Part 3

Infinite series, infinite products, and infinite fractions

CHAPTER 5

Advanced theory of infinite series

Even as the finite encloses an infinite series
And in the unlimited limits appear,
So the soul of immensity dwells in minutia
And in the narrowest limits no limit in here.
What joy to discern the minute in infinity!
The vast to perceive in the small, what divinity!
Jacob Bernoulli (1654-1705) Ars Conjectandi.

This chapter is about going in-depth into the theory and application of infinite series. One infinite series that will come up again and again in this chapter and the next chapter as well, is the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

introduced in Section 4.6. Amongst many other things, in this chapter we'll see how to write some well-known constants in terms of the Riemann zeta function; e.g. we'll derive the following neat formula for our friend $\log 2$ (§ 5.5):

$$\boxed{\log 2 = \sum_{n=2}^{\infty} \frac{1}{2^n}\, \zeta(n),}$$

another formula for our friend the Euler-Mascheroni constant (§ 5.9):

$$\boxed{\gamma = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n),}$$

and two more formulas involving our most delicious friend π (see §'s 5.10 and 5.11):

$$\boxed{\pi = \sum_{n=2}^{\infty} \frac{3^n - 1}{4^n}\, \zeta(n+1) \quad , \quad \frac{\pi^2}{6} = \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .}$$

In this chapter, we'll also derive Gregory-Leibniz-Madhava's formula (§ 5.10)

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + - \cdots ,}$$

and Machin's formula which started the "decimal place race" of computing π (§ 5.10):

$$\boxed{\pi = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \left(\frac{4}{5^{2n+1}} - \frac{1}{239^{2n+1}}\right).}$$

Leibniz's formula for $\pi/4$ is an example of an "alternating series". We study these types of series in Section 5.1. In Section 5.2 and Section 5.3 we look at the ratio and root tests, which you are probably familiar with from elementary calculus. In Section 5.4 we look at power series and prove some pretty powerful properties of power series. The formula for $\log 2$, γ, and the formula $\pi = \sum_{n=2}^\infty \frac{3^n-1}{4^n}\,\zeta(n+1)$ displayed above are proved using a famous theorem called the *Cauchy double series theorem*. This theorem, and double sequences and series in general, are the subject of Section 5.5. In Section 5.6 we investigate rearranging (that is, mixing up the order of the terms in a) series. Here's an interesting question: Does the series

$$\sum_{p \text{ is prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{23} + \frac{1}{29} + \cdots$$

converge or diverge? For the answer, see Section 5.7. In elementary calculus, you probably never seen the power series representations of tangent and secant. This is because these series are somewhat sophisticated mathematically speaking. In Section 5.8 we shall derive the power representations

$$\boxed{\tan z = \sum_{n=1}^\infty (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)\,B_{2n}}{(2n)!}\, z^{2n-1},}$$

and

$$\boxed{\sec z = \sum_{n=0}^\infty (-1)^n \frac{E_{2n}}{(2n)!}\, z^{2n}.}$$

Here, the B_{2n}'s are called "Bernoulli numbers" and the E_{2n}'s are called "Euler numbers," which are certain numbers having extraordinary properties. Although you've probably never seen the tangent and secant power series, you might have seen the logarithmic, binomial, and arctangent series:

$$\boxed{\log(1+z) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} z^n\ ,\ \ (1+z)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n} z^n\ ,\ \ \arctan z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n+1}}{2n+1}}$$

where $\alpha \in \mathbb{R}$. You most likely used calculus (derivatives and integrals) to derive these formulæ. In Section 5.9 we shall derive these formulæ without any calculus. Finally, in Sections 5.10 and 5.11 we derive many incredible and awe-inspiring formulæ involving π. In particular, in Section 5.11 we look at the famous Basel problem: Find the sum of the reciprocals of the squares of the natural numbers, $\sum_{n=1}^\infty \frac{1}{n^2}$. The answer, first given by Euler in 1734, is $\pi^2/6$.

Chapter 5 objectives: The student will be able to ...

- determine the convergence, and radius and interval of convergence, for an infinite series and power series, respectively, using various tests, e.g. Dirichlet, Abel, ratio, root, and others.
- apply Cauchy's double series theorem and know how it relates to rearrangement, and multiplication and composition of power series.
- identify series formulæ for the various elementary functions (binomial, arctangent, etc.) and for π.

5.1. Summation by parts, bounded variation, and alternating series

In elementary calculus, you studied "integration by parts," a formula I'm sure you used quite often trying to integrate tricky integrals. In this section we study a discrete version of the integration by parts formula called "summation by parts," which is used to sum tricky summations! Summation by parts has broad applications, including finding sums of powers of integers and to derive some famous convergence tests for series, the Dirichlet and Abel tests.

5.1.1. Summation by parts and Abel's lemma. Here is the famous summation by parts formula. The formula is complicated, but the proof is simple.

THEOREM 5.1 (**Summation by parts**). *For any complex sequences $\{a_n\}$ and $\{b_n\}$, we have*

$$\sum_{k=m}^{n} b_{k+1}(a_{k+1} - a_k) + \sum_{k=m}^{n} a_k(b_{k+1} - b_k) = a_{n+1}b_{n+1} - a_m b_m.$$

PROOF. Combining the two terms on the left, we obtain

$$\sum_{k=m}^{n} \Big[b_{k+1}a_{k+1} - b_{k+1}a_k + a_k b_{k+1} - a_k b_k\Big] = \sum_{k=m}^{n} \big(b_{k+1}a_{k+1} - a_k b_k\big),$$

which is a telescoping sum, leaving us with only $a_{n+1}b_{n+1} - a_m b_m$. □

As an easy corollary, we get Abel's lemma named after Niels Abel[1] (1802–1829).

COROLLARY 5.2 (**Abel's lemma**). *Let $\{a_n\}$ and $\{b_n\}$ be any complex sequences and let s_n denote the n-th partial sum of the series corresponding to the sequence $\{a_n\}$. Then for any $m < n$ we have*

$$\sum_{k=m+1}^{n} a_k b_k = s_n b_n - s_m b_m - \sum_{k=m}^{n-1} s_k(b_{k+1} - b_k).$$

PROOF. Applying the summation by parts formula to the sequences $\{s_n\}$ and $\{b_n\}$, we obtain

$$\sum_{k=m}^{n-1} b_{k+1}(s_{k+1} - s_k) + \sum_{k=m}^{n-1} s_k(b_{k+1} - b_k) = s_n b_n - s_m b_m.$$

Since $a_{k+1} = s_{k+1} - s_k$, we conclude that

$$\sum_{k=m}^{n-1} b_{k+1}a_{k+1} + \sum_{k=m}^{n-1} s_k(b_{k+1} - b_k) = s_n b_n - s_m b_m.$$

Replacing k with $k-1$ in the first sum and bringing the second sum to the right, we get our result. □

Summation by parts is a very useful tool. We shall apply it find sums of powers of integers (cf. [**189**], [**60**]); see the exercises for more applications.

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[1] *Abel has left mathematicians enough to keep them busy for 500 years. Charles Hermite (1822–1901), in "Calculus Gems"* [**161**].

5.1.2. Sums of powers of integers.

Example 5.1. Let $a_k = k$ and $b_k = k$. Then each of the differences $a_{k+1} - a_k$ and $b_{k+1} - b_k$ equals 1, so by summation by parts, we have

$$\sum_{k=1}^{n}(k+1) + \sum_{k=1}^{n} k = (n+1)(n+1) - 1 \cdot 1.$$

This sum reduces to

$$2\sum_{k=1}^{n} k = (n+1)^2 - n - 1 = n(n+1),$$

which gives the well-known result:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Example 5.2. Now let $a_k = k^2 - k = k(k-1)$ and $b_k = k - 1/2$. In this case, $a_{k+1} - a_k = (k+1)k - k(k-1) = 2k$ and $b_{k+1} - b_k = 1$, so by the summation by parts formula,

$$\sum_{k=1}^{n}\left(k + \frac{1}{2}\right)(2k) + \sum_{k=1}^{n}(k^2 - k)(1) = (n+1)n \cdot \left(n + \frac{1}{2}\right).$$

The first sum on the left contains the sum $\sum_{k=1}^{n} k$ and the second one contains the negative of the same sum. Cancelling, we get

$$3\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{2},$$

which gives the well-known result:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Example 5.3. For our final result, let $a_k = k^2$ and $b_k = (k-1)^2$. Then $a_{k+1} - a_k = (k+1)^2 - k^2 = 2k+1$ and $b_{k+1} - b_k = 2k - 1$, so by the summation by parts formula,

$$\sum_{k=1}^{n}(k+1)^2(2k+1) + \sum_{k=1}^{n} k^2(2k-1) = (n+1)^2 \cdot n^2.$$

Simplifying the left-hand side, we get

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

In Section 12.5, using integration techniques, we'll find a formula for $1^p + \cdots + n^p$ for any natural number p in terms of the Bernoulli numbers.

5.1.3. Sequences of bounded variation and Dirichlet's test. A sequence $\{a_n\}$ of complex numbers is said to be of **bounded variation** if

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty.$$

Typical examples of a sequences of of bounded variation are bounded monotone sequences of real numbers. A nice property of general sequences of bounded variation is that they always converge. We prove these facts in the following

PROPOSITION 5.3. *Any sequence of bounded variation converges. Moreover, any bounded monotone sequence is of bounded variation.*

PROOF. Let $\{a_n\}$ be of bounded variation. Given $m < n$, we can write $a_n - a_m$ as a telescoping sum:

$$a_n - a_m = (a_{m+1} - a_m) + (a_{m+2} - a_{m+1}) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) = \sum_{k=m}^{n} (a_{k+1} - a_k).$$

Hence,

$$|a_n - a_m| \le \sum_{k=m}^{n} |a_{k+1} - a_k|.$$

By assumption, the sum $\sum_{k=1}^{\infty} |a_{k+1} - a_k|$ converges, so the sum on the right-hand side of this inequality can be made arbitrarily small as $m, n \to \infty$ (Cauchy's criterion for series). Thus, $\{a_n\}$ is Cauchy and hence converges.

Now let $\{a_n\}$ be a nondecreasing and bounded sequence. We shall prove that this sequence is of bounded variation; the proof for a nonincreasing sequence is similar. In this case, we have $a_n \le a_{n+1}$ for each n, so for each n,

$$\sum_{k=1}^{n} |a_{k+1} - a_k| = \sum_{k=1}^{n} (a_{k+1} - a_k) = (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}) + (a_{n+1} - a_n) = a_{n+1},$$

since the sum telescoped. Since the sequence $\{a_n\}$ is by assumption bounded, it follows that the partial sums of the infinite series $\sum_{n=1}^{\infty} |a_{n+1} - a_n|$ are bounded, hence the series must converge by the nonnegative series test (Theorem 3.20). $\square$

Here's a useful test named after Johann Dirichlet (1805–1859).

THEOREM 5.4 (**Dirichlet's test**). *Suppose that the partial sums of the series $\sum a_n$ are uniformly bounded (although the series $\sum a_n$ may not converge). Then for any sequence $\{b_n\}$ that is of bounded variation and converges to zero, the series $\sum a_n b_n$ converges. In particular, the series $\sum a_n b_n$ converges if $\{b_n\}$ is a monotone sequence of real numbers approaching zero.*

PROOF. The trick to use Abel's lemma to rewrite $\sum a_n b_n$ in terms of an absolutely convergent series. Define $a_0 = 0$ (so that $s_0 = a_0 = 0$) and $b_0 = 0$. Then setting $m = 0$ in Abel's lemma, we can write

$$\sum_{k=1}^{n} a_k b_k = s_n b_n - \sum_{k=1}^{n-1} s_k (b_{k+1} - b_k). \tag{5.1}$$

Now we are given two facts: The first is that the partial sums $\{s_n\}$ are bounded, say by a constant C, and the second is that the sequence $\{b_n\}$ is of bounded variation and converges to zero. Since $\{s_n\}$ is bounded and $b_n \to 0$ it follows that $s_n b_n \to 0$. Since $|s_n| \le C$ for all n and $\{b_n\}$ is of bounded variation, the sum $\sum_{k=1}^{\infty} s_k (b_{k+1} - b_k)$ is absolutely convergent:

$$\sum_{k=1}^{\infty} |s_k (b_{k+1} - b_k)| \le C \sum_{k=1}^{\infty} |b_{k+1} - b_k| < \infty.$$

Therefore, taking $n \to \infty$ in (5.1) it follows that the sum $\sum a_k b_k$ converges (and equals $\sum_{k=1}^{\infty} s_k(b_{k+1} - b_k)$), and our proof is complete. $\square$

Here is an example that everyone uses to illustrate Dirichlet's test.

Example 5.4. For $x \in (0, 2\pi)$, determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n}.$$

To do so, we let $a_n = \cos nx$ and $b_n = 1/n$. Since $\{1/n\}$ is a monotone sequence converging to zero, by Dirichlet's test, if we can prove that the partial sums of $\sum \cos nx$ are bounded, then $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ converges. To establish this boundedness, we observe that $\operatorname{Re} e^{i\theta} = \cos\theta$ for any $\theta \in \mathbb{R}$, so

$$\sum_{n=1}^{m} \cos nx = \operatorname{Re} \sum_{n=1}^{m} e^{inx} = \operatorname{Re} \frac{1 - e^{i(m+1)x}}{1 - e^{inx}},$$

where we summed $\sum_{n=1}^{m} (e^{ix})^n$ via the geometric progression (2.3). Hence,

$$\left| \sum_{n=1}^{m} \cos nx \right| \leq \left| \frac{1 - e^{i(m+1)x}}{1 - e^{inx}} \right| \leq \frac{2}{|1 - e^{ix}|}.$$

Since $1 - e^{ix} = e^{ix/2}(e^{-ix/2} - e^{ix/2}) = -2ie^{ix/2}\sin(x/2)$, we see that

$$|1 - e^{ix}| = 2|\sin(x/2)| \quad \Longrightarrow \quad \left| \sum_{n=1}^{m} \sin nx \right| \leq \frac{1}{\sin(x/2)}.$$

Thus, for $x \in (0, 2\pi)$, by Dirichlet's test, given any sequence $\{b_n\}$ of bounded variation that converges to zero, the sum $\sum_{n=1}^{\infty} b_n \cos nx$ converges. In particular, $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ converges, and more generally, $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$ converges for any $p > 0$. A similar argument shows that for any $x \in (0, 2\pi)$, $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converges.

Before going to other tests, it might be interesting to note that we can determine the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ without using the fancy technology of Dirichlet's test. To this end, observe that from the addition formulas for $\sin(n \pm 1/2)x$, we have

$$\cos nx = \frac{\sin(n + 1/2)x - \sin(n - 1/2)x}{2\sin(x/2)},$$

which implies that, after gathering like terms,

$$\begin{aligned}
\sum_{n=1}^{m} \frac{\cos nx}{n} &= \frac{1}{2\sin(x/2)} \sum_{n=1}^{m} \frac{\sin(n + 1/2)x - \sin(n - 1/2)x}{n} \\
&= \frac{1}{2\sin(x/2)} \bigg(\frac{\sin(3x/2) - \sin(x/2)}{1} + \frac{\sin(5x/2) - \sin(3x/2)}{3} \\
&\qquad\qquad + \cdots + \frac{\sin(m + 1/2)x - \sin(m - 1/2)x}{m} \bigg) \\
&= \frac{1}{2\sin(x/2)} \left(-\sin(x/2) + \frac{\sin(m + 1/2)x}{m} + \sum_{n=1}^{m-1} \sin(n + 1/2)x \left(\frac{1}{n} - \frac{1}{n+1} \right) \right) \\
&= \frac{1}{2\sin(x/2)} \left(-\sin(x/2) + \frac{\sin(m + 1/2)x}{m} + \sum_{n=1}^{m-1} \sin(n + 1/2)x \frac{1}{n(n+1)} \right).
\end{aligned}$$

Therefore,

$$\frac{1}{2} + \sum_{n=1}^{m} \frac{\cos nx}{n} = \frac{\sin(m+1/2)x}{2m\sin(x/2)} + \sum_{n=1}^{m-1} \left(\frac{\sin(n+1/2)x}{2\sin(x/2)} \cdot \frac{1}{n(n+1)} \right). \tag{5.2}$$

Since the sine is always bounded by 1 and $\sum 1/n(n+1)$ converges, it follows that as $m \to \infty$, the first term on the right of (5.2) tends to zero while the summation on the right of (5.2) converges; in particular, the series in question converges, and we get the following pretty formula:

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx}{n} = \frac{1}{2\sin(x/2)} \sum_{n=1}^{\infty} \frac{\sin(n+1/2)x}{n(n+1)} \quad , \quad x \in (0, 2\pi).$$

In Example 5.39 of Section 5.9, we'll show that $\sum_{n=1}^{\infty} \frac{\cos nx}{n} = \log(2\sin(x/2))$.

5.1.4. Alternating series tests, $\log 2$**, and the irrationality of** e**.** As a direct consequence of Dirichlet's test, we immediately get the alternating series test.

Theorem 5.5 (**Alternating series test**). *If $\{a_n\}$ is a sequence of bounded variation that converges to zero, then the sum $\sum(-1)^{n-1}a_n$ converges. In particular, if $\{a_n\}$ is a monotone sequence of real numbers approaching zero, then the sum $\sum(-1)^{n-1}a_n$ converges.*

Proof. Since the partial sums of $\sum(-1)^{n-1}$ are bounded and $\{a_n\}$ is of bounded variation and converges to zero, the sum $\sum(-1)^{n-1}a_n$ converges by Dirichlet's test. □

Example 5.5. The **alternating harmonic series**

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + - \cdots$$

converges. Of course, we already knew this and we also know that the value of the alternating harmonic series equals $\log 2$ (see Section 4.6).

