1. (10 Pts) If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$ then the reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) (4 Pts) Find a basis for Col(A).

Solution: Since A row reduced to the matrix above with pivot columns 1 and 2, a basis for Col(A) can be $\{Col_1(A), Col_2(A)\}$.

(b) (4 Pts) Find a basis for Nul(A).

Solution: Since A row reduced to the matrix above, the solutions to $AX = 0$ are

$$\text{Nul}(A) = \left\{ \begin{bmatrix} r+2s \\ -2r-3s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^5 \;\middle|\; r, s \in \mathbb{R} \right\} \text{ so } \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is}$$

a basis for Nul(A).

(c) (2 Pts) Find the dimension of Row(A).

Solution: Since there are 2 non-zero rows in the $RREF(A)$, $\dim(\text{Row}(A)) = 2$.

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2. (9 Pts) Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$. Consider the following statements.

(A) The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.

(B) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent.

(C) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ spans $\mathbb{R}^n$.

(D) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ does not span $\mathbb{R}^n$.

(E) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for $\mathbb{R}^n$.

(F) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is not a basis for $\mathbb{R}^n$.

In each part, determine which of the statements (A)–(F) *must* be true about $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

There may be more than one true statement. Choose all that apply. No justification required.

Solutions:

(a) (3 Pts) $k < n$ D and F

(b) (3 Pts) $k > n$ B and F

(c) (3 Pts) $k \neq n$ F

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3. (8 Pts) Answer each part. No justification required.

Solutions:

(a) (2 Pts) If $T : \mathbb{R}^8 \to \mathbb{R}^3_2$ is a surjective linear transformation, then $\dim \operatorname{Ker}(T) = 2$

(b) (2 Pts) Can a linear map $L : \mathbb{R}^6 \to \mathcal{P}_4(t)$ be injective? <u>No</u>

(c) (2 Pts) Select the word that makes the sentence true.
If $w \notin \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$, and the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ are linearly independent, then the set $\{w, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is *never* linearly dependent.

(d) (2 Pts) If none of the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are scalar multiples of one another, is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *necessarily* linearly independent? <u>No</u>

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4. (8 Pts) Find the inverse of each matrix, or determine that it is not invertible.

Solutions:

(a) (4 Pts) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) (4 Pts) $\begin{bmatrix} -1 & 3 & 2 \\ 3 & -9 & 1 \\ 2 & -6 & 3 \end{bmatrix}$ row reduces to a matrix with a zero row, so it is not invertible.

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5. (10 Pts) Let W be the set of all $n \times n$ symmetric matrices; that is,
$W = \{A \in \mathbb{R}^n_n : A = A^T\}$.

a. (5 Pts) Show that W is a subspace of $\mathbb{R}^n_n$.

Solution:

For any $A, B \in W$ we have $A^T = A$ and $B^T = B$ so $(A+B)^T = A^T + B^T = A + B$ so $A + B \in W$.
For any $A \in W$ and $r \in \mathbb{R}$ we have $(rA)^T = r(A^T) = rA$ so $rA \in W$.
Finally, $(0^n_n)^T = 0^n_n$ so $0^n_n \in W$. So W is a **subspace** of $\mathbb{R}^n_n$.

b. (5 Pts) When $n = 2$, find a basis for W.

Solution:

When $n = 2$, a basis for $W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

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6. (10 Pts) In $\mathcal{P}_2(t)$, let

$$p_1(t) = t^2 + 1,\ p_2(t) = t^2 + t,\ p_3(t) = t + 1.$$

(a) (4 Pts) Show that the set $\mathcal{B} = \{p_1(t), p_2(t), p_3(t)\}$ is linearly independent.

Solution: We need to show that $\sum_{i=1}^{3} x_i p_i(t) = 0$ has only the trivial solution. It means $(x_1+x_2)t^2 + (x_2+x_3)t + (x_1+x_3) = 0$ so we get three equations in three variables, since each coefficient must be 0. Solving the system by row reduction or by hand gives the only solution is $x_1 = x_2 = x_3 = 0$.

(b) (2 Pts) Without further calculation explain why $\{p_1(t), p_2(t), p_3(t)\}$ is a basis for $\mathcal{P}_2(t)$.

