PRACTICE PROBLEMS (2018-1116) FOR EXAM 3, MATH 304

Problem 1: 21 points= 2+3+3+4+4+5 points
Compute the determinants of the following square matrices:

$$A = \begin{bmatrix} \sin(-x) & \cos(-x) \\ \cos(x) & \sin(x) \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & -5 & -4 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 4 & 8 \\ -3 & -4 & 2 \\ 5 & 2 & 7 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & -1 & 1 & 4 \\ 2 & 0 & 3 & 5 \\ 1 & -3 & 9 & 22 \\ -10 & -4 & -5 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 7 & 3 & 5 \\ 2 & 14 & 6 & 7 \\ -3 & -21 & 4 & 2 \\ 2 & 15 & 3 & 4 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 4 & -2 & -3 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 \\ 3 & 11 & -9 & 8 & 12 \\ 0 & 2 & 3 & 1 & 5 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & -3 & 0 & 2 & 4 \end{bmatrix}.$$

Problem 2: 21 points = 7 times 3 points

(a) If A and B and C are square matrices of the same size such that $\det(A) = 12$, $\det(B) = 4$ and $\det(C) = 3$, find $\det(A^{-1}B^2C)$.

(b) If A and B and C are square matrices of the same size such that $\det(A) = 3$, $\det(B) = -7$ and $\det(C) = 2$, find $\det(A^T B^{-1} C^3)$.

(c) If A is a square matrix such that there exists a linear dependence relation among the columns of A, find $\det(A)$.

(d) If A is a square matrix and v is an eigenvector of A of eigenvalue λ, then show that v is an eigenvector of A^2 of eigenvalue λ^2.

(e) Suppose $A, B, P \in \mathbb{R}^n_n$ are such that $B = PAP$, $\det(B) = 45$ and $\det(A) = 5$. Find all possible values for $\det(P)$.

(f) Suppose $A, B, P \in \mathbb{R}^n_n$ are such that $B^T = P^{-1}AP$ and $\det(A) = 5$. Find $\det(B)$.

(g) If $A = [a_{ij}]_{1 \le i,j \le 4} \in \mathbb{R}^4_4$ is such that we have $a_{ij} = i - j$ for all $i, j \in \{1, 2, 3, 4\}$, find $\det(A)$.

Problem 3: 12 points =4 times 3 points

Fill in the dotted parts and justify your answers:

(a) If A is a matrix of size 5×5 and determinant 1, then $\det(-2A)$ is ....

(b) Let B be a matrix of size 4×4 and determinant 1. Let C be the matrix who rows are: $\text{Row}_1(C) = \text{Row}_3(B)$, $\text{Row}_2(C) = \text{Row}_1(B) - \text{Row}_2(B)$, $\text{Row}_3(C) = -2\text{Row}_1(B) + \text{Row}_2(B)$, $\text{Row}_4(C) = \text{Row}_4(B)$. Then $\det(C)$ is ....

(c) Let D be a matrix of size 7×7. If D is antisymmetric (skew symmetric), i.e., we have $D^T = -D$, then $\det(D)$ is ....

(d) If E is a square matrix whose characteristic polynomial is $p_E(x) = x^4 + x^3 + x^2 + x + 1$ and if $F = -2E$, then the characteristic polynomial $p_F(x)$ of F is ....

Problem 4: 14 points = 7 times 2 points

Fill in the dotted part of each one of the below seven sentences with one of the following three capital letters, whose meanings are described as follows:

D A is diagonalizable.

N A is not diagonalizable.

U it cannot be decided if A is or is not diagonalizable.

(a) If the characteristic polynomial of a square matrix A is $p_A(x) = x^2 - 3$, then ....

(b) If the characteristic polynomial of a square matrix A is $p_A(x) = x^2 + 6x + 9$, then ....

(c) If the characteristic polynomial of a square matrix A is $p_A(x) = x^2 + 3x + 3$, then ....

(d) If the characteristic polynomial of a square matrix A is $p_A(x) = -x^3 + 3x$, then ....

(e) If the characteristic polynomial of a square matrix A is $p_A(x) = -x^3$, then ....

(f) If the characteristic polynomial of a square matrix A is $p_A(x) = -x^3 - 3x^2 - 3x - 1$, then ....

(g) If the characteristic polynomial of a square matrix A is $p_A(x) = -x^3 + 3x^2$, then ....

Problem 5: 7 points = 7 times 1 point

Mark each one of the following seven statements as T (true) or F (false) on the left margin of the page (no explanation is required but marking is required):

(a) If A and P are matrices of size $n \times n$ with P invertible, then $\det(P^{-1}AP) = \det(A)$.

(b) The matrices

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are similar.

