Math 304 Fall 2018 Exam 3 Solutions

1. (18 Points, 3 Pts each part) Let A, B, C, D be square matrices of the same size such that

$$\det(A) = 2, \qquad \det(B) = \sqrt{2}, \qquad \det(C) = \frac{1}{2}, \qquad \det(D) = 4.$$

   (a) Compute $\det(AD) + \det((B^2C)^T) = (\det(A))(\det(D)) + (\det(B))^2(\det(C)) = (2)(4) + (\sqrt{2})^2(\frac{1}{2}) = 8 + 1 = 9$.

   (b) $\det(D^T A^2 B^{-1}) = \det(D)(\det(A))^2(\det(B))^{-1} = (4)(2)^2(\sqrt{2})^{-1} = \frac{16}{\sqrt{2}} = 8\sqrt{2}$.

   (c) If E is a 5×5 matrix such that $\mathrm{Row}_2(E) = \mathrm{Row}_1(E) - 3\mathrm{Row}_3(E)$, compute $\det(E)$.

   Solution: $\det(E) = 0$ since there is a dependence relation among the rows of E.

   (d) If F is a 3×3 matrix such that $F^3 = 4F$, then find the possible values of $\det(F)$.

   Solution: We have $\det(F^3) = \det(4F)$ so $(\det(F))^3 = 4^3(\det(F))$ since a factor of 4 comes out of each of the three rows of $4F$. This says $\det(F)((\det(F))^2 - 4^3) = 0$ so either $\det(F) = 0$ or else $(\det(F))^2 = 4^3$ so $|\det(F)| = 2^3 = 8$. Finally, the possible values of $\det(F)$ are 0, 8, −8. These occur for $F = 0_3^3$ and $F = \pm 2I_3$.

   Another Solution: If F is not invertible, then $\det(F) = 0$. If F is invertible, then we get $F^2 = 4I_3$ after multiplying $F^3 = 4F$ by F^{-1}. So $(\det(F))^2 = 4^3 = 64$ so $\det(F) = \pm 8$.

   (e) If $G = [g_{ij}]_{1 \le i,j \le 3} \in \mathbb{R}_3^3$ is such that $g_{ij} = i^2 - j^2$ for all $i, j \in \{1, 2, 3\}$, find $\det(G)$.

   Solution: Since $g_{ij} = i^2 - j^2$ we see that $G = \begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}$ so $\det(G) = 0 + (-3)(-5)(8) + (-8)(3)(5) - 0 - 0 - 0 = 120 - 120 = 0$ by the crosshatching method. Using row operations, G is row equivalent to a matrix with a zero row, so $\det(G) = 0$. Also, $G^T = -G$, and the size of G is odd, so that gives the result.

   (f) Let H be a 4×4 matrix with determinant 2. Let J be the matrix whose rows are: $\mathrm{Row}_1(J) = \mathrm{Row}_3(H)$, $\mathrm{Row}_2(J) = 2\mathrm{Row}_1(H) - 3\mathrm{Row}_2(H)$, $\mathrm{Row}_3(J) = -5\mathrm{Row}_1(H) + \mathrm{Row}_2(H)$, and $\mathrm{Row}_4(J) = \mathrm{Row}_4(H)$. Compute $\det(J)$.

   Solution: Using $R_i(H) = Row_i(H)$, we have $\det(J) = \det \begin{bmatrix} R_3(H) \\ 2R_1(H) - 3R_2(H) \\ -5R_1(H) + R_2(H) \\ R_4(H) \end{bmatrix}$

$$= -\det \begin{bmatrix} -5R_1(H) + R_2(H) \\ 2R_1(H) - 3R_2(H) \\ R_3(H) \\ R_4(H) \end{bmatrix} = \det \begin{bmatrix} 5R_1(H) - R_2(H) \\ 2R_1(H) - 3R_2(H) \\ R_3(H) \\ R_4(H) \end{bmatrix} = \det \begin{bmatrix} R_1(H) + 5R_2(H) \\ 2R_1(H) - 3R_2(H) \\ R_3(H) \\ R_4(H) \end{bmatrix}$$

$$= \det \begin{bmatrix} R_1(H) + 5R_2(H) \\ -13R_2(H) \\ R_3(H) \\ R_4(H) \end{bmatrix} = -13\det \begin{bmatrix} R_1(H) + 5R_2(H) \\ R_2(H) \\ R_3(H) \\ R_4(H) \end{bmatrix} = -13\det \begin{bmatrix} R_1(H) \\ R_2(H) \\ R_3(H) \\ R_4(H) \end{bmatrix} = -13\det(H) =$$

$(-13)(2) = -26$.

