Math 304 Linear Algebra Fall 2018 Practice Exam 2 Feingold

SHOW ALL NECESSARY WORK FOR EACH PROBLEM

(1) (15 Points) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix} \in \mathbf{R}^4_5$, and let $L_A : \mathbf{R}^5 \to \mathbf{R}^4$ be the linear transformation determined by A, that is, $L_A(X) = AX$.

(a) (3 Points) Find a **basis** for $Row(A)$, the row space of A.
(b) (3 Points) Find a **basis** for $Ker(L_A)$, the kernel of L_A.
(c) (3 Points) Find a **basis** for $Col(A)$, the column space of A.
(d) (3 Points) Find a **basis** for $Range(L_A)$, the range of L_A.
(e) (3 Points) Use your answers to find the **dependence relations** among the columns of A.

(2) (15 Points) Let $L : \mathbf{R}^2_2 \to \mathbf{R}_4$ be the linear transformation defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [\,(a-b+5c+d) \quad (3a+2b+3d) \quad (a+b-c+d) \quad (2a+3b-5c+2d)\,]$$

(a) (3 points) Find the set of all vectors in $Ker(L)$.
(b) (2 points) Find a basis for $Ker(L)$.
(c) (1 points) Find $dim(Ker(L))$.
(d) (1 points) Is L one-to one? **Explain why!**
(e) (3 points) Find a basis for $Range(L)$.
(f) (2 points) Find $dim(Range(L))$.
(g) (2 points) Is L onto? **Explain why!**
(h) (1 points) Is L invertible? **Explain why!**

(3) (15 Points, 3 Points Each) Answer each question separately

(a) Find the elementary matrix E such that for any $A \in \mathbf{R}^3_n$, EA is the matrix obtained from A by multiplying row 2 of A by 5.
(b) If $S = \{v_1, v_2, \ldots, v_n\}$ is an **independent** set in vector space V, and $L : V \to W$ is **injective**, what is the most you can say about $L(S) = \{L(v_1), L(v_2), \ldots, L(v_n)\}$ in W?
(c) If $S = \{v_1, v_2, \ldots, v_n\}$ is a **dependent** set in vector space V, and $L : V \to W$ is any linear transformation, what is the most you can say about $L(S) = \{L(v_1), L(v_2), \ldots, L(v_n)\}$ in W?
(d) If $T = \{w_1, w_2, \ldots, w_m\}$ spans vector space W, and the last vector w_m is a linear combination of the previous vectors $T' = \{w_1, w_2, \ldots, w_{m-1}\}$, what is the most you can say about the span of T'?
(e) If $S = \{v_1, v_2, \ldots, v_n\}$ is an independent set in a vector space V and $v \in V$ is in the span of S, then what is the most you can you say about $S \cup \{v\} = \{v_1, \ldots, v_n, v\}$?

(4) (15 points, 3 points each) Answer each question separately.

(a) If $L : \mathbf{R}_3^3 \to \mathbf{R}_9$ is injective, what else must be true about L?
(b) If $L : \mathbf{R}_5 \to \mathbf{R}^9$ what are all the possibilities for $dim(Range(L))$?
(c) If $L : \mathbf{R}_3^4 \to \mathbf{R}^8$ what are all the possibilities for $dim(Ker(L))$?
(d) If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis of a vector space V, what is the most you can say about the set $\{[v_1]_S, [v_2]_S, \ldots, [v_n]_S\}$ of coordinates of those vectors?
(e) Let $T = \{[1\ 2\ 1], [2\ 5\ 5], [1\ 1\ \ -3]\}$ be a basis of $\mathbf{R}_3$ and let $v = [2\ 3\ 4] \in \mathbf{R}_3$. Find the coordinate vector $[v]_T$.

