Math 304 **Midterm 1 Solutions** **Spring 2019**

No books, no notes, no calculators. You must show work, unless the question is a true/false, multiple choice, or fill-in-the-blank question.

1. Fill in the blanks in the following definitions and statements of results from the textbook.

(a) (4 points) A linear transformation L from one vector space to another has two fundamental properties:

1. For all vectors u and v, $\underline{L(u+v)} = \underline{L(u)+L(v)}$.

2. For all vectors w and all scalars c, $\underline{L(cw)} = \underline{cL(w)}$.

Hint: The properties above are listed in Chapter 1 with the heading "The key to the whole class ...".

(b) (6 points) A matrix is said to be in "reduced row echelon form" if the following conditions are met:

0. All zero rows are below all non-zero rows.

1. In each non-zero row, the leftmost non-zero entry, called a pivot, is 1.

2. The pivot of any given row is always <u>strictly to the right of the pivot in the row above it</u>.

3. The pivot is the only <u>nonzero entry in its column</u>.

(c) (4 points) Suppose $B = (v_1, v_2, \ldots, v_n)$ is an ordered basis for a vector space V. The notation below defines a vector in V which is given by the equation:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}_B = \underline{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n}$$

(d) (3 points) The *Cauchy-Schwarz inequality* states that for any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the absolute value of the dot product, $|\mathbf{u} \cdot \mathbf{v}| \underline{\leq ||u||||v||}$

(e) (3 points) Theorem 7.5.1 states that an $n \times n$ matrix M is invertible if and only if the system of n equations in n unknowns $M\mathbf{x} = \mathbf{0}$ <u>has no solution except $\mathbf{x} = \mathbf{0}$</u>.

2. (12 points) Let V be the vector space of polynomials of degree less than or equal to 2. Let B be the ordered basis $(x^2, x, 1)$ for V. Let $L\colon V \to V$ be the linear transformation $\frac{\mathrm{d}}{\mathrm{dx}}$.

   Find ${}_BL_B$, that is, the matrix of L with respect to the basis B (used as both the input basis and output basis).

Solution: The matrix ${}_BL_B$ is, by definition (Chapter 7), the matrix whose columns are the coefficients in the expressions of the image vectors $(L(x^2), L(x), L(1))$ with respect to the vectors $(x^2, x, 1)$. We compute $L(x^2) = 2x = 0x^2 + 2 \cdot x + 0 \cdot 1$, so the entries in the first column of the matrix are, top to bottom, $0, 2, 0$. The next column has entries $0, 0, 1$ because $L(x) = 0 \cdot x^2 + 0 \cdot x + 1 \cdot 1$. The last column is all zeroes, because $L(1) = 0 \cdot x^2 + 0 \cdot x + 0 \cdot 1$. Thus we have

$$
{}_BL_B = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
$$

3. (12 points) Let S and T be linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$
S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -3 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
$$

   Find the matrix of the composition $T \circ S$ (with respect to the standard basis of $\mathbb{R}^2$), that is, the function that sends

$$
\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{to} \quad T\left(S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right)
$$

Solution: The matrix of the composition is the product of the matrices, so the matrix of $T \circ S$ is

$$
\begin{bmatrix} 6 & 6 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ -2 & -1 \end{bmatrix}
$$

4. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ be vectors such that

$$\mathbf{u} \cdot \mathbf{v} = 8, \quad \mathbf{u} \cdot \mathbf{w} = -7, \quad \mathbf{v} \cdot \mathbf{w} = 6, \quad \text{and} \; -2\mathbf{u} + 6\mathbf{v} = \mathbf{x}.$$

   (a) (3 points) Find the dot product $\mathbf{v} \cdot \mathbf{u}$.

   Solution: $\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} = 8.$

   (b) (7 points) Find the dot product $\mathbf{x} \cdot \mathbf{w}$.

   Solution: $\mathbf{x} \cdot \mathbf{w} = (-2\mathbf{u} + 6\mathbf{v}) \cdot \mathbf{w} = -2(\mathbf{u} \cdot \mathbf{w}) + 6(\mathbf{v} \cdot \mathbf{w}) = -2 \cdot (-7) + 6 \cdot 6 = 36 + 14 = 50.$

5. (10 points) Give a geometric description of the following system of equations:

$$\begin{aligned} 15x + 9y - 15z &= -6 \\ 25x + 15y - 25z &= -10 \\ -35x - 21y + 35z &= 14 \end{aligned}$$

Hints: this was a homework question. A "geometric description" is something like "these equations represent two lines in the plane, which intersect at the origin."

Solution: These three equations represent three planes in $\mathbb{R}^3$. Row reducing this matrix leaves only one nonzero row, so all the planes are identical, and the intersection is that plane. This plane does *not* pass through the origin.

6. (10 points) Given the following LU factorization of the matrix M:

$$M = \begin{bmatrix} -2 & -3 & 1 \\ 6 & 5 & 1 \\ -6 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 & 1 \\ 0 & -4 & 4 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

Use this factorization to solve the system of equations:

$$\begin{bmatrix} -2 & -3 & 1 \\ 6 & 5 & 1 \\ -6 & 3 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -30 \\ 29 \end{bmatrix}$$

Solution: First we introduce the notation:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{and} \qquad V = \begin{bmatrix} 10 \\ -30 \\ 29 \end{bmatrix}$$

Now define $W = UX$, so that $MX = V$ becomes $LUX = V$, or $LW = V$.

We solve for W (forward substitution) and obtain

$$W = \begin{bmatrix} 10 \\ 0 \\ -1 \end{bmatrix}$$

Then solve for X in $UX = W$, and obtain

$$X = \begin{bmatrix} -4 \\ -1 \\ -1 \end{bmatrix}$$