We now come to a very useful theorem for approximation purposes.

Corollary 5.6 (**Alternating series error estimate**). *If $\{a_n\}$ is a monotone sequence of real numbers approaching zero, and if s denotes the sum $\sum(-1)^{n-1}a_n$ and s_n denotes the n-th partial sum, then*

$$|s - s_n| \le |a_{n+1}|.$$

Proof. To establish the error estimate, we assume that $a_n \ge 0$ for each n, in which case we have $a_1 \ge a_2 \ge a_3 \ge a_4 \ge \cdots \ge 0$. (The case when $a_n \le 0$ is similar or can be derived from the present case by multiplying by -1.) Let's consider how $s = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is approximated by the s_n's. Observe that $s_1 = a_1$ increases from $s_0 = 0$ by the amount a_1; $s_2 = a_1 - a_2 = s_1 - a_2$ decreases from s_1 by the amount a_2; $s_3 = a_1 - a_2 + a_3 = s_2 + a_3$ increases from s_2 by the amount a_3, and so on; see Figure 5.1 for a picture of what's going on here. Studying this figure also shows why $|s - s_n| \le a_{n+1}$ holds. For this reason, we shall leave the exact proof details to the diligent and interested reader! □

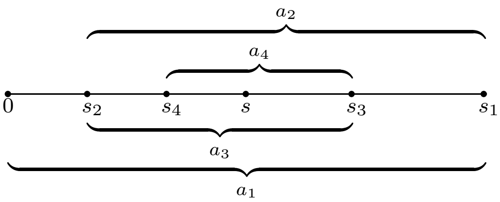

FIGURE 5.1. The partial sums $\{s_n\}$ jump forward and backward by the amounts given by the a_n's. This picture also shows that $|s - s_1| \le a_2$, $|s - s_2| \le a_3$, $|s - s_3| \le a_4$, ...

Example 5.6. Suppose that we wanted to find $\log 2$ to two decimal places (in base 10); that is, we want to find b_0, b_1, b_2 in the decimal expansion $\log 2 = b_0.b_1b_2$ where by the usual convention, b_2 is "rounded up" if $b_3 \ge 5$. We can determine these decimals by finding n such that s_n, the n-th partial sum of $\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, satisfies

$$|\log 2 - s_n| < 0.005;$$

that is,

$$\log 2 - 0.005 < s_n < \log 2 + 0.005.$$

Can you see why these inequalities guarantee that s_n has a decimal expansion starting with $b_0.b_1b_2$? Any case, according to the alternating series error estimate, we can make this this inequality hold by choosing n such that

$$|a_{n+1}| = \frac{1}{n+1} < 0.005 \quad \Longrightarrow \quad 500 < n+1 \quad \Longrightarrow \quad n = 500 \ \text{ works.}$$

With about five hours of pencil and paper work (and ten coffee breaks ☺) we find that $s_{500} = \sum_{n=1}^{500} \frac{(-1)^n}{n} = 0.69$ to two decimal places. Thus, $\log 2 = 0.69$ to two decimal places. A lot of work just to get two decimal places!

Example 5.7. (**Irrationality of e, Proof II**) Another nice application of the alternating series error estimate (or rather its proof) is a simple proof that e is irrational, cf. [**135**], [**6**]. Indeed, on the contrary, let us assume that $e = m/n$ where $m, n \in \mathbb{N}$. Then we can write

$$\frac{n}{m} = e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \quad \Longrightarrow \quad \frac{n}{m} - \sum_{k=0}^{m} \frac{(-1)^k}{k!} = \sum_{k=m+1}^{\infty} \frac{(-1)^k}{k!}.$$

Multiplying both sides by $m!/(-1)^{m+1} = \pm m!$, we obtain

$$\pm\left(n\,(m-1)! - \sum_{k=0}^{m} (-1)^k \frac{m!}{k!}\right) = \sum_{k=m+1}^{\infty} \frac{(-1)^{k-m-1} m!}{k!} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} m!}{(m+k)!}. \tag{5.3}$$

For $0 \le k \le m$, $m!/k!$ is an integer (this is because $m! = 1 \cdot 2 \cdots k \cdot (k+1) \cdots m$ contains a factor of $k!$), therefore the left-hand side of (5.3) is an integer, say $a \in \mathbb{Z}$, so that $a = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} m!}{(m+k)!}$. Thus, as seen in Figure 5.1, we have

$$0 < a < s_1 = \sum_{k=1}^{1} \frac{(-1)^{k-1} m!}{(m+k)!} = \frac{1}{m+1}.$$

Now recall that $m \in \mathbb{N}$, so $1/(m+1) \le 1/2$. Thus, a is an integer strictly between 0 and $1/2$; an obvious contradiction!

5.1.5. Abel's test for series. Now let's modify the sum $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$, say to the slightly more complicated version

$$\sum_{n=1}^{\infty} \Big(1+\frac{1}{n}\Big)^n \frac{\cos nx}{n}.$$

If we try to determine the convergence of this series using Dirichlet's test, we'll have to do some work, but if we're feeling a little lazy, we can use the following theorem, whose proof uses an "$\varepsilon/3$-trick."

THEOREM 5.7 (**Abel's test for series**). *Suppose that $\sum a_n$ converges. Then for any sequence $\{b_n\}$ of bounded variation, the series $\sum a_n b_n$ converges.*

PROOF. We shall apply Abel's lemma to establish that the sequence of partial sums for $\sum a_n b_n$ forms a Cauchy sequence, which implies that $\sum a_n b_n$ converges. For $m < n$, by Abel's lemma, we have

$$\sum_{k=m+1}^{n} a_k b_k = s_n b_n - s_m b_m - \sum_{k=m}^{n-1} s_k (b_{k+1} - b_k), \tag{5.4}$$

where s_n is the n-th partial sum of the series $\sum a_n$. Adding and subtracting $s := \sum a_n$ to s_k on the far right of (5.4), we find that

$$\begin{aligned}\sum_{k=m}^{n-1} s_k (b_{k+1} - b_k) &= \sum_{k=m}^{n-1} (s_k - s)(b_{k+1} - b_k) + s \sum_{k=m}^{n-1} (b_{k+1} - b_k) \\ &= \sum_{k=m}^{n-1} (s_k - s)(b_{k+1} - b_k) + s b_n - s b_m,\end{aligned}$$

since the sum telescoped. Replacing this into (5.4), we obtain

$$\sum_{k=m+1}^{n} a_k b_k = (s_n - s) b_n - (s_m - s) b_m - \sum_{k=m}^{n-1} (s_k - s)(b_{k+1} - b_k).$$

Let $\varepsilon > 0$. Since $\{b_n\}$ is of bounded variation, this sequence converges by Proposition 5.3, so in particular is bounded and therefore, since $s_n \to s$, $(s_n - s)b_n \to 0$ and $(s_m - s) b_m \to 0$. Thus, we can choose N such that for $n, m > N$, we have $|(s_n - s)b_n| < \varepsilon/3$, $|(s_m - s)b_m| < \varepsilon/3$, and $|s_n - s| < \varepsilon/3$. Thus, for $N < m < n$, we have

$$\begin{aligned}\left|\sum_{k=m+1}^{n} a_k b_k\right| &\le |(s_n - s)b_n| + |(s_m - s)b_m| + \sum_{k=m}^{n-1} |(s_k - s)(b_{k+1} - b_k)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \sum_{k=m}^{n-1} |b_{k+1} - b_k|.\end{aligned}$$

Finally, since $\sum |b_{k+1} - b_k|$ converges, by the Cauchy criterion for series, the sum $\sum_{k=m}^{n-1} |b_{k+1} - b_k|$ can be made less than 1 for N chosen larger if necessary. Thus, for $N < m < n$, we have $|\sum_{k=m+1}^{n} a_k b_k| < \varepsilon$. This completes our proof. $\square$

Example 5.8. Back to our discussion above, we can write

$$\sum_{n=1}^{\infty}\Big(1+\frac{1}{n}\Big)^n\frac{\cos nx}{n}=\sum a_n\, b_n,$$

where $a_n = \frac{\cos nx}{n}$ and $b_n = (1+\frac{1}{n})^n$. Since we already know that $\sum_{n=1}^{\infty} a_n$ converges and that $\{b_n\}$ is nondecreasing and bounded above (by e — see Section 3.3) and therefore is of bounded variation, Abel's test shows that the series $\sum a_n b_n$ converges.

Exercises 5.1.

1. Following Fredricks and Nelsen [**60**], we use summation by parts to derive neat identities for the Fibonacci numbers. Recall that the Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.
   (a) Let $a_n = F_{n+1}$ and $b_n = 1$ in the summation by parts formula (see Theorem 5.1) to derive the identity:
   $$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1.$$
   (b) Let $a_n = b_n = F_n$ in the summation by parts formula to get
   $$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}.$$
   (c) What a_n's and b_n's would you choose to derive the formulas:
   $$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n} \quad , \quad 1 + F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1}?$$
2. Following Fort [**59**], we relate limits of arithmetic means to summation by parts.
   (a) Let $\{a_n\}, \{b_n\}$ be sequences of complex numbers and assume that $b_n \to 0$ and $\frac{1}{n}\sum_{k=1}^{n} k\,|b_{k+1} - b_k| \to 0$ as $n \to \infty$, and that for some constant C, we have $\big|\frac{1}{n}\sum_{k=1}^{n} a_k\big| \leq C$ for all n. Prove that
   $$\frac{1}{n}\sum_{k=1}^{n} a_k b_k \to 0 \quad \text{as } n \to \infty.$$
   (b) Apply this result to $a_n = (-1)^{n-1} n$ and $b_n = 1/\sqrt{n}$ to prove that
   $$\frac{1}{n}\Big(\sqrt{1}-\sqrt{2}+\sqrt{3}-\sqrt{4}+\cdots+(-1)^{n-1}\sqrt{n}\Big)=\frac{1}{n}\sum_{k=1}^{n}(-1)^k\sqrt{k}\to 0 \quad \text{as } n\to\infty.$$
3. Determine the convergence or divergence of the following series:
   $$(a)\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\frac{1}{7}--++\cdots \quad , \quad (b)\ \sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1}-\sqrt{n}).$$
   $$(c)\ \sum_{n=2}^{\infty}\frac{\cos nx}{\log n} \quad , \quad (d)\ \frac{1}{2\cdot 1}-\frac{1}{2\cdot 2}+\frac{1}{3\cdot 3}-\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}-\frac{1}{4\cdot 6}+-\cdots$$
   $$(e)\ \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}\log\frac{2n+1}{n} \quad , \quad (f)\ \sum_{n=2}^{\infty}\cos nx\,\sin\Big(\frac{x}{n}\Big)\ (x\in\mathbb{R}) \quad , \quad (g)\ \sum_{n=2}^{\infty}(-1)^{n-1}\frac{\log n}{n}.$$

5.2. Liminfs/sups, ratio/roots, and power series

It is a fact of life that most sequences simply do not converge. In this section we introduce limit infimums and supremums, which *always* exist, either as real numbers or as $\pm\infty$. We also study their basic properties. We need these limits to study the ratio and root tests. You've probably seen these tests before in elementary calculus, but in this section we'll look at them in a slightly more sophisticated way.

5.2.1. Limit infimums and supremums. Let $a_1, a_2, a_3, \ldots$ be any sequence of real numbers bounded from above. Let us put

$$s_n := \sup_{k \geq n} a_k = \sup\{a_n, a_{n+1}, a_{n+2}, a_{n+3}, \ldots\}.$$

Note that

$$s_{n+1} = \sup\{a_{n+1}, a_{n+2}, \ldots\} \leq \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} = s_n,$$

since the set $\{a_{n+1}, a_{n+2}, \ldots\}$ is smaller than the set $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$. Thus, $s_1 \geq s_2 \geq \cdots \geq s_n \geq s_{n+1} \geq \cdots$ is an nonincreasing sequence. In particular, being a monotone sequence, the limit $\lim s_n$ is defined either a real number or (properly divergent to) $-\infty$. We define

$$\boxed{\limsup a_n := \lim s_n = \lim_{n\to\infty} \Big(\sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} \Big).}$$

This limit, which again is either a real number or $-\infty$, is called the **limit supremum** or **lim sup** of the sequence $\{a_n\}$. This name fits since $\limsup a_n$ is exactly that, a limit of supremums. If $\{a_n\}$ is not bounded from above, then we define

$$\boxed{\limsup a_n := \infty \quad \text{if } \{a_n\} \text{ is not bounded from above.}}$$

We define an **extended real number** as a real number or the symbols $\infty = +\infty$, $-\infty$. Then it is worth mentioning that lim sups *always* exist as an extended real number, unlike regular limits which may not exist.

Example 5.9. We shall compute $\limsup a_n$ where $a_n = \frac{1}{n}$. According to the definition of lim sup, we first have to find s_n:

$$s_n := \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} = \sup\left\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \frac{1}{n+3}, \ldots\right\} = \frac{1}{n}.$$

Second, we take the limit of the sequence $\{s_n\}$:

$$\limsup a_n := \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Notice that $\lim a_n$ also exists and $\lim a_n = 0$, the same as the $\limsup$. We'll come back to this observation in Example 5.11 below.

Example 5.10. Consider the sequence $\{(-1)^n\}$. In this case, we know that $\lim(-1)^n$ does not exist. To find $\limsup(-1)^n$, we first compute s_n:

$$s_n = \sup\{(-1)^n, (-1)^{n+1}, (-1)^{n+2}, \ldots\} = \sup\{+1, -1\} = 1,$$

where we used that the set $\{(-1)^n, (-1)^{n+1}, (-1)^{n+2}, \ldots\}$ is just a set consisting of the numbers $+1$ and -1. Hence,

$$\limsup(-1)^n := \lim s_n = \lim 1 = 1.$$

We can also define a corresponding $\liminf a_n$, which is a limit of infimums. To do so, assume for the moment that our generic sequence $\{a_n\}$ is bounded from below. Consider the sequence $\{\iota_n\}$ where

$$\iota_n := \inf_{k \geq n} a_k = \inf\{a_n, a_{n+1}, a_{n+2}, a_{n+3}, \ldots\}.$$

Note that

$$\iota_n = \inf\{a_n, a_{n+2}, \ldots\} \leq \inf\{a_{n+1}, a_{n+2}, \ldots\} = \iota_{n+1},$$

since the set $\{a_n, a_{n+2}, \ldots\}$ on the left of $\leq$ is bigger than the set $\{a_{n+1}, a_{n+2}, \ldots\}$. Thus, $\iota_1 \leq \iota_2 \leq \cdots \leq \iota_n \leq \iota_{n+1} \leq \cdots$ is an nondecreasing sequence. In particular, being a monotone sequence, the limit $\lim \iota_n$ is defined either a real number or (properly divergent to) ∞. We define

$$\boxed{\liminf a_n := \lim \iota_n = \lim_{n\to\infty} \Big(\inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}\Big),}$$

which exists either as a real number or $+\infty$, is called the **limit infimum** or **lim inf** of $\{a_n\}$. If $\{a_n\}$ is not bounded from below, then we define

$$\boxed{\liminf a_n := -\infty \quad \text{if } \{a_n\} \text{ is not bounded from below.}}$$

Again, as with lim sups, lim infs always exist as extended real numbers.

Example 5.11. We shall compute $\liminf a_n$ where $a_n = \frac{1}{n}$. According to the definition of $\liminf$, we first have to find ι_n:

$$\iota_n := \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\} = \inf \left\{ \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \frac{1}{n+3}, \ldots \right\} = 0.$$

Second, we take the limit of ι_n:

$$\liminf a_n := \lim_{n\to\infty} \iota_n = \lim_{n\to\infty} 0 = 0.$$

Notice that $\lim a_n$ also exists and $\lim a_n = 0$, the same as $\liminf a_n$, which is the same as $\limsup a_n$ as we saw in Example 5.9. We are thus lead to make the following conjecture: If $\lim a_n$ exists, then $\limsup a_n = \liminf a_n = \lim a_n$; this conjecture is indeed true as we'll see in Property *(2)* of Theorem 5.8.

Example 5.12. If $a_n = (-1)^n$, then

$$\inf\{a_n, a_{n+1}, a_{n+2}, \ldots\} = \sup\{(-1)^n, (-1)^{n+1}, (-1)^{n+2}, \ldots\} = \inf\{+1, -1\} = -1.$$

Hence,

$$\liminf (-1)^n := \lim -1 = -1.$$

The following theorem contains the main properties of limit infimums and supremums that we shall need in the sequel.

THEOREM 5.8 (**Properties of lim inf/sup**). *If $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers, then*

(1) $\limsup a_n = -\liminf(-a_n)$ *and* $\liminf a_n = -\limsup(-a_n)$.

(2) $\lim a_n$ *is defined, as a real number or* $\pm\infty$*, if and only if* $\limsup a_n = \liminf a_n$*, in which case,*

$$\lim a_n = \limsup a_n = \liminf a_n.$$

(3) If $a_n \leq b_n$ *for all* n *sufficiently large, then*

$$\liminf a_n \leq \liminf b_n \quad \textit{and} \quad \limsup a_n \leq \limsup b_n.$$

(4) The following inequality properties hold:

(a) $\limsup a_n < a \implies$ *there is an* N *such that* $n > N \implies a_n < a$.

(b) $\limsup a_n > a \implies$ *there exist infinitely many* n*'s such that* $a_n > a$.

(c) $\liminf a_n < a \implies$ *there exist infinitely many* n*'s such that* $a_n < a$.

