Math 304 Linear Algebra Fall 2018 Practice Exam 1

Notations: $\mathbf{R}^m_n$ is the vector space of all $m \times n$ real matrices, and $\mathbf{R}^m = \mathbf{R}^m_1$.

(1) (15 Points) Let $L_A : \mathbf{R}^4 \to \mathbf{R}^4$ be the linear function $L_A(X) = AX$ associated with the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 3 & 5 \\ 2 & 3 & -1 & 0 \end{bmatrix}.$$

(a) Find all the vectors in $\mathrm{Ker}(L_A) = \{X \in \mathbf{R}^4 \mid L_A(X) = 0\}$ in terms of some free variables.
(b) Find all the vectors in $\mathrm{Range}(L_A) = \{Y = L_A(X) \in \mathbf{R}^4 \mid X \in \mathbf{R}^4\}$ in terms of a consistency condition on the entries of $Y = [y_i]$.
(c) Determine whether L is injective and whether L is surjective.

(2) (15 Points) Let $L : \mathbf{R}^2_2 \to \mathbf{R}^2_2$ be the function $L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$
(a) Show that L is a linear transformation.
(b) Find formulas for the compositions $L^2 = L \circ L$, $L^3 = L \circ L^2$, and $L^4 = L \circ L^3$. What does this tell you about L^k for $k \geq 5$?

(3) (15 Points) Answer each question separately

(a) If $A \in \mathbf{R}^m_n$, $B \in \mathbf{R}^n_p$, and $C \in \mathbf{R}^m_p$ are matrices such that the associated functions $L_A : \mathbf{R}^n \to \mathbf{R}^m$, $L_B : \mathbf{R}^p \to \mathbf{R}^n$, and $L_C : \mathbf{R}^p \to \mathbf{R}^m$ satisfy $L_A \circ L_B = L_C$, then what is the relationship between the matrices A, B and C?
(b) If a nonzero matrix $A \in \mathbf{R}^7_4$ row reduces to B in RREF having r leading ones, what are the possible values of r?
(c) For $A \in \mathbf{R}^m_n$ what relation between m and n would guarantee that the homogeneous linear system $AX = 0$ has nontrivial solutions?
(d) What condition on the rank of $A \in \mathbf{R}^m_n$ is equivalent to the inhomogeneous linear system $AX = B$ being consistent for any choice of B?
(e) For $A, B, C \in \mathbf{R}^n_n$ write a formula for $(ABC)^T$ in terms of A^T, B^T and C^T.

(4) (15 Points) For $A \in \mathbf{R}^m_n$, let $L_A : \mathbf{R}^n \to \mathbf{R}^m$ be the linear function $L_A(X) = AX$. For $1 \leq j \leq n$ let $\mathbf{e}_j \in \mathbf{R}^n$ be the matrix with 1 in row j and 0 in all other rows.
(a) Write $\mathrm{Range}(L_A)$ as the set of all linear combinations of some specific vectors.
(b) If L_A is injective then what is the most you can say about the relation between m and n?
(c) If L_A is surjective then what is the most you can say about the relation between m and n?
(d) If $\mathrm{rank}(A) = m$ what does that tell you about L_A?
(e) If $\mathrm{rank}(A) = n$ what does that tell you about L_A?

(5) (15 Points) Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be linear with $L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $L(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Find $A \in \mathbf{R}^2_2$ such that $L = L_A$, and check that $L(X) = L_A(X) = AX$ for all $X \in \mathbf{R}^2$.

1. (a) (6 Points) To find Ker(L) we must solve a linear system by row reducing

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 3 & 5 & 0 \\ 2 & 3 & -1 & 0 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = 2r+3s \\ x_2 = -r-2s \\ x_3 = r \in \mathbf{R} \\ x_4 = s \in \mathbf{R} \end{array}.$$

$$\mathrm{Ker}(L) = \left\{ \begin{bmatrix} 2r+3s \\ -r-2s \\ r \\ s \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r, s \in \mathbf{R} \right\}.$$