You do not need to show any work for this page.

2. (10 Points, 2 pts for each part) In the space provided, write **ONE** of the following three capital letters, **D**, **N** , or **U**. Ambiguous answers will receive zero points. The meanings of the letters are described as follows:

   D A is diagonalizable.

   N A is not diagonalizable.

   U it cannot be decided if A is or is not diagonalizable.

   (a) If the characteristic polynomial of a square matrix A is $\mathcal{P}_A(x) = x^2 - 3x + 6$, then $\underline{N}$.
   Solution: $b^2-4ac = (-3)^2-4(1)(6) = 9-24 < 0$ so no real roots, A is not diagonalizable.

   (b) If the characteristic polynomial of a square matrix A is $\mathcal{P}_A(x) = x^2 + 10x + 25$, then $\underline{U}$.
   Solution: $\mathcal{P}_A(x) = (x + 5)^2$ so g_{-5} could be either 1 or 2.

   (c) If the characteristic polynomial of a square matrix A is $\mathcal{P}_A(x) = -x^3 - 1$, then $\underline{N}$.
   Solution: $\mathcal{P}_A(x) = -(x^3 + 1) = -(x + 1)(x^2 - x + 1)$ has only one real root since the quadratic factor has no real roots ($b^2 - 4ac = (-1)^2 - 4(1)(1) = 1 - 4 < 0$).

   (d) If the characteristic polynomial of a square matrix A is $\mathcal{P}_A(x) = -x^3 + x$, then $\underline{D}$.
   Solution: $\mathcal{P}_A(x) = -x^3 + x = -x(x^2 - 1) = -x(x + 1)(x - 1)$ has three distinct roots.

   (e) If the characteristic polynomial of a square matrix A is $\mathcal{P}_A(x) = x^4 - 1$, then $\underline{N}$.
   Solution: $\mathcal{P}_A(x) = x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$ has only two real roots from the linear factors, but the quadratic factor has no real roots.

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3. (14 Points, 6 for part (a), 8 for part (b)) Compute the determinant of each of the following matrices.

   (a) $$\det\begin{bmatrix}1&1&-1&-1\\3&3&4&1\\3&2&1&1\\0&1&-2&-3\end{bmatrix} = \det\begin{bmatrix}1&1&-1&-1\\0&0&7&4\\0&-1&4&4\\0&1&-2&-3\end{bmatrix} = \det\begin{bmatrix}1&1&-1&-1\\0&0&7&4\\0&0&2&1\\0&1&-2&-3\end{bmatrix} = -\det\begin{bmatrix}1&1&-1&-1\\0&1&-2&-3\\0&0&2&1\\0&0&7&4\end{bmatrix} =$$
   $$-\det\begin{bmatrix}1&1&-1&-1\\0&1&-2&-3\\0&0&2&1\\0&0&1&1\end{bmatrix} = \det\begin{bmatrix}1&1&-1&-1\\0&1&-2&-3\\0&0&1&1\\0&0&2&1\end{bmatrix} = \det\begin{bmatrix}1&1&-1&-1\\0&1&-2&-3\\0&0&1&1\\0&0&0&-1\end{bmatrix} = (1)^3(-1) = -1.$$
   This was done entirely by elementary row operations.

   (b) $$\det\begin{bmatrix}0&0&0&-3&0\\4&10&-9&8&12\\0&2&3&1&5\\0&1&1&1&1\\0&-3&0&1&0\end{bmatrix} = (-1)^{2+1}(4)\det\begin{bmatrix}0&0&-3&0\\2&3&1&5\\1&1&1&1\\-3&0&1&0\end{bmatrix} = (-4)(-1)^{1+3}(-3)\det\begin{bmatrix}2&3&5\\1&1&1\\-3&0&0\end{bmatrix} =$$
   $$(12)(-1)^{3+1}(-3)\det\begin{bmatrix}3&5\\1&1\end{bmatrix} = (-36)(3-5) = 72$$ This was done by cofactor expansions, first along column 1, then along row 1, then along row 3.

