Math 304 **Midterm 2 Solutions** **Spring 2019**

No books, no notes, no calculators. You must show work, unless the question is a true/false, multiple choice, or fill-in-the-blank question.

1. Fill in the blanks in the following definitions and statements of results from the textbook.

   (a) (4 points) If $F\colon V \to V$ is a linear transformation, we say that a vector w is an *eigenvector* of F associated to the *eigenvalue* α if two conditions are met:

   1. $\underline{w \neq 0}$
   2. $\underline{F(w) = \alpha w}$

   (b) (6 points) A list of vectors $(w_1, w_2, \dots, w_n)$ in a vector space is said to be *linearly dependent* if there are scalars $a_1, a_2, \dots, a_n$ such that the linear combination

   $$\underline{a_1 w_1 + a_2 w_2 + \cdots + a_n w_n} = \underline{0}$$

   and not all a_i are zero.

   (c) (4 points) A *basis* of a vector space V is a list $(v_1, v_2, \dots, v_n)$ of vectors in V with two properties:

   1. $(v_1, v_2, \dots, v_n)$ spans V
   2. $(v_1, v_2, \dots, v_n)$ is a linearly independent list

   (d) (10 points) Fill in the blanks in the following statments of properties of the determinant. Throughout, you may assume that A, B are $n \times n$ matrices of real numbers.

   (a) If A is the identity matrix, the determinant of A is $\underline{1}$.

(b) If B is obtained by exchanging two rows of A, then $\det B = \underline{-\det A}$.

(c) The determinant is a <u>linear</u> function of each row separately.

(d) If two rows of A are equal, then $\det A$ is $\underline{0}$.

(e) If B is obtained by subtracting a multiple of one row of A from another row of A, then $\det B = \underline{\det A}$.

(f) If A is a matrix with a row of zeroes, then $\det A = \underline{0}$.

(g) If A is not invertible, then $\det A \underline{= 0}$.

(h) How is $\det(AB)$ related to $\det(A)$ and $\det(B)$? $\det(AB) = \underline{\det(A)\det(B)}$.

(i) How is $\det(A^T)$ related to $\det(A)$? $\det(A^T) = \underline{\det A}$.

(e) (5 points) Fill in the blanks in the following statement of the "Subspace Theorem" (Theorem 9.1.1):

Let U be a non-empty subset of a vector space V. Then U is a subspace if and only if, for any two vectors v and w in $\underline{U}$, and any two <u>scalars</u> a and b, we have

$$\underline{av + bw} \in \underline{U}$$

(f) (3 points) The *characteristic polynomial* of an $n \times n$ matrix A is, by definition:

$$p_A(\lambda) = \underline{\det(\lambda I - A)}$$

2. (8 points) Find the characteristic polynomial of the matrix A below:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -3 & 6 & 3 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution: After some computing, one finds that $p_A(\lambda) = \lambda^3 - 9\lambda^2 + 27\lambda - 27$.

3. (8 points) After some computations, you find that the characteristic polynomial of the matrix

$$B = \begin{bmatrix} 7 & 6 & -2 \\ -4 & -3 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

is $p_B(\lambda) = \lambda^3 - 3\lambda^2 + 4$. Find all eigenvalues of B.

Solution: The eigenvalues are the roots of the characteristic polynomial.

Notice that -1 is a root, so $\lambda + 1$ is a factor. Compute to find $p_B(\lambda) = \lambda^3 - 3\lambda^2 + 4 = (\lambda + 1)(\lambda^2 - 4\lambda + 4) = (\lambda + 1)(\lambda - 2)^2$.

We conclude that the eigenvalues are -1 and 2.

4. (9 points) After some computation, you determine that 2 is an eigenvalue of the matrix

$$C = \begin{bmatrix} 1 & 6 & 5 \\ 0 & -4 & -3 \\ 1 & 4 & 2 \end{bmatrix}$$

Find a basis of the eigenspace of C associated to the eigenvalue 2.

Solution: The eigenspace is the null space of $C - 2I$, which is

$$C - 2I = \begin{bmatrix} 1 & 6 & 5 \\ 0 & -4 & -3 \\ 1 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 6 & 5 \\ 0 & -6 & -3 \\ 1 & 4 & 0 \end{bmatrix}$$

Now we must do a row reduction; after some work we obtain

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

The solution set is all vectors of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{where} \quad x = 2z \quad \text{and} \quad y = -\frac{1}{2}z,$$

and z can be anything. This is a one-dimensional vector space, with one possible basis being the set consisting of the single vector

$$\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

5. (9 points) Let $\mathbb{P}_2$ be the vector space of polynomials with real coefficients with degree less than or equal to 2. Let

$$B = (1, x, x^2) \qquad \text{and} \qquad C = (1, 1 + x, 1 + x + \frac{1}{2}x^2)$$

be two ordered bases of $\mathbb{P}_2$. (You may assume that B and C are bases and need not verify this.) Find the change of basis matrix P that helps us convert from the old basis B to the new basis C.

Solution: The definition of the change of basis matrix is the matrix whose columns are the coefficients in the expression of the new basis vectors in terms of the old basis vectors. There is almost no work to do here as the new basis vectors in C are already given in terms of the old basis vectors in B. Thus

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

6. (9 points) The matrix of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the standard basis E is

$$_E T_E = A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix}$$

After some computing, you find that the vectors

$$u = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad w = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors of A associated to the eigenvalues 1, 2, and 2 respectively. Let $C = (u, v, w)$ be the new basis of $\mathbb{R}^3$ formed by these three vectors, in this order.

Find ${}_C T_C$, the matrix of T with respect to the basis C (used as both the input basis and the output basis). *You need not verify statements made in the problem. In particular you may assume that u, v, and w are eigenvectors and that C is a basis.*

Solution: By definition of ${}_C T_C$, the columns of the matrix are the coefficients in the expressions of the vectors $T(u), T(v), T(w)$ in terms of u, v, w. But these vectors are eigenvectors of T. We therefore have $T(u) = 1u = 1u + 0v + 0w$, so the first column of the matrix has entries $(1, 0, 0)$ from top to bottom. Similarly we have $T(v) = 2v = 0u + 2v + 0w$, so the second column has entries $(0, 2, 0)$. And $T(w) = 3w = 0u + 0v + 3w$, so the third column has entries $(0, 0, 3)$. Thus we have:

$$
{}_C T_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}
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