(b) (6 Points) $Y = [y_i] \in \mathrm{Range}(L)$ iff the following system is consistent:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & y_1 \\ 1 & 2 & 0 & 1 & y_2 \\ -1 & 1 & 3 & 5 & y_3 \\ 2 & 3 & -1 & 0 & y_4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -2 & -3 & 2y_1 - y_2 \\ 0 & 1 & 1 & 2 & -y_1 + y_2 \\ 0 & 0 & 0 & 0 & 3y_1 - 2y_2 + y_3 \\ 0 & 0 & 0 & 0 & y_1 + y_2 - y_4 \end{array}\right] \begin{array}{c} \text{is consistent iff} \\ 0 = 3y_1 - 2y_2 + y_3 \\ \text{and} \\ 0 = y_1 + y_2 - y_4 \end{array}.$$

(c) (3 Points) L is not injective since by (a) more than one vector is sent to the zero vector, and L is not surjective since by (b) not all vectors of $\mathbf{R}_4$ are in Range(L).

---

2. (15 Points) $L : \mathbf{R}_2^2 \to \mathbf{R}_2^2$ is the function $L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$

(a) (10 Points) To show that L is a linear transformation we check both:

$$L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & a_1+a_2 \\ b_1+b_2 & c_1+c_2 \end{bmatrix} = \begin{bmatrix} 0 & a_1 \\ b_1 & c_1 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ b_2 & c_2 \end{bmatrix} = L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$L\left(r\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = L\left(\begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}\right) = \begin{bmatrix} 0 & ra \\ rb & rc \end{bmatrix} = r\begin{bmatrix} 0 & a \\ b & c \end{bmatrix} = rL\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

(b) (5 Points) We have the formulas

$$L^2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = L\left(L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right) = L\left(\begin{bmatrix} 0 & a \\ b & c \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$$

$$L^3\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = L\left(L^2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right) = L\left(\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$$

$$L^4\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = L\left(L^3\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right) = L\left(\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This means $L^k = O$ is the zero transformation for $k \geq 5$.

3. (15 Points, 3 points each)
   (a) The relationship is $AB = C$, so C is the matrix product of A and B.
   (b) If $A \in \mathbf{R}_4^7$ is not the zero matrix, the number of leading ones in its RREF could only be 1, 2, 3 or 4 since each leading one occupies a column, and there is at least one.
   (c) The relation $n > m$ guarantees that the homogeneous linear system $AX = 0$ has nontrivial solutions.
   (d) The inhomogeneous linear system $AX = B$ is consistent iff the rank of A equals m.
   (e) $(ABC)^T = C^T B^T A^T$.

---

4. (15 Points, 3 points each)
   (a) Range(L_A) consists of all vectors

$$AX = A(\sum_{j=1}^{n} x_j e_j) = \sum_{j=1}^{n} x_j A e_j = \sum_{j=1}^{n} x_j L_A(e_j)$$

   for all $x_j \in \mathbf{R}$. So Range(L_A) is the set of all linear combinations of the n vectors $L_A(\mathbf{e}_j) = A\mathbf{e}_j$ for $1 \leq j \leq n$, which are the column vectors of matrix A.
   (b) If L_A is injective then $n \leq m$ since more variables than equations would guarantee free variables.
   (c) If L_A is surjective then $m \leq n$ since more equations than variables would guarantee a row of zeros in the RREF of A, giving a consistency condition for $AX = B$.
   (d) If rank$(A) = m$ then L_A is surjective since m leading ones in the RREF means no zero rows so $AX = B$ is always consistent.
   (e) If rank$(A) = n$ then L_A is injective since n leading ones in the RREF means a leading one in each column and there are no free variables.

---

5. (15 Points) Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be linear with $L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $L(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. The $A \in \mathbf{R}_2^2$ such that $L(X) = L_A(X) = AX$ for all $X \in \mathbf{R}^2$ would have to satisfy $A\mathbf{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A\mathbf{e}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, but $A\mathbf{e}_1 = Col_1(A)$ and $A\mathbf{e}_2 = Col_2(A)$, so $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$. This works for all $X \in \mathbf{R}^2$ because L is linear, so

$$\begin{aligned} L(X) = L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= L(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1 L(\mathbf{e}_1) + x_2 L(\mathbf{e}_2) \\ &= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = AX. \end{aligned}$$