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4. (15 Points) Consider the matrix $A = \begin{bmatrix} -1 & -2 & 2 \\ 2 & 3 & -2 \\ 2 & 2 & -1 \end{bmatrix}$.

(a) (3 Pts) Compute the characteristic polynomial of A.

Solution: $\mathcal{P}_A(x) = \det \begin{bmatrix} -1-x & -2 & 2 \\ 2 & 3-x & -2 \\ 2 & 2 & -1-x \end{bmatrix} = \det \begin{bmatrix} 1-x & 0 & 1-x \\ 0 & 1-x & -1+x \\ 2 & 2 & -1-x \end{bmatrix} =$ $(x-1)^2 \det \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 2 & 2 & -1-x \end{bmatrix} = (x-1)^2 \det \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & -1-x \end{bmatrix} =$ $(x-1)^2(-1)(-1)(-x-1) = -(x-1)^2(x+1)$

(b) (3 Pts) Find all of the eigenvalues of A, with their algebraic multiplicities.

Solution: $\lambda_1 = 1$ with algebraic multiplicity $k_1 = 2$, and $\lambda_2 = -1$ with algebraic multiplicity $k_2 = 1$.

(c) (3 Pts) For each eigenvalue of A, compute its geometric multiplicity by finding a basis for the associated eigenspace of A.

Solution: For $\lambda_1 = 1$ find the eigenspace A_1 by solving $[A-1I_3|0_1^3]$: $A-1I_3 = \begin{bmatrix} -2 & -2 & 2 \\ 2 & 2 & -2 \\ 2 & 2 & -2 \end{bmatrix}$ row reduces to $C = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Solutions to $[C|0_1^3]$ are $x_1 = -r+s$, $x_2 = r$, $x_3 = s$, giving $A_1 = \left\{ r\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \mid r, s \in \mathbb{R} \right\}$, which has basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\dim(A_1) = 2 = g_1$ is the geometric multiplicity of λ_1.

For $\lambda_2 = -1$ find the eigenspace A_{-1} by solving $[A+1I_3|0_1^3]$: $A + 1I_3 = \begin{bmatrix} 0 & -2 & 2 \\ 2 & 4 & -2 \\ 2 & 2 & 0 \end{bmatrix}$ row reduces to $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Solutions to $[C|0_1^3]$ are $x_1 = -r$, $x_2 = r$, $x_3 = r$, giving $A_{-1} = \left\{ r\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \mid r \in \mathbb{R} \right\}$, which has basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $\dim(A_{-1}) = 1 = g_{-1}$ is the geometric multiplicity of λ_2.

(d) (1 Pt) Show that A is diagonalizable by computing the sum of the geometric multiplicities of its eigenvalues.

Solution: $g_1 + g_{-1} = 2 + 1 = 3 = \dim(\mathbb{R}^3)$ so A is diagonalizable.

(e) (3 Pts) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Solution: $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $P = {}_S P_T = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is the transition matrix from eigen-basis T to standard basis S, whose columns are the three eigen-basis vectors from A_1 and A_{-1}.

(f) (2 Pts) Compute A^{2018} and A^{2019}.

Solution: We find by a row reduction $P^{-1} = {}_T P_S = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$.

For any natural number m we have $A^m = PD^m P^{-1}$.

Since $D^m = \begin{bmatrix} 1^m & 0 & 0 \\ 0 & 1^m & 0 \\ 0 & 0 & (-1)^m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^m \end{bmatrix}$ equals I_3 when m is even, we get

$A^{2018} = PD^{2018}P^{-1} = PI_3P^{-1} = I_3$.

But $A^{2019} = A^{2018}A = I_3A = A$.

You must show all supporting work to receive credit.