(d) $\liminf a_n > a \implies$ *there is an* N *such that* $n > N \implies a_n > a$.

Proof. To prove *(1)* assume first that $\{a_n\}$ is not bounded from above; then $\{-a_n\}$ is not bounded from below. Hence, $\limsup a_n := \infty$ and $\liminf(-a_n) := -\infty$, which implies *(1)* in this case. Assume now that $\{a_n\}$ is bounded above. Recall from Lemma 2.29 that given any nonempty subset $A \subseteq \mathbb{R}$ bounded above, we have $\sup A = -\inf(-A)$. Hence,

$$\sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} = -\inf\{-a_n, -a_{n+1}, -a_{n+2}, -a_{n+3}, \ldots\}.$$

Taking $n \to \infty$ on both sides, we get $\limsup a_n = -\liminf(-a_n)$.

We now prove *(2)*. Suppose first that $\lim a_n$ converges to a real number L. Then given $\varepsilon > 0$, there exists an N such that

$$L - \varepsilon \leq a_k \leq L + \varepsilon, \quad \text{for all } k > N,$$

which implies that for any $n > N$,

$$L - \varepsilon \leq \inf_{k \geq n} a_k \leq \sup_{k \geq n} a_k \leq L + \varepsilon.$$

Taking $n \to \infty$ implies that

$$L - \varepsilon \leq \liminf a_n \leq \limsup a_n \leq L + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\limsup a_n = L = \liminf a_n$. Reversing these steps, we leave you to show that if $\limsup a_n = L = \liminf a_n$, then $\{a_n\}$ converges to L. We now consider *(2)* in the case that $\lim a_n = +\infty$; the case where the limit is $-\infty$ is proved similarly. Then given any real number $M > 0$, there exists an N such that

$$n > N \quad \Longrightarrow \quad M \leq a_n.$$

This implies that

$$M \leq \inf_{k \geq n} a_k \leq \sup_{k \geq n} a_k.$$

Taking $n \to \infty$ we obtain

$$M \leq \liminf a_n \leq \limsup a_n.$$

Since $M > 0$ was arbitrary, it follows that $\limsup a_n = +\infty = \liminf a_n$. Reversing these steps, we leave you to show that if $\limsup a_n = +\infty = \liminf a_n$, then $a_n \to +\infty$.

To prove *(3)* note that if $\{a_n\}$ is not bounded from below, then $\liminf a_n := -\infty$ so $\liminf a_n \leq \liminf b_n$ automatically; thus, we may assume that $\{a_n\}$ is bounded from below. In this case, observe that $a_n \leq b_n$ for all n sufficiently large implies that, for n sufficiently large,

$$\inf\{a_n, a_{n+1}, a_{n+2}, \ldots\} \leq \inf\{b_n, b_{n+1}, b_{n+2}, b_{n+3}, \ldots\}$$

Taking $n \to \infty$, and using that limits preserve inequalities, now proves *(3)*. The proof that $\limsup a_n \leq \limsup b_n$ is similar.

Because this proof is becoming unbearably unbearable ☺ we'll only prove *(a)*, *(b)* of *(4)* leaving *(c)*, *(d)* to the reader. Assume that $\limsup a_n < a$, that is,

$$\lim_{n\to\infty} \Big(\sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} \Big) < a.$$

It follows that for some N, we have

$$n > N \quad \Longrightarrow \quad \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} < a,$$

that is, the least upper bound of $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$ is strictly less than a, so we must have we have $a_n < a$ for all $n > N$. Assume now that $\limsup a_n > a$. If

$\{a_n\}$ is not bounded from above (so that $\limsup a_n = \infty$) then there must exist infinitely many n's such that $a_n > a$, for otherwise if there were only finitely many n's such that $a_n > a$, then $\{a_n\}$ would be bounded from above, which would imply that $\limsup a_n < \infty$. Assume now that $\{a_n\}$ is bounded from above. Then,

$$\lim_{n\to\infty} \Big(\sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} \Big) > a$$

implies that for some N, we have

$$n > N \quad \Longrightarrow \quad \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} > a.$$

Now if there were only finitely many n's such that $a_n > a$, then we can choose $n > N$ large enough such that $a_k \le a$ for all $k \ge n$. However, this would imply that for such n, $\sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} \le a$, a contradiction. Hence, there are infinitely many n's such that $a_n > a$. $\square$

5.2.2. Ratio/root tests, and the exponential and ζ-functions, again. In elementary calculus you should have studied the ratio test: If the limit $L_1 := \lim \left|\frac{a_{n+1}}{a_n}\right|$ *exists*, then the series $\sum a_n$ converges if $L_1 < 1$ and diverges if $L_1 > 1$ (if $L_1 = 1$, then the test is inconclusive). You also studied the root test: If the limit $L_2 := \lim |a_n|^{1/n}$ *exists*, then the series $\sum a_n$ converges if $L_2 < 1$ and diverges if $L_2 > 1$ (if $L_2 = 1$, then the test is inconclusive). Now what if the limits $\lim \left|\frac{a_{n+1}}{a_n}\right|$ or $\lim |a_n|^{1/n}$ don't exist, are there still ratio and root tests? The answer is "yes," but we have to replace lim with lim inf's and lim sup's. Before stating these new ratio/root tests, we first consider the following important lemma.

LEMMA 5.9. *If $\{a_n\}$ is a sequence of nonzero complex numbers, then*

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

PROOF. The middle inequality is automatic (because inf's are $\le$ sup's), so we just need to prove the left and right inequalities. Consider the left one; the right one is analogous and is left to the reader. If $\liminf |a_{n+1}/a_n| = -\infty$, then there is nothing to prove, so we may assume that $\liminf |a_{n+1}/a_n| > -\infty$. Given any $b < \liminf |a_{n+1}/a_n|$, we shall prove that $b < \liminf |a_n|^{1/n}$. This proves the the left side in our desired inequalities, for, if on the contrary we have $\liminf |a_n|^{1/n} < \liminf |a_{n+1}/a_n|$, then choosing $b = \liminf |a_n|^{1/n}$, we would have

$$\liminf |a_n|^{1/n} < \liminf |a_n|^{1/n},$$

an obvious contradiction. So, let $b < \liminf |a_{n+1}/a_n|$. Choose a such that $b < a < \liminf |a_{n+1}/a_n|$. Then by Property *4 (d)* in Theorem 5.8, for some N, we have

$$n > N \quad \Longrightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| > a.$$

Fix $m > N$ and let $n > m > N$. Then we can write

$$|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \cdot \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdots \left|\frac{a_{m+1}}{a_m}\right| \cdot |a_m|.$$

There are $n - m$ quotients in this equality, each of which is greater than a, so

$$|a_n| > a \cdot a \cdots a \cdot |a_m| = a^{n-m} \cdot |a_m|,$$

or

$$|a_n|^{1/n} > a^{1-m/n} \cdot |a_m|^{1/n}. \tag{5.5}$$

Since

$$\lim_{n\to\infty} a^{1-m/n} \cdot |a_m|^{1/n} = a,$$

and limit infimums preserve inequalities, we have

$$\liminf |a_n|^{1/n} \geq \liminf a^{1-m/n} \cdot |a_m|^{1/n} = \lim_{n\to\infty} a^{1-m/n} \cdot |a_m|^{1/n} = a,$$

where we used Property *(2)* of Theorem 5.8. Since $a > b$, we have $b < \liminf |a_n|^{1/n}$ and our proof is complete. □

Here's Cauchy's root test, a far-reaching generalization of the root test you learned in elementary calculus.

THEOREM 5.10 (**Cauchy's root test**). *A series* $\sum a_n$ *converges absolutely or diverges according as*

$$\limsup \left|a_n\right|^{1/n} < 1 \qquad \text{or} \qquad \limsup \left|a_n\right|^{1/n} > 1.$$

PROOF. Suppose first that $\limsup \left|a_n\right|^{1/n} < 1$. Then we can choose $0 < a < 1$ such that $\limsup \left|a_n\right|^{1/n} < a$, which, by property *4 (a)* of Theorem 5.8, implies that for some N,

$$n > N \quad \Longrightarrow \quad \left|a_n\right|^{1/n} < a,$$

that is,

$$n > N \quad \Longrightarrow \quad \left|a_n\right| < a^n.$$

Since $a < 1$, we know that the infinite series $\sum a^n$ converges; thus by the comparison theorem, the sum $\sum |a_n|$ also converges, and hence $\sum a_n$ converges as well.

Assume now that $\limsup \left|a_n\right|^{1/n} > 1$. Then by Property *4 (b)* of Theorem 5.8, there are infinitely many n's such that $\left|a_n\right|^{1/n} > 1$. Thus, there are infinitely many n's such that $|a_n| > 1$. Hence by the n-term test, the series $\sum a_n$ cannot converge. □

It is important to remark that in the other case, that is, $\limsup \left|a_n\right|^{1/n} = 1$, this test does not give information as to convergence.

Example 5.13. Consider the series $\sum 1/n$, which diverges, and observe that $\limsup |1/n|^{1/n} = \lim 1/n^{1/n} = 1$ (see Section 3.1 for the proof that $\lim n^{1/n} = 1$). However, $\sum 1/n^2$ converges, and $\limsup |1/n^2|^{1/n} = \lim(1/n^{1/n})^2 = 1$ as well, so when $\limsup |a_n|^{1/n} = 1$ it's not possible to tell whether or not the series converges.

As with the root test, in elementary calculus you learned the ratio test most likely without proof, and, accepting by faith this test as correct I'm sure that you used it quite often to determine the convergence/divergence of many types of series. Here's d'Alembert's ratio test, a far-reaching generalization of the ratio test[2].

[2] *Allez en avant, et la foi vous viendra [push on and faith will catch up with you]. Advice to those who questioned the calculus by Jean Le Rond d'Alembert (1717–1783)* [**110**]

Theorem 5.11 (**d'Alembert's ratio test**). *A series $\sum a_n$, with a_n nonzero for n sufficiently large, converges absolutely or diverges according as*

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1 \qquad \textit{or} \qquad \liminf \left|\frac{a_{n+1}}{a_n}\right| > 1.$$

Proof. If we set $L := \limsup \left|a_n\right|^{1/n}$, then by Lemma 5.9, we have

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le L \le \limsup \left|\frac{a_{n+1}}{a_n}\right|. \tag{5.6}$$

Therefore, if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $L < 1$ too, so $\sum a_n$ converges absolutely by the root test. On the other hand, if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $L > 1$ too, so $\sum a_n$ diverges by the root test. □

We remark that in the other case, that is, $\liminf \left|\frac{a_{n+1}}{a_n}\right| \le 1 \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$, this test does not give information as to convergence. Indeed, the same divergent and convergent examples used for the root test, $\sum 1/n$ and $\sum 1/n^2$, have the property that $\liminf \left|\frac{a_{n+1}}{a_n}\right| = 1 = \limsup \left|\frac{a_{n+1}}{a_n}\right|$.

Note that if $\limsup \left|a_n\right|^{1/n} = 1$ (that is, the root test fails), then setting $L = 1$ in (5.6), we see that the ratio test also fails. Thus,

$$\text{root test fails} \quad \Longrightarrow \quad \text{ratio test fails}. \tag{5.7}$$

Therefore, if the root test fails one cannot hope to appeal to the ratio test.

Let's now consider some examples.

Example 5.14. First, our old friend:

$$\exp(z) := \sum_{n=1}^{\infty} \frac{z^n}{n!},$$

which we already knows converges, but for the fun of it, let's apply the ratio test. Observe that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}}\right| = |z| \cdot \frac{n!}{(n+1)!} = \frac{|z|}{n+1}.$$

Hence,

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$$

Thus, the exponential function $\exp(z)$ converges absolutely for all $z \in \mathbb{C}$. This proof was a little easier than the one in Section 3.7, but then again, back then we didn't have the up-to-day technology of the ratio test that we have now. Here's an example that fails.

Example 5.15. Consider the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \qquad \operatorname{Re} z > 1.$$

If $z = x + iy$ is separated into its real and imaginary parts, then

$$\left|a_n\right|^{1/n} = \left|\frac{1}{n^z}\right|^{1/n} = \Big(\frac{1}{n^x}\Big)^{1/n} = \Big(\frac{1}{n^{1/n}}\Big)^x.$$

Since $\lim n^{1/n} = 1$, it follows that

$$\lim |a_n|^{1/n} = 1$$

so the root test fails to give information, which also implies that the ratio test fails as well. Of course, using the comparison test as we did in the proof of Theorem 4.34 we already know that $\zeta(z)$ converges for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$.

It's easy to find examples of series for which the ratio test fails but the root test succeeds.

Example 5.16. A general class of examples that foil the ratio test are (see Problem 4)

$$a + b + a^2 + b^2 + a^3 + b^3 + a^4 + b^4 + \cdots \quad , \quad 0 < b < a < 1; \tag{5.8}$$

here, the odd terms are given by $a_{2n-1} = a^n$ and the even terms are given by $a_{2n} = b^n$. For concreteness, let us consider the series

$$\frac{1}{2} + \frac{1}{3} + \Big(\frac{1}{2}\Big)^2 + \Big(\frac{1}{3}\Big)^2 + \Big(\frac{1}{2}\Big)^3 + \Big(\frac{1}{3}\Big)^3 + \Big(\frac{1}{2}\Big)^4 + \Big(\frac{1}{3}\Big)^4 + \cdots .$$

Since

$$\Big|\frac{a_{2n}}{a_{2n-1}}\Big| = \Big|\frac{(1/3)^n}{(1/2)^n}\Big| = \Big(\frac{2}{3}\Big)^n$$

and

$$\Big|\frac{a_{2n+1}}{a_{2n}}\Big| = \Big|\frac{(1/2)^{n+1}}{(1/3)^n}\Big| = \Big(\frac{3}{2}\Big)^n \cdot \frac{1}{2},$$

It follows that $\liminf |a_{n+1}/a_n| = 0 < 1 < \infty = \limsup |a_{n+1}/a_n|$, so the ratio test does not give information. On the other hand, since

$$|a_{2n-1}|^{1/(2n-1)} = \big((1/2)^n\big)^{1/(2n-1)} = \Big(\frac{1}{2}\Big)^{\frac{n}{2n-1}}$$

and

$$|a_{2n}|^{1/(2n)} = \big((1/3)^{n-1}\big)^{1/(2n)} = \Big(\frac{1}{3}\Big)^{\frac{n-1}{2n}}$$

it follows that $\limsup |a_n|^{1/n} = (1/2)^{1/2} < 1$, so the series converges by the root test.

Thus, in contrast to (5.7),

$$\text{ratio test fails} \quad \not\Longrightarrow \quad \text{root test fails.}$$

However, in the following lemma we show that if the ratio test fails such that the *true* limit $\lim |\frac{a_{n+1}}{a_n}| = 1$, then the root test fails as well.

Lemma 5.12. *If* $|\frac{a_{n+1}}{a_n}| \to L$ *with* L *an extended real number, then* $|a_n|^{1/n} \to L$.

Proof. By Lemma 5.9, we know that

$$\liminf \Big|\frac{a_{n+1}}{a_n}\Big| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \Big|\frac{a_{n+1}}{a_n}\Big|.$$

By Theorem 5.8, a limit exists if and only if the lim inf and the lim sup have the same limit, so the outside quantities in these inequalities equal L. It follows that $\liminf |a_n|^{1/n} = \limsup |a_n|^{1/n} = L$ as well, and hence $\lim |a_n|^{1/n} = L$. $\square$

Let's do one last (important) example:

Example 5.17. Consider the series

$$1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)\,(2n+1)}. \tag{5.9}$$

Applying the ratio test, we have

$$\frac{a_{n+1}}{a_n} = \frac{(2n+1)(2n+1)}{(2n+2)(2n+3)} = \frac{4n^2+8n+1}{4n^2+10n+6} = \frac{1 + \frac{2}{n} + \frac{1}{4n^2}}{1 + \frac{5}{2n} + \frac{3}{2n^2}}. \tag{5.10}$$

Therefore, $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$, so the ratio and root test give no information! What can we do? We'll see that Raabe's test in Section 5.3 will show that (5.9) converges. Later on (Section 11.6) we'll see that the value of the series (5.9) equals $\pi/2$.

5.2.3. Power series. Our old friend

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is an example of a **power series**, by which we mean a series of the form

$$\sum_{n=0}^{\infty} a_n\, z^n, \quad \text{where} \quad z \in \mathbb{C}, \qquad \text{or} \qquad \sum_{n=0}^{\infty} a_n\, x^n, \quad \text{where} \quad x \in \mathbb{R},$$

where $a_n \in \mathbb{C}$ for all n (in particular, the a_n's may be real). However, we shall focus on power series of the complex variable z *although essentially everything we mention works for real variables* x.

Example 5.18. Besides the exponential function, other familiar examples of power series include the trigonometric series, $\sin z = \sum_{n=0}^{\infty} (-1)^n z^{2n+1}/(2n+1)!$, $\cos z = \sum_{n=0}^{\infty} (-1)^n z^{2n}/(2n)!$.