(c) The matrices

$$C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

are similar.

(d) If A and B are 3×3 matrices such that $\det(A) = 1$ and $\det(B) = 0$, then $\text{rank}(A) > \text{rank}(B)$.

(e) If A is a 6×6 matrix of rank 5 and characteristic polynomial $p_A(x) = x^2(x-1)(x-2)(x-3)(x-4)$, then A is not diagonalizable.

(f) If A is a square matrix of size $n \times n$ with $n \geq 2$ such that all its entries are odd integers, then $\det(A)$ is odd.

(g) If A is a square matrix of size 3×3 and rank 2, then x^2 divides the characteristic polynomial $p_A(x)$ of A.

Problem 6: 7 points = 7 times 1 point

Mark each one of the following seven statements as T (true) or F (false) on the left margin of the page (no explanation is required but marking is required):

(a) If B is a square matrix of size 3×3 such that all its entries are even integers, then $\det(B)$ is divisible by 16.

(b) If A is a square matrix which is not diagonalizable, then A^2 is not diagonalizable.

(c) If B is a square matrix of size 6×6 and rank 3, then x^3 divides the characteristic polynomial $p_B(x)$ of B.

(d) If A and B are square matrices of the same size, then $\det(A+B) = \det(A) + \det(B)$.

(e) There exists an invertible matrix whose determinant is 0.

(f) If A and B are square matrices whose characteristic polynomials are equal to $x^2 - 3$, then A and B are similar.

(g) If A and B are square matrices whose characteristic polynomials are equal to $x^2 - 4x + 4$, then A and B are similar.

Problem 7: 4 points

Compute the characteristic polynomial of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ -2 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}.$$

Problem 8: 6 points= 2 points + 4 points

We consider the following matrix

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}.$$

(a) Compute the characteristic polynomial $p_A(x)$ of A.

(b) Decide if A is or is not diagonalizable. If it is, then find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Problem 9: 12 points = 4 times 3 points

We consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of A.

(b) Find all the eigenvalues of A with their algebraic and geometric multiplicities.

(c) For each eigenvalue of A, find a basis for the associated (corresponding) eigenspace of A.

(d) Show that A is diagonalizable. Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Problem 10: 6 points

We consider the following two matrices:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 5 & -4 \\ 0 & 6 & -5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Decide if A and B are or are not similar. Show all the work necessary to come to the correct conclusion (answer).

Problem 11: 8 points= 2 times 4 points

We consider the following two matrices:

$$A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}.$$

(a) Find all natural numbers m and n such that $A^m = B^n$. If none exist, explain why they do not exist.

(b) Find all natural numbers m and n such that A^m and B^n are similar. If none exist, explain why they do not exist.

Problem 12: 16 points = 4 times 4 points

(a) Show that for each square matrix A, the characteristic polynomials of A and A^T are equal.

(b) Show that if A and B are two similar square matrices, then the characteristic polynomials of A and B are equal.

(c) Let A and P be square matrices of size $n \times n$. We assume that P is invertible. Let λ be an eigenvalue of A. Let E_λ be the eigenspace of A corresponding to the eigenvalue λ. Show that $P(E_\lambda) = \{Pv | v \in E_\lambda\}$ is the eigenspace of PAP^{-1} corresponding to the eigenvalue λ.

(d) Let V be a vector space of dimension n. Let $L : V \to V$ be a linear transformation. Let X and Y be two ordered bases for V. Show that the matrix representations ${}_XL_X$ and ${}_YL_Y$ are similar.

Problem 13: 15 points= 5 points +3 points + 7 points

For each $A \in \mathbb{R}^n_n$ given below define $L = L_A : \mathbb{R}^n \to \mathbb{R}^n$ by $L(X) = AX$. So $A = {}_SL_S$ is the matrix representing L with respect to the standard basis S of $\mathbb{R}^n$. Answer the following questions.

(a) Find the characteristic polynomial $p_A(x) = \det(A - xI_n)$ of A.

(b) Find the eigenvalues of A (the roots of $p_A(x)$) and their algebraic multiplicities.

(c) Is A is diagonalizable? If it is not, then give reasons why. If it is, then find a basis T of $\mathbb{R}^n$ and a diagonal matrix $D = {}_T L_T$ representing L from T to T. Also find the change of basis (transition) matrix $P = {}_S I_T$.

(i)

$$A = \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix}.$$

(ii)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

(iii)

$$A = \begin{bmatrix} 8 & 0 & -3 \\ 0 & 5 & 0 \\ 6 & 0 & -1 \end{bmatrix}.$$

Problem 14: 8 points= 4 times 2 points

Consider the matrix with real entries

$$A = \begin{bmatrix} 1 & t \\ 1 & -1 \end{bmatrix}.$$

(a) Find the eigenvalues of A (they will depend on the value of $t \in \mathbb{R}$).