5. (12 Points, 3 pts for each part) In the parts below, you may ONLY USE the following 2×2 matrices:

$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

$\mathcal{P}_A(x) = x^2$, $\mathcal{P}_B(x) = (x-1)^2$, $\mathcal{P}_C(x) = (x-1)^2$, $\mathcal{P}_D(x) = x^2+1$, $\mathcal{P}_E(x) = x^2-1$, $\mathcal{P}_F(x) = x^2$.

(a) Pick two distinct matrices that have the same characteristic polynomial.

Solution: A and F, or B and C.

(b) Pick a matrix that is not diagonalizable, but whose square is diagonalizable.

Solution: A works since $A^2 = 0_2^2$ is diagonal, but A is not diagonalizable. B does not work since it is diagonal. C does not work since $C^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable. D works since it is not diagonalizable (no real eigenvalues), and $D^2 = -I_2$ is diagonal. E does not work since it is diagonalizable (two distinct real eigenvalues). F works since it is not diagonalizable (geometric multiplicity is only 1), but $F^2 = 0_2^2$ is diagonal. In summary, correct choices could be A, D or F.

(c) Pick a matrix which has no (real) eigenvalues.

Solution: Only D has no real eigenvalues.

(d) Pick three matrices so that no two among them are similar.

Solution: Any three with all different characteristic polynomials, for example, $\{A, B, D\}$, $\{A, B, E\}$, $\{B, D, E\}$, $\{B, D, F\}$, $\{C, D, E\}$, $\{C, D, F\}$, $\{D, E, F\}$. Additional choices are possible since B and C are not similar since B is diagonal but C is not diagonalizable.

You do not need to show any work for this page.

(6 points, 1 point for each part) Write the entire word **True** or **False** in the space provided. Ambiguous answers will receive zero points.

6. (a) If A is a 4×4 matrix with rank 2, then x^2 is a factor of the characteristic polynomial of A, $\mathcal{P}_A(x)$. $\underline{True}$.

   Solution: $Rank(A) = 2$ means $g_0 = \dim(A_0) = 4 - 2 = 2 \leq k_0$ so x^2 is a factor in $\mathcal{P}_A(x)$.

   (b) If A is a 6×6 matrix of rank 4 with characteristic polynomial $\mathcal{P}_A(x) = x^2(x + 1)(x + 2)(x + 3)(x + 4)$, then A is not diagonalizable. $\underline{False}$.

   Solution: $Rank(A) = 4$ means $g_0 = \dim(A_0) = 6 - 4 = 2$. All other eigenvalues (-1, -2, -3, -4) have algebraic multiplicity equal 1 so geometric multiplicity equals 1, so get a total of 6 basis eigenvectors, and A is diagonalizable.

   (c) If B is an upper triangular, 3×3 matrix such that each one of its non-zero entries is an integer which is divisible by 3, then $\det(B)$ is divisible by 27. $\underline{True}$.

   Solution: $\det(B) = 3^3 \det(C)$ where C is obtained from B by factoring out 3 from each row of B. The divisibility of all entries of B by 3 means all entries of C are integers, so $\det(C)$ is an integer, and $3^3 = 27$, so $\det(B)$ is divisible by 27.

   (d) If A and B are similar 4×4 matrices, and if the matrices B and $3I_4$ are similar, then $A = B$. $\underline{True}$.

   Solution: $A = PBP^{-1}$ for some invertible P, and $B = Q(3I_4)Q^{-1} = 3QQ^{-1} = 3I_4$, so $A = P3I_4P^{-1} = 3PP^{-1} = 3I_4 = B$.

   (e) If matrices A and B are similar, and matrices B and C are similar, then A must be similar to C. $\underline{True}$.

   Solution: $A = PBP^{-1}$ for some invertible P, and $B = QCQ^{-1}$ for some invertible Q, so $A = PQCQ^{-1}P^{-1} = (PQ)C(PQ)^{-1}$ with PQ invertible (product of invertibles is invertible), so A is similar to C.

   (f) There can only be one eigenvector associated with an eigenvalue. $\underline{False}$.

   Solution: If $Av = \lambda v$ for nonzero vector v then for any non-zero real number, c, we have $A(cv) = \lambda(cv)$, so cv is also a nonzero eigenvector for A with eigenvalue λ.