NAME (Printed): ____

Math 304-5,6 Linear Algebra Fall 2018 Quiz 7 Feingold

INSTRUCTIONS: Show all calculations and reasons needed to justify your answers.

Let $L : \mathbf{R}^3 \to \mathbf{R}^2$ be given by

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b+2c \\ -a-2b+5c \end{bmatrix}.$$

Let $S = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbf{R}^3$ and let $T = \{f_1, f_2\}$ be the standard basis of $\mathbf{R}^2$. Let other ordered bases be

$$S' = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T' = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}.$$

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(1) (2 Pts) Find the matrix ${}_T[L]_S$ representing L from S to T.

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(2) (3 Pts) Find the matrix ${}_{T'}[L]_{S'}$ representing L from S' to T' **without using transition matrices**. (Do it directly.)

(3) (3 Pts) Find the transition matrices ${}_S P_{S'}$ and ${}_T Q_{T'}$ and show that ${}_{T'}[L]_{S'} = ({}_T Q_{T'})^{-1}\ {}_T[L]_S\ ({}_S P_{S'})$.

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(1) (2 Pts) Find the matrix ${}_T[L]_S$ representing L from S to T.

Solution: ${}_T[L]_S = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 5 \end{bmatrix}$ is easy to get since $S = \{e_1, e_2, e_3\}$ and T are standard. $L(e_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $L(e_2) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $L(e_3) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. $[T \mid L(S)] = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & -2 & 5 \\ \multicolumn{2}{c|}{T} & \multicolumn{3}{c}{L(S)} \end{array}\right]$ is already reduced, so the right side is ${}_T[L]_S$.

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(2) (3 Pts) Find the matrix ${}_{T'}[L]_{S'}$ representing L from S' to T' **without using transition matrices**. (Do it directly.)

Solution: $L(v_1) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $L(v_2) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $L(v_3) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Row reduce

$$\left[\begin{array}{cc|ccc} 1 & 2 & 4 & 3 & 2 \\ 1 & 3 & 2 & 3 & -3 \\ \multicolumn{2}{c|}{T'} & \multicolumn{3}{c}{L(S')} \end{array}\right] \text{ to } \left[\begin{array}{cc|ccc} 1 & 0 & 8 & 3 & 12 \\ 0 & 1 & -2 & 0 & -5 \\ \multicolumn{2}{c|}{I_2} & \multicolumn{3}{c}{{}_{T'}[L]_{S'}} \end{array}\right] \text{ so } {}_{T'}[L]_{S'} = \begin{bmatrix} 8 & 3 & 12 \\ -2 & 0 & -5 \end{bmatrix}$$

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(3) (3 Pts) Find the transition matrices ${}_SP_{S'}$ and ${}_TQ_{T'}$ and show that ${}_{T'}[L]_{S'} = ({}_TQ_{T'})^{-1}\ {}_T[L]_S\ ({}_SP_{S'})$.

Solution: ${}_SP_{S'} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and ${}_TQ_{T'} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ since S and T are standard.

To get ${}_{T'}Q_T = ({}_TQ_{T'})^{-1}$, reduce $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ \multicolumn{2}{c|}{T'} & \multicolumn{2}{c}{T} \end{array}\right]$ to $\left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \\ \multicolumn{2}{c|}{I_2} & \multicolumn{2}{c}{{}_{T'}Q_T} \end{array}\right]$

$$({}_TQ_{T'})^{-1}{}_T[L]_S({}_SP_{S'}) = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 5 & 7 & -4 \\ -2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 3 & 12 \\ -2 & 0 & -5 \end{bmatrix} = {}_{T'}[L]_{S'} \text{ checks.}$$