The convergence of power series is quite easy to analyze. First, $\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$ certainly converges if $z = 0$. For $|z| > 0$ we can use the root test: Observe that (see Problem 7 for the proof that we can take out $|z|$)

$$\limsup \left|a_n z^n\right|^{1/n} = \limsup |z|\,|a_n|^{1/n} = |z|\, \limsup |a_n|^{1/n}.$$

Therefore, $\sum a_n z^n$ converges (absolutely) or diverges according as

$$|z| \cdot \limsup |a_n|^{1/n} < 1 \quad \text{or} \quad |z| \cdot \limsup |a_n|^{1/n} > 1.$$

Therefore, if we define $0 \le R \le \infty$ by

$$\boxed{R := \frac{1}{\limsup |a_n|^{1/n}}} \tag{5.11}$$

where by convention, we put $R := +\infty$ when $\limsup |a_n|^{1/n} = 0$ and $R := 0$ when $\limsup |a_n|^{1/n} = +\infty$, then it follows that $\sum a_n z^n$ converges (absolutely) or diverges according to $|z| < R$ or $|z| > R$; when $|z| = R$, anything can happen. According to Figure 5.2, it is quite fitting to call R the **radius of convergence**. Let us summarize our findings in the following theorem named after Cauchy (whom we've already met many times) and Jacques Hadamard (1865–1963).[3]

[3] *The shortest path between two truths in the real domain passes through the complex domain. Jacques Hadamard (1865–1963). Quoted in The Mathematical Intelligencer 13 (1991).*

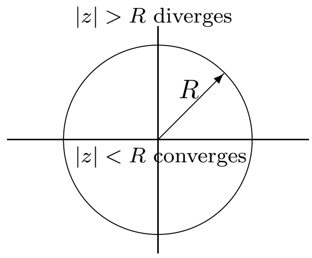

FIGURE 5.2. $\sum a_n z^n$ converges (absolutely) or diverges according as $|z| < R$ or $|z| > R$.

THEOREM 5.13 (**Cauchy-Hadamard theorem**). *If R is the radius of convergence of the power series $\sum a_n z^n$, then the series is absolutely convergent for $|z| < R$ and is divergent for $|z| > R$.*

One final remark. Suppose that the a_n's are nonzero for n sufficiently large and $\lim |\frac{a_n}{a_{n+1}}|$ exists. Then by Lemma 5.12, we have

$$\boxed{R = \lim \left| \frac{a_n}{a_{n+1}} \right|.} \tag{5.12}$$

This formula for the radius of convergence might, in some cases, be easier to work with than the formula involving $|a_n|^{1/n}$.

EXERCISES 5.2.

1. Find the lim inf/sups of the following sequences:

$$(a)\ a_n = \frac{2 + (-1)^n}{4} \quad , \quad (b)\ a_n = (-1)^n \Big(1 - \frac{1}{n}\Big) \quad , \quad (c)\ a_n = 2^{(-1)^n} \quad , \quad (d)\ a_n = 2^{n(-1)^n}.$$

   (e) If $\{r_n\}$ is a list of all rationals in $(0, 1)$, prove $\liminf r_n = 0$ and $\limsup r_n = 1$.

2. Investigate the following series for convergence (in (c), $z \in \mathbb{C}$):

$$(a)\ \sum_{n=1}^{\infty} \frac{(n+1)(n+2)\cdots(n+n)}{n^n}\ ,\ (b)\ \sum_{n=1}^{\infty} \frac{(n+1)^n}{n!}\ ,\ (c)\ \sum_{n=1}^{\infty} \frac{n^z}{n!}\ ,\ (d)\ \sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}}.$$

3. Determine the radius of convergence for the following series:

$$(a)\ \sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} z^n \quad , \quad (b)\ \sum_{n=1}^{\infty} \Big(\frac{n}{n+1}\Big)^n z^n \quad , \quad (d)\ \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} z^n \quad , \quad (d)\ \sum_{n=1}^{\infty} \frac{z^n}{n^p},$$

   where in the last sum, $p \in \mathbb{R}$. If $z = x \in \mathbb{R}$, state all $x \in \mathbb{R}$ such that the series (a), (b), (c) converge. For (c), your answer should depend on p.

4. (a) Investigate the series (5.8) for convergence using both the ratio and the root tests.
   (b) Here is another class of examples:

$$1 + a + b^2 + a^3 + b^4 + a^5 + b^6 + \cdots \quad , \quad 0 < a < b < 1.$$

   Show that the ratio test fails but the root test works.

5. Lemma 5.12 is very useful to determine certain limits which aren't obvious at first glance. Using this lemma, derive the following limits:

$$(a)\ \lim \frac{n}{(n!)^{1/n}} = e \ \ ,\ \ (b)\ \lim \frac{n+1}{(n!)^{1/n}} = e \ \ ,\ \ (c)\ \lim \frac{n}{[(n+1)(n+2)\cdots(n+n)]^{1/n}} = \frac{e}{4}.$$

   Suggestion: For (a), let $a_n = n^n/n!$. Prove that $\lim \frac{a_{n+1}}{a_n} = e$ and hence $\lim a_n^{1/n} = e$ as well. As a side remark, recall that (a) is called (the "weak") Stirling's formula, which we introduced in (3.27) and proved in Problem 4 of Exercises 3.3.

6. In this problem we investigate the interesting power series
$$\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n, \qquad z \in \mathbb{C}.$$
   (a) Prove that this series has radius of convergence $R = e$.
   (b) If $|z| = e$, then the ratio and root test both fail. However, if $|z| = e$, then prove that the infinite series diverges.
   (c) Investigate the convergence/divergence of $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$, where $z \in \mathbb{C}$.
7. Here are some lim inf/sup problems. Let $\{a_n\}, \{b_n\}$ be sequences of real numbers.
   (a) Prove that if $c > 0$, then $\liminf(ca_n) = c \liminf a_n$ and $\limsup(ca_n) = c \limsup a_n$. Here, we take the "obvious" conventions: $c \cdot \pm\infty = \pm\infty$.
   (b) Prove that if $c < 0$, then $\liminf(ca_n) = c \limsup a_n$ and $\limsup(ca_n) = c \liminf a_n$.
   (c) If $\{a_n\}, \{b_n\}$ are bounded, prove that $\liminf a_n + \liminf b_n \le \liminf(a_n + b_n)$.
   (d) If $\{a_n\}, \{b_n\}$ are bounded, prove that $\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$.
8. If $a_n \to L$ where L is a positive real number, prove that $\limsup(a_n \cdot b_n) = L \limsup b_n$ and $\liminf(a_n \cdot b_n) = L \liminf b_n$. Here are some steps if you want them:
   (i) Show that you can get the lim inf statement from the lim sup statement, hence we can focus on the lim sup statement. We shall prove that $\limsup(a_n b_n) \le L \limsup b_n$ and $L \limsup b_n \le \limsup(a_n b_n)$.
   (ii) Show that the inequality $\limsup(a_n b_n) \le L \limsup b_n$ follows if the following statement holds: If $\limsup b_n < b$, then $\limsup(a_n b_n) < L\, b$.
   (iii) Now prove that if $\limsup b_n < b$, then $\limsup(a_n b_n) < L\, b$. Suggestion: If $\limsup b_n < b$, then choose a such that $\limsup b_n < a < b$. Using Property *4 (a)* of Theorem 5.8 and the definition of $L = \lim a_n > 0$, prove that there is an N such that $n > N$ implies $b_n < a$ and $a_n > 0$. Conclude that for $n > N$, $a_n b_n < a a_n$. Finally, take lim sups of both sides of $a_n b_n < a a_n$.
   (iv) Show that the inequality $L \limsup b_n \le \limsup(a_n b_n)$ follows if the following statement holds: If $\limsup(a_n b_n) < L\, b$, then $\limsup b_n < b$; then prove this statement.
9. Let $\{a_n\}$ be a sequence of real numbers. We prove that there are monotone subsequences of $\{a_n\}$ that converge to $\liminf a_n$ and $\limsup a_n$. Proceed as follows:
   (i) Using Theorem 3.13, show that it suffices to prove that there are subsequences converging to $\liminf a_n$ and $\limsup a_n$
   (ii) Show that it suffices to that there is a subsequence converging to $\liminf a_n$.
   (iii) If $\liminf a_n = \pm\infty$, prove there is a subsequence converging to $\liminf a_n$.
   (iv) Now assume that $\liminf a_n = \lim_{n\to\infty} \big(\inf\{a_n, a_{n+1}, \ldots\}\big) \in \mathbb{R}$. By definition of limit, show that there is an n so that $a - 1 < \inf\{a_n, a_{n+1}, \ldots\} < a + 1$. Show that we can choose an n_1 so that $a - 1 < a_{n_1} < a + 1$. Then show there an $n_2 > n_1$ so that $a - \frac{1}{2} < a_{n_2} < a + \frac{1}{2}$. Continue this process.

5.3. A potpourri of ratio-type tests and "big $\mathcal{O}$" notation

In the previous section, we left it in the air whether or not the series
$$1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)\,(2n+1)}$$
converges (both the ratio and root tests failed). In this section we'll develop some new technologies that are able to detect the convergence of this series and other series for which the ratio and root tests *fail* to give information.

5.3.1. Kummer's test. The fundamental enhanced version of the ratio test is named after Ernst Kummer (1810–1893), from which we'll derive a potpourri of other ratio-type tests.

THEOREM 5.14 (**Kummer's test**). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers where the sum $\sum b_n$ diverges, and define*

$$\kappa_n = \frac{1}{b_n}\frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}}.$$

Then $\sum a_n$ converges or diverges according as $\liminf \kappa_n > 0$ or $\limsup \kappa_n < 0$. In particular, if κ_n tends to some definite limit, κ, then $\sum a_n$ converges to diverges according as $\kappa > 0$ or $\kappa < 0$.

PROOF. If $\liminf \kappa_n > 0$, then by Property *4 (d)* of Theorem 5.8, given any positive number a less than this limit infimum, there is an N such that

$$n > N \quad \Longrightarrow \quad \frac{1}{b_n}\frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}} > a.$$

Thus,

$$n > N \quad \Longrightarrow \quad \frac{1}{b_n}a_n - \frac{1}{b_{n+1}}a_{n+1} > a\, a_{n+1}. \tag{5.13}$$

Let $m > N$ and let $n > m > N$. Then (5.13) implies that

$$\sum_{k=m}^{n} a\, a_{k+1} < \sum_{k=m}^{n}\Big(\frac{1}{b_k}a_k - \frac{1}{b_{k+1}}a_{k+1}\Big) = \frac{1}{b_m}a_m - \frac{1}{b_{n+1}}a_{n+1},$$

since the sum telescoped. Therefore, as $\frac{1}{b_{n+1}}a_{n+1} > 0$, we have $\sum_{k=m}^{n} a\, a_{k+1} < \frac{1}{b_m}a_m$, or more succinctly,

$$\sum_{k=m}^{n} a_{k+1} < C$$

where $C = \frac{1}{a}\frac{1}{b_m}a_m$ is a constant independent of n. Since $n > m$ is completely arbitrary it follows that the partial sums of $\sum a_n$ always remain bounded by a fixed constant, so the sum must converge.

Assume now that $\limsup \kappa_n < 0$. Then by property *4 (a)* of Theorem 5.8, there is an N such that for all $n > N$, $\kappa_n < 0$, that is,

$$n > N \quad \Longrightarrow \quad \frac{1}{b_n}\frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}} < 0, \quad \text{that is,} \quad \frac{a_n}{b_n} < \frac{a_{n+1}}{b_{n+1}}.$$

Thus, for $n > N$, $\frac{a_n}{b_n}$ is increasing with n. In particular, fixing $m > N$, for all $n > m$, we have $C < a_n/b_n$, where $C = a_m/b_m$ is a constant independent of n. Thus, for all $n > m$, we have $Cb_n < a_n$ and since the sum $\sum b_n$ diverges, the comparison test implies that $\sum a_n$ diverges too. □

Note that d'Alembert's ratio test is just Kummer's test with $b_n = 1$ for each n.

5.3.2. Raabe's test and "big $\mathcal{O}$" notation. The following test, attributed to Joseph Ludwig Raabe (1801–1859), is just Kummer's test with the b_n's making up the harmonic series: $b_n = 1/n$.

THEOREM 5.15 (**Raabe's test**). *A series $\sum a_n$ of positive terms converges or diverges according as*

$$\liminf\ n\Big(\frac{a_n}{a_{n+1}} - 1\Big) > 0 \qquad \textit{or} \qquad \limsup\ n\Big(\frac{a_n}{a_{n+1}} - 1\Big) < 0.$$

In order to effectively apply Raabe's test, it is useful to first introduce some very handy notation. For a nonnegative function g, when we write $f = \mathcal{O}(g)$ ("**big O**" of g), we simply mean that $|f| \leq Cg$ for some constant C. In words, the big $\mathcal{O}$ notation just represents "a function that is in absolute value less than or equal to a constant times". This big $\mathcal{O}$ notation was introduced by Paul Bachmann (1837–1920) but became well-known through Edmund Landau (1877–1938) [**182**].

Example 5.19. For $x \geq 0$, we have

$$\frac{x^2}{1+x} = \mathcal{O}(x^2)$$

because $x^2/(1+x) \leq x^2$ for $x \geq 0$. Thus, for $x \geq 0$,

$$\frac{1}{1+x} = 1 - x + \frac{x^2}{1+x} \quad \Longrightarrow \quad \frac{1}{1+x} = 1 - x + \mathcal{O}(x^2). \tag{5.14}$$

In this section, we are mostly interested in using the big $\mathcal{O}$ notation when dealing with natural numbers.

Example 5.20. For $n \in \mathbb{N}$,

$$\frac{2}{n} + \frac{1}{4n^2} = \mathcal{O}\left(\frac{1}{n}\right), \tag{5.15}$$

because $\frac{2}{n} + \frac{1}{4n^2} \leq \frac{2}{n} + \frac{1}{4n} = \frac{C}{n}$ where $C = 2 + 1/4 = 9/4$.

Three important properties of the big $\mathcal{O}$ notation are (1) if $f = \mathcal{O}(ag)$ with $a \geq 0$, then $f = \mathcal{O}(g)$, and if $f_1 = \mathcal{O}(g_1)$ and $f_2 = \mathcal{O}(g_2)$, then (2) $f_1 f_2 = \mathcal{O}(g_1\, g_2)$ and (3) $f_1 + f_2 = \mathcal{O}(g_1 + g_2)$. To prove these properties, observe that if $|f| \leq C(ag)$, then $|f| \leq C'g$, where $C' = aC$, and that $|f_1| \leq C_1 g_1$ and $|f_2| \leq C_2 g_2$ imply

$$|f_1 f_2| \leq \big(C_1 C_2\big)\, g_1 g_2 \quad \text{and} \quad |f_1 + f_2| \leq (C_1 + C_2)\,(g_1 + g_2);$$

hence, our three properties.

Example 5.21. Thus, in view of (5.15), we have $\mathcal{O}\big(\frac{2}{n} + \frac{1}{4n^2}\big)^2 = \mathcal{O}\left(\frac{1}{n} \cdot \frac{1}{n}\right) = \mathcal{O}\left(\frac{1}{n^2}\right)$. Therefore, using (the right-hand part of) (5.14), we obtain

$$\begin{aligned} \frac{1}{1 + \left(\dfrac{2}{n} + \dfrac{1}{4n^2}\right)} &= 1 - \frac{2}{n} - \frac{1}{4n^2} + \mathcal{O}\left(\frac{2}{n} + \frac{1}{4n^2}\right)^2 = 1 - \frac{2}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) + \mathcal{O}\left(\frac{1}{n^2}\right) \\ &= 1 - \frac{2}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \end{aligned}$$

since $\mathcal{O}(2/n^2) = \mathcal{O}(1/n^2)$.

Here we can see the very "big" advantage of using the big $\mathcal{O}$ notation: it hides a lot of complicated junk information. For example, the left-hand side of the equation is exactly equal to (see the left-hand part of (5.14))

$$\frac{1}{1 + \left(\dfrac{2}{n} + \dfrac{1}{4n^2}\right)} = 1 - \frac{2}{n} + \left[-\frac{1}{4n^2} + \frac{\left(\frac{2}{n} + \frac{1}{4n^2}\right)^2}{1 + \frac{2}{n} + \frac{1}{4n^2}} \right],$$

so the big $\mathcal{O}$ notation allows us to summarize the complicated material on the right as the very simple $\mathcal{O}\big(\frac{1}{n^2}\big)$.

Example 5.22. Consider our "mystery" series

$$1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)\,(2n+1)}$$

already considered in (5.9). We saw that the ratio and root tests failed for this series; however, it turns out that Raabe's test works. To see this, let a_n denote the n-th term in the "mystery" series. Then from (5.10), we see that

$$\begin{aligned}\frac{a_n}{a_{n+1}} = \frac{1+\frac{5}{2n}+\frac{3}{2n^2}}{1+\frac{2}{n}+\frac{1}{4n^2}} &= \left(1+\frac{5}{2n}+\frac{3}{2n^2}\right)\left(1-\frac{2}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right)\\ &= \left(1+\frac{5}{2n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right)\left(1-\frac{2}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right).\end{aligned}$$

Multiplying out the right-hand side, using the properties of big $\mathcal{O}$, we get

$$\frac{a_n}{a_{n+1}} = 1+\frac{5}{2n}-\frac{2}{n}+\mathcal{O}\left(\frac{1}{n^2}\right) = 1+\frac{1}{2n}+\mathcal{O}\left(\frac{1}{n^2}\right).$$

Hence,

$$n\left(\frac{a_n}{a_{n+1}}-1\right) = \frac{1}{2}+\mathcal{O}\left(\frac{1}{n}\right) \quad\Longrightarrow\quad \lim n\left(\frac{a_n}{a_{n+1}}-1\right) = \frac{1}{2} > 0,$$

so by Raabe's test, the "mystery" sum converges. In Section 11.6 we'll see that the sum is equal to $\pi/2$.

5.3.3. De Morgan and Bertrand's test. We next study a test due to Augustus De Morgan (1806–1871) and Joseph Bertrand (1822–1900). For this test, we let $b_n = 1/n\log n$ in Kummer's test.

