NAME (Printed): ______________________________ Section: ______

Math 304-5,6 Linear Algebra Fall 2018 Quiz 8 Feingold

INSTRUCTIONS: Show all calculations and reasons needed to justify your answers.

Let $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis of $\mathbf{R}^2$ and let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be given by

$$L \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ -a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{so} \quad {}_S[L]_S = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

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(1) (4 Pts) Find an eigen-basis, $T = \{w_1, w_2\}$, such that the matrix ${}_T[L]_T$ representing L from T to T is a diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. That means $L(w_1) = \lambda_1 w_1$ and $L(w_2) = \lambda_2 w_2$ for some $\lambda_1, \lambda_2 \in \mathbf{R}$ with $T = \{w_1, w_2\}$ independent.

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(2) (3 Pts) Find the transition matrices ${}_S P_T$ and ${}_T P_S$ and verify that ${}_T P_S \; {}_S[L]_S \; {}_S P_T = {}_T[L]_T = D$.

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(1) (4 Pts) Find an eigen-basis, $T = \{w_1, w_2\}$, such that the matrix ${}_T[L]_T$ representing L from T to T is a diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. That means $L(w_1) = \lambda_1 w_1$ and $L(w_2) = \lambda_2 w_2$ for some $\lambda_1, \lambda_2 \in \mathbf{R}$ with $T = \{w_1, w_2\}$ independent.

Solution: We need to find nonzero vectors such that $L \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ -a \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$, that is, $-b = \lambda a$ and $-a = \lambda b$, so $-b = \lambda(-\lambda b) = -\lambda^2 b$ so $\lambda^2 - 1 = 0$ since $b \neq 0$. The only possible values of λ are $\lambda_1 = 1$ and $\lambda_2 = -1$. If $L(w) = 1w$, it means $\begin{bmatrix} -b \\ -a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ so $b = -a$, giving a 1-dimensional eigenspace with basis $\left\{ w_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. If $L(w) = -1w$, it means $\begin{bmatrix} -b \\ -a \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix}$ so $b = a$, giving a 1-dimensional eigenspace with basis $\left\{ w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. So the independent set $T = \{w_1, w_2\}$ is the desired eigen-basis.

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(2) (3 Pts) Find the transition matrices ${}_SP_T$ and ${}_TP_S$ and verify that ${}_TP_S \; {}_S[L]_S \; {}_SP_T = {}_T[L]_T = D$.

Solution: Since S is standard, ${}_SP_T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is the matrix whose columns are w_1 and w_2. Also, ${}_TP_S = ({}_SP_T)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ can be found by row reduction of $[T|S]$ or by using the formula for inverting a 2×2 matrix. We already know ${}_S[L]_S$, so we just verify that

$${}_TP_S \; {}_S[L]_S \; {}_SP_T = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} =$$

$$\frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D \text{ checks.}$$

Using the direct method to find ${}_T[L]_T$ we would row reduce

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ \multicolumn{2}{c|}{T} & \multicolumn{2}{c}{L(T)} \end{array} \right] \text{ to } \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ \multicolumn{2}{c|}{I_2} & \multicolumn{2}{c}{{}_T[L]_T} \end{array} \right] \text{ so } {}_T[L]_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$
