \le n$, $L_A(\mathbf{e}_j) = Col_j(A)$ is the j^{th} column of A.

(b) What relation between m and n would guarantee that L_A is not injective?

Solution: If $n > m$ then L_A is not injective since there would be free variables in the solution to $L_A(X) = O$, giving a nontrivial kernel.

(c) What relation between m and n would guarantee that L_A is not surjective?

Solution: If $n < m$ then L_A is not surjective since more equations than variables would guarantee a row of zeros in the RREF of A, giving a consistency condition for $AX = B$.

(d) If $\text{rank}(A) = n$ what does that tell you about L_A?

Solution: If $\text{rank}(A) = n$ then L_A is injective since n leading ones in the RREF means a leading one in each column and there are no free variables.

(e) If $\text{rank}(A) = m$ what does that tell you about L_A?

Solution: If $\text{rank}(A) = m$ then L_A is surjective since m leading ones in the RREF means no zero rows so $AX = B$ is always consistent.

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(5) (15 Points) Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be linear with

$$L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad L(\mathbf{e}_2) = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \quad \text{and} \quad L(\mathbf{e}_3) = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

Find $A \in \mathbf{R}^3_3$ such that $L = L_A$, and check that $L(X) = L_A(X) = AX$ for all $X \in \mathbf{R}^3$.

Solution: The $A \in \mathbf{R}^3_3$ such that $L(X) = L_A(X) = AX$ for all $X \in \mathbf{R}^3$ would have to satisfy $A\mathbf{e}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $A\mathbf{e}_2 = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ and $A\mathbf{e}_3 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, but $A\mathbf{e}_j = Col_j(A)$ for $j = 1, 2, 3$, so $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \\ 3 & 4 & 2 \end{bmatrix}$. This works for all $X \in \mathbf{R}^3$ because L is linear, so

$$\begin{aligned} L(X) = L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= L(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = x_1 L(\mathbf{e}_1) + x_2 L(\mathbf{e}_2) + x_3 L(\mathbf{e}_3) \\ &= x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = AX. \end{aligned}$$