THEOREM 5.16 (**De Morgan and Bertrand's test**). *Let $\{a_n\}$ be a sequence of positive numbers and define α_n by the equation*

$$\frac{a_n}{a_{n+1}} = 1+\frac{1}{n}+\frac{\alpha_n}{n\log n}.$$

Then $\sum a_n$ converges or diverges according as $\liminf \alpha_n > 1$ or $\limsup \alpha_n < 1$.

PROOF. If we let $b_n = 1/n\log n$ in Kummer's test, then

$$\begin{aligned}\kappa_n = \frac{1}{b_n}\frac{a_n}{a_{n+1}}-\frac{1}{b_{n+1}} &= n\log n\left(1+\frac{1}{n}+\frac{\alpha_n}{n\log n}\right)-(n+1)\log(n+1)\\ &= \alpha_n+(n+1)\Big[\log n-\log(n+1)\Big].\end{aligned}$$

Since

$$(n+1)\Big[\log n-\log(n+1)\Big] = \log\left(1-\frac{1}{n+1}\right)^{n+1} \to \log e^{-1} = -1,$$

we have

$$\liminf \kappa_n = \liminf \alpha_n - 1 \quad\text{and}\quad \limsup \kappa_n = \limsup \alpha_n - 1.$$

Invoking Kummer's test now completes the proof. □

5.3.4. Gauss's test. Finally, to end our potpourri of tests, we conclude with Gauss' test:

THEOREM 5.17 (**Gauss' test**). *Let $\{a_n\}$ be a sequence of positive numbers and suppose that we can write*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\xi}{n} + \mathcal{O}\left(\frac{1}{n^p}\right),$$

where ξ is a constant and $p > 1$. Then $\sum a_n$ converges or diverges according as $\xi \le 1$ or $\xi > 1$.

PROOF. The hypotheses imply that

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \xi + n\mathcal{O}\left(\frac{1}{n^p}\right) = \xi + \mathcal{O}\left(\frac{1}{n^{p-1}}\right) \to \xi$$

as $n \to \infty$, where we used that $p-1 > 0$. Thus, Raabe's test shows that series $\sum a_n$ converges for $\xi > 1$ and diverges for $\xi < 1$. For the case $\xi = 1$, let $\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + f_n$ where $f_n = \mathcal{O}\left(\frac{1}{n^p}\right)$. Then we can write

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + f_n = 1 + \frac{1}{n} + \frac{\alpha_n}{n\log n},$$

where $\alpha_n = f_n\, n\log n$. If we let $p = 1+\delta$, where $\delta > 0$, then we know that $\frac{\log n}{n^\delta} \to 0$ as $n \to \infty$ by Problem 8 in Exercises 4.6, so

$$\alpha_n = f_n\, n\log n = \mathcal{O}\left(\frac{1}{n^{1+\delta}}\right) n\log n = \mathcal{O}\left(\frac{\log n}{n^\delta}\right) \quad\Longrightarrow\quad \lim \alpha_n = 0.$$

Thus, De Morgan and Bertrand's test shows that the series $\sum a_n$ diverges. $\square$

Example 5.23. Gauss' test originated with Gauss' study of the hypergeometric series:

$$1 + \frac{\alpha\cdot\beta}{1\cdot\gamma} + \frac{\alpha(\alpha-1)\cdot\beta(\beta-1)}{2!\cdot\gamma(\gamma+1)} + \frac{\alpha(\alpha-1)(\alpha-2)\cdot\beta(\beta-1)(\beta-2)}{3!\cdot\gamma(\gamma+1)(\gamma+2)} + \cdots,$$

where α, β, γ are positive real numbers. With

$$a_n = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\cdot\beta(\beta-1)(\beta-2)\cdot(\beta-n+1)}{n!\cdot\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)},$$

for $n \ge 1$, we have

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} = \frac{n^2+(\gamma+1)n+\gamma}{n^2+(\alpha+\beta)n+\alpha\beta} = \frac{1+\dfrac{\gamma+1}{n}+\dfrac{\gamma}{n^2}}{1+\dfrac{\alpha+\beta}{n}+\dfrac{\alpha\beta}{n^2}}.$$

Using the handy formula from (5.14),

$$\frac{1}{1+x} = 1 - x + \frac{x^2}{1+x},$$

we see that (after some algebra)

$$\begin{aligned}\frac{a_n}{a_{n+1}} &= \left(1+\frac{\gamma+1}{n}+\frac{\gamma}{n^2}\right)\left[1-\frac{\alpha+\beta}{n}-\frac{\alpha\beta}{n^2}+\mathcal{O}\left(\frac{1}{n^2}\right)\right] \\ &= 1+\frac{\gamma+1-\alpha-\beta}{n}+\mathcal{O}\left(\frac{1}{n^2}\right).\end{aligned}$$

Thus, the hypergeometric series converges if $\gamma > \alpha + \beta$ and diverges if $\gamma \leq \alpha + \beta$.

EXERCISES 5.3.

1. Determine whether or not the following series converge.

$$(a)\ \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n (n+1)!} \quad , \quad (b)\ \sum_{n=1}^{\infty} \frac{3 \cdot 6 \cdot 9 \cdots (3n)}{7 \cdot 10 \cdot 13 \cdots (3n+4)},$$

$$(c)\ \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \quad , \quad (d)\ \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n+2)}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n)}.$$

For $\alpha, \beta \neq 0, -1, -2, \ldots$,

$$(e)\ \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{n!} \quad , \quad (f)\ \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}.$$

If $\alpha, \beta, \gamma, \kappa, \lambda \neq 0, -1, -2, \ldots$, then prove that the following monster

$$(g)\ \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)\gamma(\gamma+1)\cdots(\gamma+n-1)}{n!\,\kappa(\kappa+1)\cdots(\kappa+n-1)\lambda(\lambda+1)\cdots(\lambda+n-1)}$$

converges for $\kappa + \lambda - \alpha - \beta - \gamma > 0$.

2. Using Raabe's test, prove that $\sum 1/n^p$ converges for $p > 1$ and diverges for $p < 1$.
3. (**Logarithmic test**) We prove a useful test called the **logarithmic test**: If $\sum a_n$ is a series of positive terms, then this series converges or diverges according as

$$\liminf \Big(n \log \frac{a_n}{a_{n+1}}\Big) > 1 \quad \text{or} \quad \limsup \Big(n \log \frac{a_n}{a_{n+1}}\Big) < 1.$$

To prove this, proceed as follows.

(i) Suppose first that $\liminf \big(n \log \frac{a_n}{a_{n+1}}\big) > 1$. Show that there is an $a > 1$ and an N such that

$$n > N \quad \Longrightarrow \quad a < n \log \frac{a_n}{a_{n+1}} \quad \Longrightarrow \quad \frac{a_{n+1}}{a_n} < e^{-a/n}.$$

(ii) Using $\big(1 + \frac{1}{n}\big)^n < e$ from (3.26), the p-test, and the limit comparison test (see Problem 7 in Exercises 3.6) previous problem, prove that $\sum a_n$ converges.

(iii) Similarly, prove that if $\limsup \big(n \log \frac{a_n}{a_{n+1}}\big) < 1$, then $\sum a_n$ diverges.

(iv) Using the logarithmic test, determine the convergence/diverence of

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^n}{n!}.$$

5.4. Some pretty powerful properties of power series

The title of this section speaks for itself. As stated already, we focus on power series of a complex variable z, but all the results stated in this section have corresponding statements for power series of a real variable x.

5.4.1. Continuity and the exponential function (again). We first prove that power series are always continuous (within their radius of convergence).

LEMMA 5.18. *If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, then $\sum_{n=1}^{\infty} n\, a_n z^{n-1}$ also has radius of convergence R.*

PROOF. (See Problem 3 for another proof of this lemma using properties of lim sup.) For $z \neq 0$, $\sum_{n=1}^{\infty} n\, a_n z^{n-1}$ converges if and only if $z \cdot \sum_{n=1}^{\infty} n\, a_n z^{n-1} = \sum_{n=1}^{\infty} n\, a_n z^n$ converges, so we just have to show that $\sum_{n=1}^{\infty} n\, a_n z^n$ has radius of convergence R. Since $|a_n| \leq n|a_n|$, by comparison, if $\sum_{n=1}^{\infty} n\, |a_n|\, |z|^n$ converges, then $\sum_{n=1}^{\infty} |a_n|\, |z|^n$ also converges, so the radius of convergence of the series $\sum_{n=1}^{\infty} n\, a_n z^n$

can't be larger than R. To prove that the radius of convergence is at least R, fix z with $|z| < R$; we need to prove that $\sum_{n=1}^{\infty} n\,|a_n|\,|z|^n$ converges. To this end, fix ρ with $|z| < \rho < R$ and note that $\sum_{n=1}^{\infty} n\,(|z|/\rho)^n$ converges, by e.g. the root test:

$$\lim \left| n\left(\frac{|z|}{\rho}\right)^n \right|^{1/n} = \lim n^{1/n} \cdot \frac{|z|}{\rho} = \frac{|z|}{\rho} < 1.$$

Since $\sum_{n=1}^{\infty} |a_n|\rho^n$ converges (because $\rho < R$, the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$), by the n-th term test, $|a_n|\rho^n \to 0$ as $n \to \infty$. In particular, $|a_n|\rho^n \le M$ for some constant M, hence

$$n\,|a_n|\,|z|^n = n\,|a_n|\,\rho^n \cdot \left(\frac{|z|}{\rho}\right)^n \le M \cdot n\left(\frac{|z|}{\rho}\right)^n.$$

Since $M \sum n\,(|z|/\rho)^n$ converges, by the comparison test, it follows that $\sum n\,|a_n|\,|z|^n$ also converges. This completes our proof. $\square$

Theorem 5.19 (**Continuity theorem for power series**). *A power series is continuous within its radius of convergence.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R; we need to show that $f(z)$ is continuous at each point $c \in \mathbb{C}$ with $|c| < R$. So, let us fix such a c. Since

$$z^n - c^n = (z-c)\,q_n(z), \quad \text{where} \quad q_n(z) = z^{n-1} + z^{n-2}\,c + \cdots + z\,c^{n-2} + c^{n-1},$$

which is proved by multiplying out $(z-c)\,q_n(z)$, we can write

$$f(z) - f(c) = \sum_{n=1}^{\infty} a_n (z^n - c^n) = (z-c)\sum_{n=0}^{\infty} a_n q_n(z).$$

To make the sum $\sum_{n=0}^{\infty} a_n q_n(z)$ small in absolute value we proceed as follows. Fix r such that $|c| < r < R$. Then for $|z-c| < r - |c|$, we have

$$|z| \le |z-c| + |c| < r - |c| + |c| = r.$$

Thus, as $|c| < r$, for $|z-c| < r - |c|$ we see that

$$|q_n(z)| \le \underbrace{r^{n-1} + r^{n-2}\,r + \cdots + r\,r^{n-2} + r^{n-1}}_{n \text{ terms}} = nr^{n-1}.$$

By our lemma, $\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}$ converges, so if $C := \sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}$, then

$$|f(z) - f(c)| \le |z-c| \sum_{n=1}^{\infty} |a_n|\,|q_n(z)| \le |z-c| \sum_{n=1}^{\infty} |a_n|\,nr^{n-1} = C|z-c|,$$

which implies that $\lim_{z\to c} f(z) = f(c)$; that is, f is continuous at $z = c$. $\square$

5.4.2. Abel's limit theorem. Abel's limit theorem has to do with the following question. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence R; this implies, in particular, that $f(x)$ is defined for all $-R < x < R$ and, by Theorem 5.19, is continuous on the interval $(-R, R)$. Let us suppose that $f(R) = \sum_{n=0}^{\infty} a_n R^n$ converges. In particular, $f(x)$ is defined for all $-R < x \le R$. Question: Is f continuous on the interval $(-R, R]$, that is, is it true that

$$\lim_{x\to R-} f(x) = f(R)? \tag{5.16}$$

The answer to this question is "yes" and it follows from the following more general theorem due to Neils Abel; however, Abel's theorem is mostly used for the real variable case $\lim_{x\to R-} f(x) = f(R)$ that we just described.

THEOREM 5.20 (**Abel's limit theorem**). *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ have radius of convergence R and let $z_0 \in \mathbb{C}$ with $|z_0| = R$ where the series $f(z_0) = \sum_{n=0}^\infty a_n z_0^n$ converges. Then*

$$\lim_{z\to z_0} f(z) = f(z_0)$$

where the limit on the left is taken in such a way that $|z| < R$ and that the ratio $\frac{|z_0-z|}{R-|z|}$ remains bounded by a fixed constant.

PROOF. By considering the limit of the function $g(z) = f(z_0 z) - f(z_0)$ as $z \to 1$ in such a way that $|z| < 1$ and that the ratio $|1-z|/(1-|z|)$ remains bounded by a fixed constant, we may henceforth assume that $z_0 = 1$ and that $f(z_0) = 0$ (the diligent student will check the details of this statement). If $s_n = a_0 + a_1 + \cdots + a_n$, then (because we're supposing $R = 1$) by assumption $0 = f(1) = \sum_{n=0}^\infty a_n = \lim s_n$. Now observe that $a_n = s_n - s_{n-1}$, so

$$\begin{aligned}
\sum_{k=0}^n a_k z^k &= a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \\
&= s_0 + (s_1 - s_0)z + (s_2 - s_1)z^2 + \cdots + (s_n - s_{n-1})z^n \\
&= s_0(1-z) + s_1(z - z^2) + \cdots + s_{n-1}(z^{n-1} - z^n) + s_n z^n \\
&= s_0(1-z) + s_1(1-z)z + \cdots + s_{n-1}(1-z)z^{n-1} + s_n z^n \\
&= (1-z)\big(s_0 + s_1 z + \cdots + s_{n-1} z^{n-1}\big) + s_n z^n
\end{aligned}$$

Thus, $\sum_{k=0}^n a_k z^k = (1-z)\sum_{k=0}^n s_k z^k + s_n z^n$. Since $s_n \to 0$ and $|z| < 1$ it follows that $s_n z^n \to 0$. Therefore, taking $n \to \infty$, we obtain

$$f(z) = \sum_{n=0}^\infty a_n z^n = (1-z)\sum_{n=0}^\infty s_n z^n,$$

which implies that

$$|f(z)| \le |1-z| \sum_{n=0}^\infty |s_n|\,|z|^n.$$

Let us now take $z \to 1$ in such a way that $|z| < 1$ and $|1-z|/(1-|z|) < C$ where $C > 0$. Let $\varepsilon > 0$ be given and, since $s_n \to 0$, we can choose an integer N such that $n > N \Longrightarrow |s_n| < \varepsilon/(2C)$. Define $K := \sum_{n=0}^N |s_n|$. Then we can write

$$\begin{aligned}
|f(z)| &\le |1-z| \sum_{n=0}^N |s_n|\,|z|^n + |1-z| \sum_{n=N}^\infty |s_n|\,|z|^n \\
&< |1-z| \sum_{n=0}^N |s_n| \cdot 1^n + |1-z| \sum_{n=N}^\infty \frac{\varepsilon}{2C}\,|z|^n \\
&= K|1-z| + \frac{\varepsilon}{2C}|1-z| \sum_{n=0}^\infty |z|^n \\
&= K|1-z| + \frac{\varepsilon}{2C}\frac{|1-z|}{1-|z|} < K|1-z| + \frac{\varepsilon}{2}.
\end{aligned}$$

Thus, with $\delta := \varepsilon/(2K)$, we have

$$|z-1| < \delta \ \text{ with } \ |z| < 1 \ \text{ and } \ \frac{|1-z|}{1-|z|} < C \quad \Longrightarrow \quad |f(z)| < \varepsilon.$$

This completes our proof. $\square$

Notice that for $z = x$ with $0 < x < R$, we have

$$\frac{|R-z|}{R-|z|} = \frac{|R-x|}{R-|x|} = \frac{R-x}{R-x} = 1$$

which is, in particular, bounded by 1, so (5.16) holds under the assumptions stated. Once we prove this result at $x = R$, we can prove a similar result at $x = -R$: If $f(x) = \sum_{n=0}^\infty a_n x^n$ has radius of convergence R and $f(-R) = \sum_{n=0}^\infty a_n (-R)^n$ converges, then

$$\lim_{x \to -R+} f(x) = f(-R).$$

To prove this, consider the function $g(x) = f(-x)$, then apply (5.16) to g.

5.4.3. The identity theorem. The identity theorem is perhaps one of the most useful properties of power series. The identity theorem says, very roughly, that if two power series are identical at "sufficiently many" points, then in fact, the power series are identical everywhere!

THEOREM 5.21 (**Identity theorem**). *Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ have positive radii of convergence and suppose that $f(c_k) = g(c_k)$ for some nonzero sequence $c_k \to 0$. Then the power series $f(z)$ and $g(z)$ must be identical; that is $a_n = b_n$ for every $n = 0, 1, 2, 3, \ldots$.*

PROOF. We begin by proving that for each $m = 0, 1, 2, \ldots$, the series

$$f_m(z) := \sum_{n=m}^\infty a_n z^{n-m} = a_m + a_{m+1} z + a_{m+2} z^2 + a_{m+3} z^3 + \cdots$$

has the same radius of convergence as f. Indeed, since we can write

$$f_m(z) = z^{-m} \sum_{n=m}^\infty a_n z^n$$

for $z \neq 0$, the power series $f_m(z)$ converges if and only if $\sum_{n=m}^\infty a_n z^n$ converges, which in turn converges if and only if $f(z)$ converges. It follows that $f_m(z)$ and $f(z)$ have the same radius of convergence; in particular, by the continuity theorem for power series, $f_m(z)$ is continuous at 0. Similarly, for each $m = 0, 1, 2, \ldots$, $g_m(z) := \sum_{n=m}^\infty b_n z^{n-m}$ has the same radius of convergence as $g(z)$; in particular, $g_m(z)$ is continuous at 0. These continuity facts concerning f_m and g_m are the important facts that will be used below.