(b) Check that for precisely one particular value of t (and say which one), the matrix A is similar to every matrix of the form

$$B = \begin{bmatrix} 2 & * \\ 0 & -2 \end{bmatrix}$$

(where $*$ represents an arbitrary real number).

(c) Find all values $t \in \mathbb{R}$ for which the matrix A is not diagonalizable.

(d) Find all values $t \in \mathbb{R}$ for which the matrix A is not diagonalizable but it has an eigenvalue.

Problem 15: 7 points = 2 points + 5 points

Let A be a 2×2 matrix with eigenvalues 1 and $\frac{1}{2}$ and corresponding eigenvectors $v_1 = [1\ 1]^T$ and $v_2 = [-1\ 1]^T$ (so $Av_1 = v_1$ and $Av_2 = \frac{1}{2}v_2$). For each integer $k \geq 0$, let $w_k = A^k w_0$, where $w_0 = [5\ 3]^T$.

(a) Find w_1.

(b) For each integer $k \geq 0$, find $a_k, b_k \in \mathbb{R}$ such that $w_k = a_k v_1 + b_k v_2$.

Problem 16: 10 points = 5 times 2 points

We consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Is 5 an eigenvalue of A? If yes, find the eigenspace of A for the eigenvalue 5.

(b) Is 1 an eigenvalue of A? If yes, find the eigenspace of A for the eigenvalue 1.

(c) Calculate A^2. Find the eigenspace of A^2 for the eigenvalue 1.

(d) Is it possible to find a basis of $\mathbb{R}^3$ consisting entirely of eigenvectors of A?

(e) Is it possible to find a basis of $\mathbb{R}^3$ consisting entirely of eigenvectors of A^2?

Problem 17: 6 points = 3 times 2 points
Give examples of:

(a) two square matrices which have the same characteristic polynomial but are not similar;

(b) a square matrix B which is not diagonalizable, but B^2 is diagonalizable;

(c) a square matrix which has no eigenvalue.

Problem 18: 15 points = 5 times 3 points
We consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Find a basis for each eigenspace of A.

(c) What are the algebraic multiplicity and geometric multiplicity of each eigenvalue?

(d) Is A diagonalizable? If it is, write down an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(e) Compute A^{2019}.

Problem 19: 15 points = 1 point + 10 points + 4 points
Let

$$A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 4 & -1 & 2 & 0 \\ -4 & 6 & -3 & 0 \\ 2 & 0 & 2 & 1 \end{bmatrix}.$$

(a) Check that $v = [1\ 0\ -2\ -1]^T$ is an eigenvector of A.

(b) Find a diagonal matrix D and an invertible matrix P, such that $A = PDP^{-1}$. **(c)** Compute A^n for all $n \in \mathbb{N}$.

Problem 20: 10 points
Find a matrix P such that for

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix}$$

the matrix $D = P^{-1}AP$ is diagonal.

SOME ANSWERS and SOLUTIONS (at least one for each problem)

P1 (a) $\det(A) = -1$.
P2 (c) $\det(A) = 0$.
P3 (c) $\det(D) = 0$.
P3 (d) We have $p_F(x) = \det(F - xI_4) = \det(-2E - xI_4) = \det(-2(E + \frac{x}{2}I_4)) = (-2)^4 \det(E + \frac{x}{2}I_4) = 16p_E(-\frac{x}{2}) = 16(\frac{1}{16}x^4 - \frac{1}{8}x^3 + \frac{1}{4}x^2 - \frac{1}{2}x + 1) = x^2 - 2x^3 + 4x^2 - 8x + 16$.
P4 (a) D.

P5 (a) T.
P6 (e) F.
P7 (a) $p_A(x) = -x^3 + x^2 + 5x + 11$.
P8 (a) $p_A(x) = x^2 - x - 6$.
P9 (a) $p_A(x) = -x(x-1)(x-2)$.

P10 We have $p_A(x) = p_B(x) = -(1+x)(x-1)^2$. Both A and B have eigenvalue 1 with geometric multiplicity 2 and eigenvalue -1 with geometric multiplicity 1. As $3 = 2+1$, we get that A and B are diagonalizable. Thus both A and B are similar to the diagonal matrix D whose entries on the main diagonal are -1, 1 and 1. We get that there exist invertible matrices P and Q such that $A = PDP^{-1}$ and $B = QDQ^{-1}$. It follows that $D = P^{-1}AP$ and therefore $B = QP^{-1}APQ^{-1} = RAR^{-1}$, where $R = QP^{-1}$ is invertible. We conclude that A and B are similar.

