MATH 478: SOLUTION TO PROBLEM 3.1.2 (C)

1. Example Solution

Problem (3.1.2 (c))**.** Using the ϵ-N definition of limit, prove that $\lim[\sqrt{n^2+n} - n] = \frac{1}{2}$.

Solution of Problem 3.1.2 (c). Our proof will have 5 steps: first we will do some algebra to put the expression for which we are computing the limit into a more convenient form. Then, using this more convenient form, we obtain a new expression which we must show is bounded by ϵ in order to prove that the limit is 1/2. Next we develop bounds for the numerator and denominator of this new expression. Then we combine these bounds to get a bound for the new expression as a whole. Finally we are able to choose an N associated to a given ϵ and prove that the limit is 1/2, as claimed.

Step 1. We put the expression $\sqrt{n^2+n} - n$ into a more convenient form. Observe that

$$\sqrt{n^2+n} - n = \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} = \frac{n^2+n-n^2}{\sqrt{n^2+n} + n} = \frac{n}{\sqrt{n^2+n} + n}.$$

So in order to solve the problem it suffices to prove that $\lim[\frac{n}{\sqrt{n^2+n}+n}] = \frac{1}{2}$. In other words, we wish to show that given $\epsilon > 0$, $\exists N$ such that $\forall n > N$,

$$\left| \frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2} \right| < \epsilon. \tag{1}$$

Step 2. We rewrite the left-hand-side of the inequality (1) as

$$\frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2} = \frac{2n - (\sqrt{n^2+n}+n)}{2(\sqrt{n^2+n}+n)} = \frac{n - \sqrt{n^2+n}}{2(\sqrt{n^2+n}+n)} \tag{2}$$

Step 3. Now we obtain bounds on the numerator and denominator of the expression on the far right of (2). Observe that

$$n^2 < n^2 + n < n^2 + 2n + 1 = (n+1)^2.$$

Taking square roots we have $n < \sqrt{n^2+n} < n+1$. Thus we have $|n - \sqrt{n^2+n}| < 1$ and $\sqrt{n^2+n} + n > 2n$.

Step 4. We use the bounds obtained in Step 3 to bound the whole fraction on the left-hand side of (1):

$$\left|\frac{n-\sqrt{n^2+n}}{2(\sqrt{n^2+n}+n)}\right| < \frac{1}{2(2n)} = \frac{1}{4n}.$$

Step 5. Now we can show that $\lim[\frac{n}{\sqrt{n^2+n}+n}] = \frac{1}{2}$. Given $\epsilon > 0$, we take $N = \frac{1}{4\epsilon}$. Now if $n > N$, we have $n > \frac{1}{4\epsilon}$ and so $\frac{1}{4n} < \epsilon$. We therefore have

$$\left|\frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2}\right| = \left|\frac{n-\sqrt{n^2+n}}{2(\sqrt{n^2+n}+n)}\right| < \frac{1}{4n} < \epsilon,$$

as desired. Notice that the equality above is an application of Step 2, and that the first inequality is an application of Step 4.

Finally, note that Step 5 actually computes the desired limit, as we showed in Step 1. □

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