Now to our proof. We are given that

$$a_0 + a_1 c_k + a_2 c_k^2 + \cdots = b_0 + b_1 c_k + b_2 c_k^2 + \cdots \quad \text{that is, } f(c_k) = g(c_k) \tag{5.17}$$

for all k. In particular, taking $k \to \infty$ in the equality $f(c_k) = g(c_k)$, using that $c_k \to 0$ and that f and g are continuous at 0, we obtain $f(0) = g(0)$, or $a_0 = b_0$. Cancelling $a_0 = b_0$ and dividing by $c_k \neq 0$ in (5.17), we obtain

$$a_1 + a_2 c_k + a_3 c_k^2 + \cdots = b_1 + b_2 c_k + b_3 c_k^2 + \cdots \quad \text{that is, } f_1(c_k) = g_1(c_k) \tag{5.18}$$

for all k. Taking $k \to \infty$ and using that $c_k \to 0$ and that f_1 and g_1 are continuous at 0, we obtain $f_1(0) = g_1(0)$, or $a_1 = b_1$. Cancelling $a_1 = b_1$ and dividing by $c_k \neq 0$ in (5.18), we obtain

$$a_2 + a_3 c_k + a_4 c_k^2 + \cdots = b_2 + b_3 c_k + b_4 c_k^2 + \cdots \quad \text{that is, } f_2(c_k) = g_2(c_k) \tag{5.19}$$

for all k. Taking $k \to \infty$, using that $c_k \to 0$ and that f_2 and g_2 are continuous at 0, we obtain $f_2(0) = g_2(0)$, or $a_2 = b_2$. Continuing by induction we get $a_n = b_n$ for all $n = 0, 1, 2, \ldots$, which is exactly what we wanted to prove. □

Corollary 5.22. *If $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ have positive radii of convergence and $f(x) = g(x)$ for all $x \in \mathbb{R}$ with $|x| < \varepsilon$ for some $\varepsilon > 0$, then $a_n = b_n$ for every n; in other words, f and g are actually the same power series.*

Proof. To prove this, observe that since $f(x) = g(x)$ for all $x \in \mathbb{R}$ such that $|x| < \varepsilon$, then $f(c_k) = g(c_k)$ for all k sufficiently large where $c_k = 1/k$; the identity theorem now implies $a_n = b_n$ for every n. □

Using the identity theorem we can deduce certain properties of series.

Example 5.24. Suppose that $f(z) = \sum a_n z^n$ is an **odd function** in the sense that $f(-z) = -f(z)$ for all z within its radius of convergence. In terms of power series, the identity $f(-z) = -f(z)$ is

$$\sum a_n (-1)^n z^n = \sum -a_n z^n.$$

By the identity theorem, we must have $(-1)^n a_n = -a_n$ for each n. Thus, for n even we must have $a_n = -a_n$ or $a_n = 0$, and for n odd, we must have $-a_n = -a_n$, a tautology. In conclusion, we see that f is odd if and only if all coefficients of even powers vanish:

$$f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1};$$

that is, f is odd if and only if f has only odd powers in its series expansion.

Exercises 5.4.

1. Prove that $f(z) = \sum a_n z^n$ is an **even function** in the sense that $f(-z) = f(z)$ for all z within its radius of convergence if and only if f has only even powers in its expansion, that is, f takes the form $f(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}$.
2. Recall that the binomial coefficient is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $0 \le k \le n$. Prove the highly nonobvious result:
$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}.$$
Suggestion: Apply the binomial formula to $(1+z)^{m+n}$, which equals $(1+z)^m \cdot (1+z)^n$. Prove that
$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$
3. Prove that $\sum_{n=1}^{\infty} n\,|a_n|\,r^n$ converges, where the notation is as in the proof of Theorem 5.19, using the root test. You will need Problem 8 in Exercises 5.2.
4. (**Abel!summability**) We say that a series $\sum a_n$ is **Abel!summable** to L if the power series $f(x) := \sum a_n x^n$ is defined for all $x \in [0,1)$ and $\lim_{x \to 1-} f(x) = L$.
(a) Prove that if $\sum a_n$ converges to $L \in \mathbb{C}$, then $\sum a_n$ is also Abel summable to L.

(b) Derive the following amazing formulas (properly interpreted!):

$$\boxed{\begin{aligned} &1-1+1-1+1-1+-\cdots =_a \frac{1}{2}, \\ &1+2-3+4-5+6-7+-\cdots =_a \frac{1}{4}, \end{aligned}}$$

where $=_a$ mean "is Abel summable to". You will need Problem 5 in Exercises 3.5.

5. In this problem we continue our fascinating study of Abel summability. Let $a_0, a_1, a_2, \ldots$ be a positive nonincreasing sequence tending to zero (in particular, $\sum(-1)^{n-1}a_n$ converges by the alternating series test). Define $b_n := a_0 + a_1 + \cdots + a_n$. We shall prove the neat formula

$$b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + - \cdots =_a \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n a_n.$$

(i) Let $f(x) = \sum_{n=0}^{\infty}(-1)^n b_n\, x^n$. Prove that f has radius of convergence 1. Suggestion: Use the ratio test.

(ii) Let

$$\begin{aligned} f_n(x) &= \sum_{k=0}^{n}(-1)^k b_k\, x^k \\ &= a_0 - (a_0+a_1)x + (a_0+a_1+a_2)x^2 - \cdots + (-1)^n(a_0+a_1+\cdots+a_n)x^n \end{aligned}$$

be the n-th partial sum of $f(x)$. Prove that

$$\begin{aligned} f_n(x) = \frac{1}{1+x}\big(a_0 - a_1x + a_2x^2 - a_3x^3 + \cdots + (-1)^n a_n x^n\big) \\ + (-1)^n\frac{x^{n+1}}{1+x}\big(a_0 + a_2 + a_3 + \cdots + a_n\big). \end{aligned}$$

(iii) Prove that[4]

$$f(x) = \frac{1}{1+x}\sum_{n=0}^{\infty}(-1)^n a_n x^n.$$

Finally, from this formula prove the desired result.

(iv) Establish the remarkable formula

$$\boxed{1 - \left(1+\frac{1}{2}\right) + \left(1+\frac{1}{2}+\frac{1}{3}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) + - \cdots =_a \frac{1}{2}\log 2}.$$

6. Suppose that $f(z) = \sum a_n z^n$ has radius of convergence 1, where $\sum a_n$ is a divergent series of positive real numbers. Prove that $\lim_{x\to 1-} f(x) = +\infty$.

5.5. Double sequences, double series, and a ζ-function identity

After studying single integrals in elementary calculus, you probably took a course where you studied "double integrals". In a similar way, now that we have a thorough background in "single infinite series," we now move to the topic of "double infinite series". The main result of this section is Cauchy's double series theorem, which we'll use quite often in the sequel.

[4]In the next section, we'll learn how to prove this identity in a much quicker way using the technologically advanced Cauchy's double series theorem.

5.5.1. Double sequences and series and Pringsheim's theorem. We begin by studying double sequences. Recall that a complex sequence is really just a function $s : \mathbb{N} \to \mathbb{C}$ where we usually denote $s(n)$ by s_n. By analogy, we define a **double sequence** of complex numbers as a function $s : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C}$. We usually denote $s(m,n)$ by s_{mn} and the corresponding double sequence by $\{s_{mn}\}$.

Example 5.25. For $m,n \in \mathbb{N}$,

$$s_{mn} = \frac{m \cdot n}{(m+n)^2}$$

defines a double sequence $\{s_{mn}\}$.

Whenever we talk about sequences, the idea of convergence is bound to follow. Let $\{s_{mn}\}$ be a double sequence of complex numbers. We say that the double sequence $\{s_{mn}\}$ **converges** if there is a complex number L having the property that given any $\varepsilon > 0$ there is a real number N such that

$$m,n > N \quad \Longrightarrow \quad |L - s_{mn}| < \varepsilon,$$

in which case we write $L = \lim s_{mn}$.

Care has to be taken when dealing with double sequences because sometimes sequences that look convergent are actually not.

Example 5.26. The nice looking double sequence $s_{mn} = mn/(m+n)^2$ does not converge. To see this, observe that if $m = n$, then

$$s_{mn} = \frac{n \cdot n}{(n+n)^2} = \frac{n^2}{4n^2} = \frac{1}{4}.$$

However, if $m = 2n$, then

$$s_{mn} = \frac{2n \cdot n}{(2n+n)^2} = \frac{2n^2}{9n^2} = \frac{2}{9}.$$

Therefore it is impossible for s_{mn} to approach any single number no matter how large we take m,n.

Given a double sequence $\{s_{mn}\}$ it is convenient to look at the **iterated limits**:

$$\lim_{m\to\infty} \lim_{n\to\infty} s_{mn} \qquad \text{and} \qquad \lim_{n\to\infty} \lim_{m\to\infty} s_{mn}. \tag{5.20}$$

For $\lim_{m\to\infty} \lim_{n\to\infty} s_{mn}$ on the left, we mean to first take $n \to \infty$ and second to take $m \to \infty$, reversing the order for $\lim_{n\to\infty} \lim_{m\to\infty} s_{mn}$. In general, the iterated limits (5.20) may have no relationship!

Example 5.27. Consider the double sequence $s_{mn} = mn/(m+n^2)$. We have

$$\lim_{n\to\infty} s_{mn} = \lim_{n\to\infty} \frac{mn}{m+n^2} = 0 \quad \Longrightarrow \quad \lim_{m\to\infty} \lim_{n\to\infty} s_{mn} = \lim_{m\to\infty} 0 = 0.$$

On the other hand,

$$\lim_{m\to\infty} s_{mn} = \lim_{m\to\infty} \frac{mn}{m+n^2} = n \quad \Longrightarrow \quad \lim_{n\to\infty} \lim_{m\to\infty} s_{mn} = \lim_{n\to\infty} n = \infty.$$

Here are a couple questions:

(I) If both iterated limits (5.20) exist and are equal, say to a number L, is it true that the regular double limit $\lim s_{mn}$ exists and $\lim s_{mn} = L$?

(II) If $L = \lim s_{mn}$ exists, then is it true that both iterated limits (5.20) exist and are equal to L:

$$L = \lim_{m\to\infty} \lim_{n\to\infty} s_{mn} = \lim_{n\to\infty} \lim_{m\to\infty} s_{mn}? \tag{5.21}$$

It may shock you, but the answer to both of these questions is "no".

Example 5.28. For a counter example to Question I, consider our first example $s_{mn} = mn/(m+n)^2$. We know that $\lim s_{mn}$ does not exist, but observe that

$$\lim_{n\to\infty} s_{mn} = \lim_{n\to\infty} \frac{mn}{(m+n)^2} = 0 \quad \Longrightarrow \quad \lim_{m\to\infty} \lim_{n\to\infty} s_{mn} = \lim_{m\to\infty} 0 = 0.$$

and

$$\lim_{m\to\infty} s_{mn} = \lim_{m\to\infty} \frac{mn}{(m+n)^2} = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \lim_{m\to\infty} s_{mn} = \lim_{n\to\infty} 0 = 0,$$

so both iterated limits converge. For a counter example to Question II, see limit (d) in Problem 1.

However, if a double sequence converges and both iterated sequences converge, then they all must equal the same number. This is the content of the following theorem, named after Alfred Pringsheim (1850–1941).

Theorem 5.23 (**Pringsheim's theorem for sequences**). *If a double sequence converges and both iterated sequences converge, then the equality* (5.21) *holds.*

Proof. Let $\varepsilon > 0$. Then there is an N such that for all $m, n > N$, we have $|L - s_{mn}| < \varepsilon/2$. Taking $n \to \infty$ implies that

$$\big|L - \lim_{n\to\infty} s_{mn}\big| \le \frac{\varepsilon}{2},$$

and then taking $m \to \infty$, we get

$$\big|L - \lim_{m\to\infty} \lim_{n\to\infty} s_{mn}\big| \le \frac{\varepsilon}{2} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this establishes the first equality in (5.21). A similar argument establishes the equality with the limits of m and n reversed. $\square$

Recall that if $\{a_n\}$ is a sequence of complex numbers, then we say that $\sum a_n$ converges if the sequences $\{s_n\}$ converges, where $s_n := \sum_{k=1}^n a_k$. By analogy, we define a double series of complex numbers as follows. Let $\{a_{mn}\}$ be a double sequence of complex numbers and let

$$s_{mn} := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij},$$

called the m,n-th partial sum of $\sum a_{mn}$. We say that the double series $\sum a_{mn}$ **converges** if the double sequence $\{s_{mn}\}$ of partial sums converges. If $\sum a_{mn}$ exists, we can ask whether or not

$$\sum a_{mn} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}\ ? \tag{5.22}$$

$$
\begin{array}{|ccccc}
a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\
a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\
a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\qquad\qquad
\begin{array}{|ccccc}
a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\
a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\
a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

FIGURE 5.3. In the first array we are "summing by rows" and in the second array we are "summing by columns".

Here, with $s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$, the **iterated series** on the right are defined as

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} := \lim_{m\to\infty} \lim_{n\to\infty} s_{mn} \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} := \lim_{n\to\infty} \lim_{m\to\infty} s_{mn}.$$

Thus, (5.22) is just the equality (5.21) with $s = \sum a_{mn}$. Hence, Pringsheim's theorem for sequences immediately implies the following.

THEOREM 5.24 (**Pringsheim's theorem for series**). *If a double series converges and both iterated series converge, then the equality* (5.22) *holds.*

We can "visualize" the iterated sums in (5.22) as follows. First, we arrange the a_{mn}'s in an infinite array as shown in Figure 5.3. Then for fixed $m \in \mathbb{N}$, the sum $\sum_{n=1}^{\infty} a_{mn}$ is summing all the numbers in the m-th row shown on the left picture in Figure 5.3. For example, if $m = 1$, then $\sum_{n=1}^{\infty} a_{1n}$ is summing all the numbers in the first row shown on the left picture in Figure 5.3. The summation $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}$ is summing over all the rows (that have already been summed). Similarly, $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}$ is summing over all the columns. In the next section, we shall generalize summing by rows and columns to "summing by curves".

5.5.2. Absolutely convergent double series and "summing by curves". In analogy with single series, we say that a double series $\sum a_{mn}$ **converge absolutely** if the series of absolute values $\sum |a_{mn}|$ converges. In order to study absolutely convergent double series it is important to understand double series of nonnegative terms, which is the content of the following lemma.

LEMMA 5.25 (**Nonnegative double series lemma**). *If $\sum a_{mn}$ converges where $a_{mn} \geq 0$ for all m, n, then both iterated series converge, and*

$$\sum a_{mn} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}.$$

Moreover, given $\varepsilon > 0$ there is an N such that

$$k > N \quad \Longrightarrow \quad \sum_{i=1}^{\infty} \sum_{j=k}^{\infty} a_{ij} < \varepsilon \quad \textit{and} \quad \sum_{i=k}^{\infty} \sum_{j=1}^{\infty} a_{ij} < \varepsilon.$$

PROOF. Assume that the series $\sum a_{mn}$ converges and let $\varepsilon > 0$. Since $\sum a_{mn}$ converges, setting $s := \sum a_{mn}$ and

$$s_{mn} := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}, \tag{5.23}$$

by definition of convergence we can choose N such that

$$m, n > N \implies |s - s_{mn}| < \frac{\varepsilon}{2}. \tag{5.24}$$

Given $i \in \mathbb{N}$, choose $m \geq i$ such that $m > N$ and let $n > N$. Then in view of (5.24) we have

$$\sum_{j=1}^{n} a_{ij} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = s_{mn} < s + \frac{\varepsilon}{2}.$$

Therefore, the partial sums of $\sum_{j=1}^{\infty} a_{ij}$ are bounded above by a fixed constant and hence (by the nonnegative series test — see Theorem 3.20), for any $i \in \mathbb{N}$, the sum $\sum_{j=1}^{\infty} a_{ij}$ exists. In particular,

$$\lim_{n\to\infty} s_{mn} = \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{\infty} a_{ij};$$

here, we can interchange the limit with the first sum because the first sum is finite. Now taking $n \to \infty$ in (5.24), we see that

$$m > N \implies \left| s - \sum_{i=1}^{m} \sum_{j=1}^{\infty} a_{ij} \right| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, by definition of convergence, $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ exists with limit $s = \sum a_{mn}$:

$$\sum a_{mn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}.$$

Similarly, using the second form of s_{mn} in (5.23), one can use an analogous argument to show that $\sum a_{mn} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$.

We now prove the last statement of our lemma. To this end, we observe that for any $k \in \mathbb{N}$, we have

$$s = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{k} a_{ij} + \sum_{j=k+1}^{\infty} a_{ij} \right) = \sum_{i=1}^{\infty} \sum_{j=1}^{k} a_{ij} + \sum_{i=1}^{\infty} \sum_{j=k+1}^{\infty} a_{ij},$$

which implies that

$$\sum_{i=1}^{\infty} \sum_{j=k+1}^{\infty} a_{ij} = s - \sum_{i=1}^{\infty} \sum_{j=1}^{k} a_{ij} = s - \lim_{m\to\infty} s_{mk}. \tag{5.25}$$

By the first part of this proof, we know that $s = \lim_{k\to\infty} \lim_{m\to\infty} s_{mk}$, so taking $k \to \infty$ in (5.25), we see that the left-hand side of (5.25) tends to zero as $k \to \infty$, so it follows that for some N_1,

$$k > N_1 \implies \sum_{i=1}^{\infty} \sum_{j=k}^{\infty} a_{ij} < \varepsilon.$$

A similar argument shows that there is an N_2 such that

$$k > N_2 \implies \sum_{i=k}^{\infty} \sum_{j=1}^{\infty} a_{ij} < \varepsilon.$$

Setting N as the largest of N_1 and N_2 completes the proof. □

$$
\begin{array}{|ccccc}
\hline
a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\
a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\
a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\qquad\qquad
\begin{array}{|ccccc}
\hline
a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\
a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\
a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

FIGURE 5.4. “Summing by squares” and “summing by triangles”.