(5) (15 Pts) Let $L : \mathbf{R}_4 \to \mathbf{R}^2$ be given by

$$L\left([a \quad b \quad c \quad d]\right) = \begin{bmatrix} 2a - b + c - 3d \\ -a + 4b - 2c - d \end{bmatrix}.$$

Let S be the standard basis of $\mathbf{R}_4$ and let T be the standard basis of $\mathbf{R}^2$. Let other ordered bases be

$$S' = \{[1\ 1\ 0\ 1], [1\ 0\ 0\ 1], [0\ 0\ 1\ 1], [0\ 1\ 1\ 0]\} \quad \text{and} \quad T' = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}.$$

(a) (3 pts) Find the matrix ${}_T[L]_S$ representing L from S to T.
(b) (3 pts) Find the matrix ${}_{T'}[L]_{S'}$ representing L from S' to T' **without using transition matrices**. (Do it directly.)
(c) (9 pts) Find the transition matrices ${}_SP_{S'}$ and ${}_TQ_{T'}$ and show that ${}_{T'}[L]_{S'} = ({}_TQ_{T'})^{-1}\ {}_T[L]_S\ ({}_SP_{S'})$.

1. (a) (3 Points)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix} \text{ reduces to } \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ so}$$

$\{[1 \quad 0 \quad -1 \quad -2 \quad -3], [0 \quad 1 \quad 2 \quad 3 \quad 4]\}$ is a basis for $Row(A)$.

(b) (3 Points) A basis for $Ker(A)$ would be found by row reducing A augmented by a column of zeros, interpreting the solutions in $\mathbf{R}^5$, and separating the free variables to get three independent vectors which span it:

$$\left\{ \begin{bmatrix} r+2s+3t \\ -2r-3s-4t \\ r \\ s \\ t \end{bmatrix} = r\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbf{R}^5 \;\middle|\; r,s,t \in \mathbf{R} \right\}$$

so a basis for $Ker(L_A)$ is the set of those three vectors in $\mathbf{R}^5$.

(c) (3 Points) One basis for $Col(A)$ consists of the first two columns of A because only those pivot columns have leading ones in the RREF row equivalent to A. Other correct answers can be obtained by linear combinations of those two columns. So answers are:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

(d) (3 Points) $Range(L_A) = Col(A)$ so this question has the same answer as part (c).

(e) (3 Points) Each basis vector in $Ker(L_A)$ gives a dependence relation among the columns of A. The three dependence relations obtained that way are:

$$1Col_1(A) - 2Col_2(A) + 1Col_3(A) = \theta,$$

$$2Col_1(A) - 3Col_2(A) + 1Col_4(A) = \theta,$$

$$3Col_1(A) - 4Col_2(A) + 1Col_5(A) = \theta,$$

where $\theta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbf{R}^4$.

2. (15 points) To find $Ker(L)$, row reduce

$$\left[\begin{array}{cccc|c} 1 & -1 & 5 & 1 & 0 \\ 3 & 2 & 0 & 3 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ 2 & 3 & -5 & 2 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} a = -2r - s \\ b = 3r \\ c = r \in \mathbf{R} \\ d = s \in \mathbf{R} \end{array}.$$

(a) (3 points) $Ker(L) = \left\{ \begin{bmatrix} -2r - s & 3r \\ r & s \end{bmatrix} \in \mathbf{R}_2^2 \;\middle|\; r, s \in \mathbf{R} \right\}$.

(b) (2 points) $\left\{ \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of $Ker(L)$.

(c) (1 point) $dim(Ker(L)) = 2$.

(d) (1 point) L is not one-to-one since $Ker(L)$ is nontrivial.

(e) (3 points) $Range(L) =$
$\{[\,(a - b + 5c + d) \quad (3a + 2b + 3d) \quad (a + b - c + d) \quad (2a + 3b - 5c + 2d)\,] \in \mathbf{R}_4 \mid$

$a, b, c, d \in \mathbf{R}\}$ is spanned by the set of vectors
$\{[1\ 3\ 1\ 2],\ [-1\ 2\ 1\ 3],\ [5\ 0\ -1\ -5],\ [1\ 3\ 1\ 2]\}$.

The two free variables in $Ker(L)$ mean the last two vectors are redundant vectors, so a basis for $Range(L)$ is $\{[1\ 3\ 1\ 2],\ [-1\ 2\ 1\ 3]\}$.

(f) (2 points) $dim(Range(L)) = 2$.

(g) (2 points) L is not onto, $Range(L) \neq \mathbf{R}_4$.

(h) (1 points) L is not invertible since it is not bijective.

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3. (15 Points, 3 points each)

(a) Find the elementary matrix E such that for any $A \in \mathbf{R}_n^3$, EA is the matrix obtained from A by multiplying row 2 of A by 5. That elementary matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is obtained by doing that elementary row operation to I_3.