P11 (a) We have $p_A(x) = x^2 - 1 = (x-1)(x+1)$ and thus there exists an invertible matrix P and a diagonal matrix D with the entries 1 and -1 on the main diagonal such that $A = PDP^{-1}$. If m is odd, then $D^m = D$ and if m is even, then $D^m = I_2$. Thus, if m is odd, then $A^m = PD^mP^{-1} = PDP^{-1} = A$ and if m is even, then $A^m = PD^mP^{-1} = PI_2P^{-1} = PP^{-1} = I_2$.

Similarly, $p_B(x) = x^2 - 1 = (x-1)(x+1)$ and there exists an invertible matrix Q such that $B = QDQ^{-1}$ (same D). Moreover, similarly, if n is odd, then $B^n = B$ and if n is even, then $B^n = I_2$.

As A, B, I_2 are distinct, we conclude that we have $A^m = B^n$ if and only if n and m are both even (natural numbers).

P11 (b) If n and m are both odd, then $A^m = A$ and $B^n = B$ are similar as both are similar to D (compare with P10). If n and m are both even, then $A^m = B^n = I_2$ are similar (being identical). If m is odd and n is even, then $A^m = A$ is not similar to $B^n = I_2$ (note that I_2 is only similar to itself). If m is even and n is odd, then $A^n = I_2$ is not similar to $B^n = B$. Thus, the answer is: A^m and B^n are similar if and only if n and m have the same parity (i.e., 2 divides $n - m$).

P12 (b) See textbook (Lemma 5.3.17 on page 232).

P13 (b) for the matrix of (ii) We have $p_A(x) = (2-x)(x-1)^2$.

P14 (a) We have $p_A(x) = x^2 - 1 - t$, and therefore the eigenvalues of A are the solutions of the equation $x^2 = t + 1$. If $-1 - t > 0$, i.e., if $t < -1$, then A has no eigenvalue. If $-1 - t = 0$, i.e., $t = -1$, then the only eigenvalue of A is 0. If $-1 - t < 0$, i.e., $t > -1$, then A has exactly two eigenvalues: $\sqrt{t+1}$ and $-\sqrt{t+1}$.

P14 (b) We have $p_B(x) = x^2 - 4$ and $p_A(x) = x^2 - 1 - t$. Only for $t = 3$ we get that $p_A(x) = p_B(x)$. As $p_B(x) = (x-2)(x+2)$ has two distinct zeros 2 and -2, we get that only for $t = 3$ the matrices A and B are similar (both are similar to the diagonal matrix whose entries on the main diagonal are 2 and -2; compare with $P10$).

P14 (c) If $t < -1$, then A is not diagonalizable as it has no eigenvalue. If $t > -1$, then A has two distinct eigenvalues and therefore it is diagonalizable. If $t = -1$, then the only eigenvalue 0 of A has geometric multiplicity -1 and therefore A is not diagonalizable. Final answer: $t \leq -1$ (or $t \in (-\infty, -1]$).

P14 (d) From (a) and (c) we get that the answer is $t = -1$.

P15 (a) + (b) We have $w_0 = 4v_1 - v_2$. Thus $w_1 = Aw_0 = 4Av_1 - Av_2 = 4v_1 - \frac{1}{2}v_2 = [4.5\ 3.5]^T$. We have $w_k = A^k w_0 = A^k(4v_1 - v_2) = 4A^k v_1 - A^k v_2 = 4v_1 - 2^{-k}v_2$. Therefore $a_k = 1$ and $b_k = -2^{-k}$.

P16 (a) No, as for $p_A(x) = -(x-1)^2(x+1)$ we have $p_A(5) \neq 0$.

P17 (a+b) Let $A = 0_2^2$ and let B be such that $\text{Row}_1(B) = [0\ 1]$ and $\text{Row}_2(B) = [0\ 0]$. We have $p_A(x) = p_B(x) = x^2$. Moreover, $\dim(\text{Null}(A)) = 2$ while $\dim(\text{Null}(B)) = 1$ and thus for the eigenvalue 0 the geometric multiplicities of A and B are distinct. Therefore A and B are not similar, as desired. As for the eigenvalue 0, the algebraic multiplicity 2 of B is greater than the geometric multiplicity 1 of B, B is not diagonalizable. But $B^2 = A = O_2^2$ is diagonal and thus diagonalizable.

P18 (a) -1, 0, and 1, as $p_A(x) = -x(x-1)(x+1)$.

P19 (a) We have $Av = -v$. Thus v is an eigenvector of A of eigenvalue -1.

P20 See textbook for a full solution (Exercise S5-9 (h) on page 237).