Before presenting the “sum by curves theorem” (Theorem 5.26 below) it might be helpful to give a couple examples of this theorem to help in understanding what it says. Let $\sum a_{mn}$ be an absolutely convergent series.

Example 5.29. Let

$$S_k = \{(m,n)\,;\, 1 \le m \le k\ ,\ 1 \le n \le k\},$$

which represents a $k \times k$ square of numbers; see the left-hand picture in Figure 5.4 for 1×1, 2×2, 3×3, and 4×4 examples. We denote by $\sum_{(m,n)\in S_k} a_{mn}$ the sum of those a_{mn}'s within the $k \times k$ square S_k. Explicitly,

$$\sum_{(m,n)\in S_k} a_{mn} = \sum_{m=1}^{k} \sum_{n=1}^{k} a_{mn}.$$

The sum by curves theorem implies that

$$\sum a_{mn} = \lim_{k\to\infty} \sum_{(m,n)\in S_k} a_{mn} = \lim_{k\to\infty} \sum_{m=1}^{k} \sum_{n=1}^{k} a_{mn}. \tag{5.26}$$

As we already noted, $\sum_{(m,n)\in S_k} a_{mn}$ involves summing the a_{mn}'s within a $k \times k$ square; for this reason, (5.26) is referred to as “summing by squares”.

Example 5.30. Now let

$$S_k = T_1 \cup \cdots \cup T_k \quad , \quad \text{where} \ \ T_\ell = \{(m,n)\,;\, m+n = \ell+1\}.$$

Notice that $T_\ell = \{(m,n)\,;\, m+n = \ell+1\} = \{(1,\ell),(2,\ell-1),\ldots,(\ell,1)\}$ represents the ℓ-th diagonal in the right-hand picture in Figure 5.4; for instance, $T_3 = \{(1,3),(2,2),(3,1)\}$ is the third diagonal in Figure 5.4. Then

$$\sum_{(m,n)\in S_k} a_{mn} = \sum_{\ell=1}^{k} \sum_{(m,n)\in T_\ell} a_{mn}$$

is the sum of the a_{mn}'s that are within the triangle consisting of the first k diagonals. The sum by curves theorem implies that

$$\sum a_{mn} = \lim_{k\to\infty} \sum_{(m,n)\in S_k} a_{mn} = \lim_{k\to\infty} \sum_{\ell=1}^{k} \sum_{(m,n)\in T_\ell} a_{mn},$$

or using that $T_\ell = \{(1,\ell),(2,\ell-1),\ldots,(\ell,1)\}$, we have

$$\sum a_{mn} = \sum_{k=1}^{\infty} \big(a_{1,k} + a_{2,k-1} + \cdots + a_{k,1}\big). \tag{5.27}$$

We refer to (5.27) as “summing by triangles”.

More generally, we can "sum by curves" as long as the curves increasingly fill up the array like the squares or triangles shown in Figure 5.4.

THEOREM 5.26 (**Sum by curves theorem**). *An absolutely convergent series* $\sum a_{mn}$ *itself converges. Moreover, if* $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots \subseteq \mathbb{N}\times\mathbb{N}$ *is a nondecreasing sequence of finite sets having the property that for any* m, n *there is a* k *such that*

$$\{1,2,\ldots,m\} \times \{1,2,\ldots,n\} \subseteq S_k \subseteq S_{k+1} \subseteq S_{k+2} \subseteq \cdots, \tag{5.28}$$

then the sequence $\{s_k\}$ *converges, where* s_k *is the finite sum*

$$s_k := \sum_{(m,n)\in S_k} a_{mn},$$

and furthermore,

$$\sum a_{mn} = \lim s_k.$$

PROOF. We first prove that the sequence $\{s_k\}$ converges by showing that the sequence is Cauchy. Indeed, let $\varepsilon > 0$ be given. By assumption, $\sum |a_{mn}|$ converges, so by the nonnegative double series lemma we can choose $N \in \mathbb{N}$ such that

$$k > N \quad \Longrightarrow \quad \sum_{i=1}^{\infty}\sum_{j=k}^{\infty} |a_{ij}| < \varepsilon \quad \text{and} \quad \sum_{i=k}^{\infty}\sum_{j=1}^{\infty} |a_{ij}| < \varepsilon. \tag{5.29}$$

By the property (5.28) of the sets $\{S_k\}$ there is an N' such that

$$\{1,2,\ldots,N\} \times \{1,2,\ldots,N\} \subseteq S_{N'} \subseteq S_{N'+1} \subseteq S_{N'+2} \subseteq \cdots \tag{5.30}$$

Let $k > \ell > N'$. Then, since $S_\ell \subseteq S_k$, we have

$$|s_k - s_\ell| = \left| \sum_{(i,j)\in S_k} a_{ij} - \sum_{(i,j)\in S_\ell} a_{ij} \right| = \left| \sum_{(i,j)\in S_k\setminus S_\ell} a_{ij} \right| \le \sum_{(i,j)\in S_k\setminus S_\ell} |a_{ij}|.$$

Since $\ell > N'$, by (5.30), S_ℓ contains $\{1,2,\ldots,N\} \times \{1,2,\ldots,N\}$. Hence,

$$S_k \setminus S_\ell \ \text{ is a subset of } \ \mathbb{N} \times \{N+1, N+2, \ldots\} \text{ or } \ \{N+1, N+2, \ldots\} \times \mathbb{N}.$$

For concreteness, assume that the first case holds; the second case can be dealt with in a similar way. In this case, by the property (5.29), we have

$$|s_k - s_\ell| \le \sum_{(i,j)\in S_k\setminus S_\ell} |a_{ij}| \le \sum_{i=1}^{\infty} \sum_{j=N+1}^{\infty} |a_{ij}| < \varepsilon.$$

This shows that $\{s_k\}$ is Cauchy and hence converges.

We now show that $\sum a_{mn}$ converges with sum equal to $\lim s_k$. Let $\varepsilon > 0$ be given and choose N such that (5.29) holds with ε replaced by $\varepsilon/2$. Fix natural numbers $m, n > N$. By the property (5.28) and the fact that $s_k \to s := \lim s_k$ we can choose a $k > N$ such that

$$\{1,2,\ldots,m\} \times \{1,2,\ldots,n\} \subseteq S_k$$

and $|s_k - s| < \varepsilon/2$. Now observe that

$$|s_k - s_{mn}| = \left| \sum_{(i,j)\in S_k} a_{ij} - \sum_{(i,j)\in\{1,\ldots,m\}\times\{1,\ldots,n\}} a_{ij} \right| \le \sum_{(i,j)\in S_k\setminus(\{1,\ldots,m\}\times\{1,\ldots,n\})} |a_{ij}|.$$

Notice that $S_k \setminus (\{1,\ldots,m\} \times \{1,\ldots,n\})$ is a subset of $\mathbb{N} \times \{n+1, n+2, \ldots\}$ or $\{m+1, m+2, \ldots\} \times \mathbb{N}$. For concreteness, assume that the first case holds; the second case can be dealt with in a similar manner. In this case, by the property (5.29) (with ε replaced with $\varepsilon/2$), we have

$$|s_k - s_{mn}| \le \sum_{(i,j)\in S_k\setminus(\{1,\ldots,m\}\times\{1,\ldots n\})} |a_{ij}| < \sum_{i=1}^{\infty}\sum_{j=n+1}^{\infty} |a_{ij}| < \frac{\varepsilon}{2}.$$

Hence,

$$|s_{mn} - s| \le |s_{mn} - s_k| + |s_k - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that $\sum a_{mn} = s$ and completes our proof. □

We now come to Cauchy's double series theorem, the most important result of this section.

5.5.3. Cauchy's double series theorem. Instead of summing by curves, in many applications we are interested in summing by rows or by columns.

THEOREM 5.27 (**Cauchy's double series theorem**). *A series $\sum a_{mn}$ is absolutely convergent if and only if*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}| < \infty \quad \textit{or} \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |a_{mn}| < \infty,$$

in which case

$$\sum a_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}$$

in the sense that both iterated sums converge and are equal to the sum of the series.

PROOF. Assume that the sum $\sum a_{mn}$ converges absolutely. Then by the nonnegative double series lemma, we know that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}| = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |a_{mn}| = \sum |a_{mn}|.$$

We shall prove that the iterated sums $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}$ and $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}$ converge and equal $s := \sum a_{mn}$, which exists by the sum by curves theorem. Let s_{mn} denote the partial sums of $\sum a_{mn}$. Let $\varepsilon > 0$ and choose N such that

$$m, n > N \quad \Longrightarrow \quad |s - s_{mn}| < \frac{\varepsilon}{2}. \tag{5.31}$$

Since

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}| < \infty,$$

this implies, in particular, that for any $m \in \mathbb{N}$, the sum $\sum_{n=1}^{\infty} |a_{mn}|$ converges, and hence for any $m \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_{mn} = \lim_{n\to\infty} s_{mn}$ converges. Thus, taking $n \to \infty$ in (5.31), we obtain

$$m > N \quad \Longrightarrow \quad |s - \lim_{n\to\infty} s_{mn}| \le \frac{\varepsilon}{2} < \varepsilon.$$

But this means that $s = \lim_{m\to\infty}\lim_{n\to\infty} s_{mn}$; that is,

$$s = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}.$$

A similar argument gives this equality with the sums reversed.

Now assume that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}| = t < \infty.$$

We will show that $\sum a_{mn}$ is absolutely convergent; a similar proof shows that if $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|a_{mn}| < \infty$, then $\sum a_{mn}$ is absolutely convergent. Let $\varepsilon > 0$. Then the fact that $\sum_{i=1}^{\infty}\big(\sum_{j=1}^{\infty}|a_{ij}|\big) < \infty$ implies, by the Cauchy criterion for series, there is an N such that for all $m > N$,

$$m > N \quad \Longrightarrow \quad \sum_{i=m+1}^{\infty}\Big(\sum_{j=1}^{\infty}|a_{ij}|\Big) < \frac{\varepsilon}{2}.$$

Let $m, n > N$. Then for any $k > m$, we have

$$\left|\sum_{i=1}^{k}\sum_{j=1}^{\infty}|a_{ij}| - \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|\right| \le \sum_{i=m+1}^{k}\sum_{j=1}^{\infty}|a_{ij}| \le \sum_{i=m+1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| < \frac{\varepsilon}{2}.$$

Taking $k \to \infty$ shows that for all $m, n > N$,

$$\left|t - \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|\right| \le \frac{\varepsilon}{2} < \varepsilon,$$

which proves that $\sum |a_{mn}|$ converges, and completes the proof of our result. $\square$

Corollary 5.28. *If $\{a_{mn}\}$ is a double sequence of nonnegative numbers, then*

$$\sum a_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{mn}$$

in the sense that either sum converges or diverges together and when one, and hence all, converge, their values are equal.

Now for examples.

Example 5.31. For our first example, consider the sum $\sum 1/(m^p n^q)$ where $p, q \in \mathbb{R}$. Since in this case,

$$\sum_{n=1}^{\infty}\frac{1}{m^p n^q} = \frac{1}{m^p}\cdot\Big(\sum_{n=1}^{\infty}\frac{1}{n^q}\Big),$$

it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^p n^q} = \Big(\sum_{m=1}^{\infty}\frac{1}{m^p}\Big)\cdot\Big(\sum_{n=1}^{\infty}\frac{1}{n^q}\Big).$$

Therefore, by Cauchy's double series theorem and the p-test, $\sum 1/(m^p n^q)$ converges if and only if both $p, q > 1$.

Example 5.32. The previous example can help us with other examples such as $\sum 1/(m^4 + n^4)$. Observe that

$$(m^2 - n^2)^2 \ge 0 \quad \Longrightarrow \quad m^4 + n^4 - 2m^2n^2 \ge 0 \quad \Longrightarrow \quad \frac{1}{m^4 + n^4} \le \frac{1}{2m^2n^2}.$$

Since $\sum 1/(m^2n^2)$ converges, by an easy generalization of our good ole comparison test (Theorem 3.27) to double series, we see that $\sum 1/(m^4 + n^4)$ converges too.

Example 5.33. For an application of Cauchy's theorem and the sum by curves theorem, we look at the double sum $\sum z^{m+n}$ for $|z| < 1$. For such z, this sum converges absolutely because

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|z|^{m+n} = \sum_{m=0}^{\infty}|z|^m \cdot \frac{1}{1-|z|} = \frac{1}{(1-|z|)^2} < \infty,$$

where we used the geometric series test (twice): If $|r| < 1$, then $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. So $\sum z^{m+n}$ converges absolutely by Cauchy's double series theorem, and

$$\sum z^{m+n} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z^{m+n} = \sum_{m=0}^{\infty} z^m \cdot \frac{1}{1-z} = \frac{1}{(1-z)^2}.$$

On the other hand, by our sum by curves theorem, we can determine $\sum z^{m+n}$ by summing over curves; we shall choose to sum over triangles. Thus, if we set

$$S_k = T_0 \cup T_1 \cup T_2 \cup \cdots \cup T_k \quad , \quad \text{where } \ T_\ell = \{(m,n)\,;\, m+n=\ell\ ,\ m,n \geq 0\},$$

then

$$\sum z^{m+n} = \lim_{k\to\infty}\sum_{(m,n)\in S_k} z^{m+n} = \lim_{k\to\infty}\sum_{\ell=0}^{k}\sum_{(m,n)\in T_\ell} z^{m+n}.$$

Since $T_\ell = \{(m,n)\,;\, m+n=\ell\} = \{(0,\ell),(1,\ell-1),\ldots,(\ell,0)\}$, we have

$$\sum_{(m,n)\in T_\ell} z^{m+n} = z^{0+\ell} + z^{1+(\ell-1)} + z^{2+(\ell-2)} + \cdots + z^{\ell+0} = (\ell+1)z^\ell.$$

Thus, $\sum z^{m+n} = \sum_{k=0}^{\infty}(k+1)z^k$. However, we already proved that $\sum z^{m+n} = 1/(1-z)^2$, so

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}. \tag{5.32}$$

Example 5.34. Another very neat application of Cauchy's double series theorem is to derive nonobvious identities. For example, let $|z| < 1$ and consider the series

$$\sum_{n=1}^{\infty}\frac{z^n}{1+z^{2n}} = \frac{z}{1+z^2} + \frac{z^2}{1+z^4} + \frac{z^3}{1+z^6} + \cdots;$$

we'll see why this converges in a moment. Observe that (since $|z| < 1$)

$$\frac{1}{1+z^{2n}} = \sum_{m=0}^{\infty}(-1)^m z^{2mn},$$

by the familiar geometric series test: If $|r| < 1$, then $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ with $r = -z^{2n}$. Therefore,

$$\sum_{n=1}^{\infty}\frac{z^n}{1+z^{2n}} = \sum_{n=1}^{\infty} z^n \cdot \sum_{m=0}^{\infty}(-1)^m z^{2mn} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m z^{(2m+1)n}$$

We claim that the double sum $\sum(-1)^m z^{(2m+1)n}$ converges absolutely. To prove this, observe that

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}|z|^{(2m+1)n} = \sum_{n=1}^{\infty}|z|^n\sum_{m=0}^{\infty}|z|^{2nm} = \sum_{n=1}^{\infty}\frac{|z|^n}{1-|z|^{2n}}.$$

Since $\frac{1}{1-|z|^{2n}} \le \frac{1}{1-|z|}$ (this is because $|z|^{2n} \le |z|$ for $|z| < 1$), we have

$$\frac{|z|^n}{1-|z|^{2n}} \le \frac{1}{1-|z|} \cdot |z|^n.$$

Since $\sum |z|^n$ converges, by the comparison theorem, $\sum_{n=1}^\infty \frac{|z|^n}{1-|z|^{2n}}$ converges too. Hence, Cauchy's double series theorem applies, and

$$\begin{aligned}\sum_{n=1}^\infty \sum_{m=0}^\infty (-1)^m z^{(2m+1)n} &= \sum_{m=0}^\infty \sum_{n=1}^\infty (-1)^m z^{(2m+1)n}\\ &= \sum_{m=0}^\infty (-1)^m \sum_{n=1}^\infty z^{(2m+1)n}\\ &= \sum_{m=0}^\infty (-1)^m \frac{z^{2m+1}}{1-z^{2m+1}}.\end{aligned}$$

Thus,

$$\sum_{n=1}^\infty \frac{z^n}{1+z^{2n}} = \sum_{m=0}^\infty (-1)^m \frac{z^{2m+1}}{1-z^{2m+1}};$$

that is, we have derived the striking identity between even and odd powers of z:

$$\frac{z}{1+z^2} + \frac{z^2}{1+z^4} + \frac{z^3}{1+z^6} + \cdots = \frac{z}{1-z} - \frac{z^3}{1-z^3} + \frac{z^5}{1-z^5} - + \cdots .$$

There are more beautiful series like this found in the exercises (see Problem 4 or better yet, Problem 5). We just touch on one more because it's so nice:

5.5.4. A neat ζ-function identity. Recall that the ζ-function is defined by $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$, which converges absolutely for $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$. Here's a beautiful theorem from Flajolet and Vardi [**58, 176**].