Solution: $\dim(\mathcal{P}_2(t)) = 3$ so any three independent vectors in $\mathcal{P}_2(t)$ form a basis of it.

(c) (4 Pts) If $q(t) = t^2 + 2$, find the coordinate vector $[q(t)]_{\mathcal{B}}$.

Solution: We need to solve $\sum_{i=1}^{3} x_i p_i(t) = q(t)$, that is,

$$(x_1 + x_2)t^2 + (x_2 + x_3)t + (x_1 + x_3) = t^2 + 2$$

which gives the linear system: $x_1 + x_2 = 1$, $x_2 + x_3 = 0$, $x_1 + x_3 = 2$. Solve it by row reducing

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{array}\right] \text{ to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \text{ so } [q(t)]_{\mathcal{B}} = \begin{bmatrix} 3/2 \\ -1/2 \\ 1/2 \end{bmatrix}.$$

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7. (5 Pts) For what values of r is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent? The vectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ r \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} r \\ 3 \\ 3 \end{bmatrix}.$$

Solution: The set is dependent when $r = 0$ or when $r = 3$. To see this, row reduce $\begin{bmatrix} 1 & 1 & r \\ r & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$ to $\begin{bmatrix} 1 & 1 & r \\ r & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. It has a zero row when $r = 0$. When $r \neq 0$ it reduces further to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & (3-r) \end{bmatrix}$, which has rank 2 when $r = 3$, but has rank 3 otherwise.

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8. (5 Pts) Let

$$p_1(t) = 1 + 2t + 3t^2, \quad p_2(t) = 3 - t, \quad p_3(t) = 2 + 2t^2.$$

Then $\mathcal{T} = \{p_1(t), p_2(t), p_3(t)\}$ is a basis of $\mathcal{P}_2(t)$. Suppose $q(t) \in \mathcal{P}_2(t)$ has the following coordinates with respect to $\mathcal{T}$: $[q(t)]_{\mathcal{T}} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$. Find the polynomial $q(t)$.

Solution: From those coordinates with respect to $\mathcal{T}$ we have that

$$q(t) = -2(1+2t+3t^2)+1(3-t)-1(2+2t^2) = -2-4t-6t^2+3-t-2-2t^2 = -1-5t-8t^2.$$

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9. (10 Pts) Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

(a) (5 Pts) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear function defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x - y + 3z \\ x + 2y - 4z \end{bmatrix}.$$

Find the matrix for L relative to the bases $\mathcal{B}$ and $\mathcal{C}$. That is, find ${}_{\mathcal{C}}L_{\mathcal{B}}$.

Solution: First we find $L(\mathcal{B})$: $L\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, $L\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Row reduce $\left[\begin{array}{cc|ccc} 1 & 1 & 4 & 2 & -1 \\ 1 & 2 & -1 & -2 & 2 \\ \multicolumn{2}{c|}{\scriptstyle \mathcal{C}} & \multicolumn{3}{c}{\scriptstyle L(\mathcal{B})} \end{array}\right]$ to $\left[\begin{array}{cc|ccc} 1 & 0 & 9 & 6 & -4 \\ 0 & 1 & -5 & -4 & 3 \\ \multicolumn{2}{c|}{\scriptstyle I_2} & \multicolumn{3}{c}{\scriptstyle {}_{\mathcal{C}}L_{\mathcal{B}}} \end{array}\right]$ so ${}_{\mathcal{C}}L_{\mathcal{B}} = \begin{bmatrix} 9 & 6 & -4 \\ -5 & -4 & 3 \end{bmatrix}$

(b) (5 Pts) If $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis for $\mathbb{R}^3$, find ${}_{\mathcal{B}}P_{\mathcal{E}}$ = the change of basis matrix *from* $\mathcal{E}$ *to* $\mathcal{B}$.

Solution: Row reduce $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ \multicolumn{3}{c|}{\scriptstyle \mathcal{B}} & \multicolumn{3}{c}{\scriptstyle \mathcal{E}} \end{array}\right]$ to $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ \multicolumn{3}{c|}{\scriptstyle I_3} & \multicolumn{3}{c}{\scriptstyle {}_{\mathcal{B}}P_{\mathcal{E}}} \end{array}\right]$ so ${}_{\mathcal{B}}P_{\mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.