(b) $L(S)$ is independent in W (L injective takes an independent set to an independent set.)

(c) $L(S)$ is dependent in W (Any linear L takes a dependent set to a dependent set.)

(d) The span of T' is W. (Removing a redundant vector leaves the span unchanged.)

(e) $S \cup \{v\}$ is dependent. (A set with a redundant vector is dependent.)

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4. (15 Points, 3 points each)

(a) Use $\dim(V) = \dim(Ker(L)) + \dim(Range(L))$. L injective means $\dim(Ker(L)) = 0$ so $\dim(V) = \dim(Range(L))$. But $\dim(V) = \dim(\mathbf{R}_3^3) = 9 = \dim(\mathbf{R}_9) = \dim(W)$, so $\dim(W) = 9 = \dim(Range(L))$, so $W = Range(L)$, so L is onto. L is injective and surjective, so bijective, invertible, an isomorphism.

(b) $0 \leq dim(Range(L)) \leq 5$.

(c) $4 \leq dim(Ker(L)) \leq 12$.

(d) Those coordinates are the standard basis vectors of $\mathbf{R}^n$.

(e) Row reduce $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 5 & 1 & 3 \\ 1 & 5 & -3 & 4 \\ & T & & v \end{array}\right]$ to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 19 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -5 \\ & I_3 & & [v]_T \end{array}\right]$ so $[v]_T = \begin{bmatrix} 19 \\ -6 \\ -5 \end{bmatrix}$. Check that $19\ [1\ 2\ 1] - 6\ [2\ 5\ 5] - 5\ [1\ 1\ -3] = [2\ 3\ 4]$.

5. (15 Points)

(a) (3 Pts) ${}_T[L]_S = \begin{bmatrix} 2 & -1 & 1 & -3 \\ -1 & 4 & -2 & -1 \end{bmatrix}$ is easy to get since S and T are standard. $L([1\ 0\ 0\ 0]) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $L([0\ 1\ 0\ 0]) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $L([0\ 0\ 1\ 0]) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$,
$L([0\ 0\ 0\ 1]) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$. $[T \mid L(S)] = \left[\begin{array}{cc|cccc} 1 & 0 & 2 & -1 & 1 & -3 \\ 0 & 1 & -1 & 4 & -2 & -1 \\ \multicolumn{2}{c|}{\scriptstyle T} & \multicolumn{4}{c}{\scriptstyle L(S)} \end{array}\right]$ is already reduced, so the right side is ${}_T[L]_S$.

(b) (3 Pts) $L([1\ 1\ 0\ 1]) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $L([1\ 0\ 0\ 1]) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, $L([0\ 0\ 1\ 1]) = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$, $L([0\ 1\ 1\ 0]) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Row reduce $\left[\begin{array}{cc|cccc} 2 & 3 & -2 & -1 & -2 & 0 \\ 1 & 2 & 2 & -2 & -3 & 2 \\ \multicolumn{2}{c|}{\scriptstyle T'} & \multicolumn{4}{c}{\scriptstyle L(S')} \end{array}\right]$ to $\left[\begin{array}{cc|cccc} 1 & 0 & -10 & 4 & 5 & -6 \\ 0 & 1 & 6 & -3 & -4 & 4 \\ \multicolumn{2}{c|}{\scriptstyle I_2} & \multicolumn{4}{c}{\scriptstyle {}_{T'}[L]_{S'}} \end{array}\right]$

(c) (9 Pts) ${}_SP_{S'} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ and ${}_TQ_{T'} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ since S and T are the standard bases.
To get ${}_{T'}Q_T = ({}_TQ_{T'})^{-1}$, reduce

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \multicolumn{2}{c|}{\scriptstyle T'} & \multicolumn{2}{c}{\scriptstyle T} \end{array}\right] \text{ to } \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ \multicolumn{2}{c|}{\scriptstyle I_2} & \multicolumn{2}{c}{\scriptstyle {}_{T'}Q_T} \end{array}\right]$$

$$({}_TQ_{T'})^{-1}{}_T[L]_S({}_SP_{S'}) = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 & -3 \\ -1 & 4 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 7 & -14 & 8 & -3 \\ -4 & 9 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 4 & 5 & -6 \\ 6 & -3 & -4 & 4 \end{bmatrix} = {}_{T'}[L]_{S'} \text{ checks.}$$