THEOREM 5.29. *If $f(z) = \sum_{n=2}^\infty a_n z^n$ and $\sum_{n=2}^\infty |a_n|$ converges, then*

$$\boxed{\sum_{n=1}^\infty f\Big(\frac{1}{n}\Big) = \sum_{n=2}^\infty a_n\, \zeta(n).}$$

PROOF. We first write

$$\sum_{n=1}^\infty f\Big(\frac{1}{n}\Big) = \sum_{n=1}^\infty \sum_{m=2}^\infty a_m \frac{1}{n^m}.$$

Now if we set $C := \sum_{m=2}^\infty |a_m| < \infty$, then

$$\sum_{n=1}^\infty \sum_{m=2}^\infty \Big|a_m \frac{1}{n^m}\Big| \le \sum_{n=1}^\infty \sum_{m=2}^\infty |a_m| \frac{1}{n^2} \le C \sum_{n=1}^\infty \frac{1}{n^2} < \infty.$$

Hence, by Cauchy's double series theorem, we can switch order of summation:

$$\sum_{n=1}^\infty f\Big(\frac{1}{n}\Big) = \sum_{n=1}^\infty \sum_{m=2}^\infty a_m \frac{1}{n^m} = \sum_{m=2}^\infty a_m \sum_{n=1}^\infty \frac{1}{n^m} = \sum_{m=2}^\infty a_m\, \zeta(m),$$

which completes our proof. □

Using this theorem we can derive the pretty formula (see Problem 7):

$$\boxed{\log 2 = \sum_{n=2}^{\infty} \frac{1}{2^n}\,\zeta(n).} \tag{5.33}$$

Not only is this formula pretty, it converges to $\log 2$ much faster than the usual series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (from which (5.33) is derived by the help of Theorem 5.29); see [**58, 176**] for a discussion of such convergence issues.

EXERCISES 5.5.

1. Determine the convergence of the limits and the iterated limits for the double sequences

$$(a)\ s_{mn} = \frac{1}{m} + \frac{1}{n} \quad , \quad (b)\ s_{mn} = \frac{m}{m+n} \quad , \quad s_{mn} = \Big(\frac{n+1}{n+2}\Big)^m,$$
$$(d)\ s_{mn} = (-1)^{m+n}\Big(\frac{1}{m} + \frac{1}{n}\Big) \quad , \quad (e)\ s_{mn} = \frac{1}{1+(m-n)^2}.$$

2. Determine the convergence, iterated convergence, and absolute convergence, for the double series

$$(a)\ \sum_{m,n\geq 1} \frac{(-1)^{mn}}{mn} \quad , \quad (b)\ \sum_{m,n\geq 1} \frac{(-1)^n}{(m+n^p)(m+n^p-1)}\ ,\ p>1 \quad , \quad (c)\ \sum_{m\geq 2, n\geq 1} \frac{1}{m^n}$$

Suggestion: For (b), show that $\sum_{m=1}^{\infty} \frac{1}{(m+n^p)(m+n^p-1)}$ telescopes.

3. (mn**-term test for double series**) Show that if $\sum a_{mn}$ converges, then $a_{mn} \to 0$. Suggestion: First verify that $a_{mn} = s_{mn} - s_{m-1,n} - s_{m,n-1} + s_{m-1,n-1}$.

4. Let $|z| < 1$. Using Cauchy's double series theorem, derive the beautiful identities

$$(a)\ \frac{z}{1+z^2} + \frac{z^3}{1+z^6} + \frac{z^5}{1+z^{10}} + \cdots = \frac{z}{1-z^2} - \frac{z^3}{1-z^6} + \frac{z^5}{1-z^{10}} - + \cdots,$$
$$(b)\ \frac{z}{1+z^2} - \frac{z^2}{1+z^4} + \frac{z^3}{1+z^6} - + \cdots = \frac{z}{1+z} - \frac{z^3}{1+z^3} + \frac{z^5}{1+z^5} - + \cdots,$$
$$(c)\ \frac{z}{1+z} - \frac{2z^2}{1+z^2} + \frac{3z^3}{1+z^3} - + \cdots = \frac{z}{(1+z)^2} - \frac{z^2}{(1+z^2)^2} + \frac{z^3}{(1+z^3)^2} - + \cdots.$$

Suggestion: For (c), you need the formula $1/(1-z)^2 = \sum_{n=1}^{\infty} nz^{n-1}$ found in (5.32).

5. (**Number theory series**) Here are some pretty formulas involving number theory!
   (1) For $n \in \mathbb{N}$, let $\tau(n)$ denote the number of positive divisors of n (that is, the number of positive integers that divide n). For example, $\tau(1) = 1$ and $\tau(4) = 3$ (because $1, 2, 4$ divide 4). Prove that

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} \tau(n) z^n \quad , \quad |z| < 1. \tag{5.34}$$

   Suggestion: Write $1/(1-z^n) = \sum_{m=0}^{\infty} z^{mn} = \sum_{m=1}^{\infty} z^{n(m-1)}$, then prove that the left-hand side of (5.34) equals $\sum z^{mn}$. Finally, use Theorem 5.26 with the set S_k given by $S_k = T_1 \cup \cdots \cup T_k$ where $T_k = \{(m,n) \in \mathbb{N}\times\mathbb{N}\,;\, m\cdot n = k\}$.
   (2) For $n \in \mathbb{N}$, let $\sigma(n)$ denote the sum of the positive divisors of n. For example, $\sigma(1) = 1$ and $\sigma(4) = 1 + 2 + 4 = 7$). Prove that

$$\sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)^2} = \sum_{n=1}^{\infty} \sigma(n) z^n \quad , \quad |z| < 1.$$

6. Here is a neat problem. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Determine a set of $z \in \mathbb{C}$ for which the following formula is valid:

$$\sum_{n=1}^{\infty} b_n f(z^n) = \sum_{n=1}^{\infty} a_n g(z^n).$$

From this formula, derive the following pretty formulas:

$$\sum_{n=1}^{\infty} f(z^n) = \sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n} \quad , \quad \sum_{n=1}^{\infty} (-1)^{n-1} f(z^n) = \sum_{n=1}^{\infty} \frac{a_n z^n}{1+z^n},$$

and my favorite:

$$\boxed{\sum_{n=1}^{\infty} \frac{f(z^n)}{n!} = \sum_{n=1}^{\infty} a_n e^{z^n}.}$$

7. In this problem we derive (5.33).
   (i) Prove that $\log 2 = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} = \sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$, where $f(z) = \frac{z^2}{2(2-z)}$.
   (ii) Show that $f(z) = \sum_{n=2}^{\infty} \frac{z^n}{2^n}$ and from this and Theorem 5.29 prove (5.33).
8. (Cf. [**58, 176**]) Prove the following extension of Theorem 5.29: If $f(z) = \sum_{n=2}^{\infty} a_n\, z^n$ and for some $N \in \mathbb{N}$, $\sum_{n=2}^{\infty} \frac{|a_n|}{N^n}$ converges, then

$$\boxed{\sum_{n=N}^{\infty} f\Big(\frac{1}{n}\Big) = \sum_{n=2}^{\infty} a_n \left\{\zeta(n) - \Big(1 + \frac{1}{2^n} + \cdots + \frac{1}{(N-1)^n}\Big)\right\},}$$

where the sum $\left(1 + \frac{1}{2^n} + \cdots + \frac{1}{(N-1)^n}\right)$ is (by convention) zero if $N = 1$.

5.6. Rearrangements and multiplication of power series

We already know that the associative law holds for infinite series. That is, we can group the terms of an infinite series in any way we wish and the resulting series still converges with the same sum (see Theorem 3.23). A natural question that you may ask is whether or not the commutative law holds for infinite series. That is, suppose that $s = a_1 + a_2 + a_3 + \cdots$ exists. Can we commute the a_n's in any way we wish and still get the same sum? For instance, is it true that

$$s = a_1 + a_2 + a_4 + a_3 + a_6 + a_8 + a_5 + a_{10} + a_{12} + \cdots ?$$

For general series, the answer is, quite shocking at first, "no!"

5.6.1. Rearrangements.

A sequence $\nu_1, \nu_2, \nu_3, \ldots$ of natural numbers such that every natural number occurs exactly once in this list is called a **rearrangement** of the natural numbers.

Example 5.35. $1, 2, 4, 3, 6, 8, 5, 10, 12, \ldots$, where we follow every odd number by two adjacent even numbers, is a rearrangement.

A **rearrangement** of a series $\sum_{n=1}^{\infty} a_n$ is a series $\sum_{n=1}^{\infty} a_{\nu_n}$ where $\{\nu_n\}$ is a rearrangement of $\mathbb{N}$.

Example 5.36. Let us rearrange the alternating harmonic series

$$\log 2 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + - \cdots$$

using the rearrangement $1, 2, 4, 3, 6, 8, 5, 10, 12, \ldots$ we've already mentioned:

$$s = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + -- \\ \cdots + \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} + \cdots ,$$

provided of course that this sum converges. Here, the bottom three terms represent the general formula for the k-th triplet of a positive term followed by two negative ones. To see that this sum converges, let s_n denote its n-th partial sum. Then we can write $n = 3k + \ell$ where ℓ is either 0, 1, or 2, and so

$$s_n = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + - - \cdots + \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} + r_n,$$

where r_n consists of the next ℓ $(= 0, 1, 2)$ terms of the series for s_n. Note that $r_n \to 0$ as $n \to \infty$. In any case, we can write

$$\begin{aligned} s_n &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + - - \cdots + \left(\frac{1}{2k-1} - \frac{1}{4k-2}\right) - \frac{1}{4k} + r_n \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + - \cdots + \frac{1}{4k-2} - \frac{1}{4k} + r_n \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots + \frac{1}{2k-1} - \frac{1}{2k}\right) + r_n. \end{aligned}$$

Taking $n \to \infty$, we see that

$$s = \frac{1}{2}\log 2.$$

Thus, the rearrangement s has a different sum than the original series!

In summary, rearrangements of series can, in general, have different sums that the original series. In fact, it turns out that a convergent series can be rearranged to get a different value if and only if the series is not absolutely convergent. The "only if" portion is proved in Theorem 5.31 and the "if" portion is proved in

5.6.2. Riemann's rearrangement theorem.

THEOREM 5.30 (**Riemann's rearrangement theorem**). *If a series $\sum a_n$ of real numbers converges, but not absolutely, then there are rearrangements of the series that can be made to converge to $\pm\infty$ or any real number whatsoever.*

PROOF. We shall prove that there are rearrangements of the series that converge to any real number whatsoever; following the argument for this case, you should be able to handle the $\pm\infty$ cases yourself.

Step 1: We first show that the series corresponding to the positive and negative terms in $\sum a_n$ each diverge. Let $b_1, b_2, b_3, \ldots$ denote the terms in the sequence $\{a_n\}$ that are nonnegative, in the order in which they occur, and let $c_1, c_2, c_3, \ldots$ denote the absolute values of the terms in $\{a_n\}$ that are negative, again, in the order in which they occur. We claim that both series $\sum b_n$ and $\sum c_n$ diverge. To see this, observe that

$$\sum_{k=1}^{n} a_k = \sum_i b_i - \sum_j c_j, \tag{5.35}$$

where the right-hand sums are only over those natural numbers i, j such that b_i and c_j occur in the left-hand sum. The left-hand side converges as $n \to \infty$ by assumption, so if either sum $\sum_{n=1}^{\infty} b_n$ or $\sum_{n=1}^{\infty} c_n$ of nonnegative numbers converges, then the equality (5.35) would imply that the other sum converges. But this would then imply that

$$\sum_{k=1}^{n} |a_k| = \sum_i b_i + \sum_j c_j$$

converges as $n \to \infty$, which does not. Hence, both sums $\sum b_n$ and $\sum c_n$ diverge.

Step 2: We produce a rearrangement. Let $\xi \in \mathbb{R}$. We shall produce a rearrangement

$$b_1 + \cdots + b_{m_1} - c_1 - \cdots - c_{n_1} + b_{m_1+1} + \cdots + b_{m_2} \\ - c_{n_1+1} - \cdots - c_{n_2} + b_{m_2+1} + \cdots + b_{m_3} - c_{n_2+1} - \cdots \tag{5.36}$$

such that such that its partial sums converge to ξ. We do so as follows. Let $\{\beta_n\}$ and $\{\gamma_n\}$ denote the partial sums for $\sum b_n$ and $\sum c_n$, respectively. Since $\beta_n \to \infty$, for n sufficiently large, $\beta_n > \xi$. We define m_1 as the smallest natural number such that

$$\beta_{m_1} > \xi.$$

Note that β_{m_1} differs from ξ by at most b_{m_1}. Since $\gamma_n \to \infty$, for n sufficiently large, $\beta_{m_1} - \gamma_n < \xi$. We define n_1 to be the smallest natural number such that

$$\beta_{m_1} - \gamma_{n_1} < \xi.$$

Note that the left-hand side differs from ξ by at most c_{n_1}. Now define m_2 as the smallest natural number greater than m_1 such that

$$\beta_{m_2} - \gamma_{n_1} > \xi.$$

As before, such a number exists because $\beta_n \to \infty$, and the left-hand side differs from ξ by at most b_{m_2}. We define the number n_2 as the smallest natural number greater than n_1 such that

$$\beta_{m_2} - \gamma_{n_2} < \xi,$$

where the left-hand side differs from ξ by at most c_{n_2}. Continuing this process, we produce sequences $m_1 < m_2 < m_3 < \cdots$ and $n_1 < n_2 < n_3 < \cdots$ such that for every k,

$$\beta_{m_k} - \gamma_{n_{k-1}} > \xi,$$

where the left-hand side differs from ξ by at most b_{m_k}, and

$$\beta_{m_k} - \gamma_{n_k} < \xi,$$

where the left-hand side differs from ξ by at most c_{n_k}.

Step 3: We now show that the series (5.36), which is just a rearrangement of $\sum a_n$, converges to ξ. Let

$$\beta'_k := b_1 + \cdots + b_{m_1} - c_1 - \cdots - c_{n_1} + b_{m_1+1} + \cdots + b_{m_2} - \\ \cdots - c_{n_{k-2}+1} - \cdots - c_{n_{k-1}} + b_{m_{k-1}+1} + \cdots + b_{m_k} = \beta_{m_k} - \gamma_{n_{k-1}}$$

and

$$\gamma'_k := b_1 + \cdots + b_{m_1} - c_1 - \cdots - c_{n_1} + b_{m_1+1} + \cdots + b_{m_2} - \\ \cdots + b_{m_{k-1}+1} + \cdots + b_{m_k} - c_{n_{k-1}+1} - \cdots - c_{n_k} = \beta_{m_k} - \gamma_{n_k}.$$

Then any given partial sum t of (5.36) is of the following two sorts:

$$t = b_1 + \cdots + b_{m_1} - c_1 - \cdots - c_{n_1} + b_{m_1+1} + \cdots + b_{m_2} - \\ \cdots - c_{n_{k-2}+1} - \cdots - c_{n_{k-1}} + b_{m_{k-1}+1} + \cdots + b_\ell,$$

where $\ell \le m_k$, in which case, $\gamma'_{k-1} < t \le \beta'_k$; otherwise,

$$t = b_1 + \cdots + b_{m_1} - c_1 - \cdots - c_{n_1} + b_{m_1+1} + \cdots + b_{m_2} - \\ \cdots + b_{m_{k-1}+1} + \cdots + b_{m_k} - c_{n_{k-1}+1} - \cdots - c_\ell,$$

where $\ell \leq n_k$, in which case, $\gamma'_k \leq t < \beta'_{k+1}$. Now by construction, β'_k differs from ξ by at most b_{m_k} and γ'_k differs from ξ by at most c_{n_k}. Therefore, the fact that $\gamma'_{k-1} < t \leq \beta'_k$ or $\gamma'_k \leq t < \beta'_{k+1}$ imply that

$$\xi - c_{n_{k-1}} < t < \xi + b_{n_k} \quad \text{or} \quad \xi - c_{n_k} < t < \xi + b_{n_{k+1}}.$$

By assumption, $\sum a_n$ converges, so $b_{n_k}, c_{n_k} \to 0$, hence the partial sums of (5.36) must converge to ξ. This completes our proof. □

We now prove that a convergent series can be rearranged to get a different value only if the series is not absolutely convergent. Actually, we shall prove the contrapositive: If a series is absolutely convergent, then any rearrangement has the same value as the original sum. This is a consequence of the following theorem.

THEOREM 5.31 (**Dirichlet's theorem**). *All rearrangements of an absolutely convergent series of complex numbers converge with the same sum as the original series.*

PROOF. Let $\sum a_n$ converge absolutely. We shall prove that any rearrangement of this series converges to the same value as the sum itself. To see this, let $\nu_1, \nu_2, \nu_3, \ldots$ be any rearrangement of the natural numbers and define

$$a_{mn} = \begin{cases} a_m & \text{if } m = \nu_n, \\ 0 & \text{else.} \end{cases}$$

Then by definition of a_{mn}, we have

$$a_m = \sum_{n=1}^{\infty} a_{mn} \quad \text{and} \quad a_{\nu_n} = \sum_{m=1}^{\infty} a_{mn}.$$

Moreover,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}| = \sum_{m=1}^{\infty} |a_m| < \infty,$$

so by Cauchy's double series theorem,

$$\sum_{m=1}^{\infty} a_m = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty} a_{\nu_n}.$$

□

We now move to the important topic of multiplication of series.

5.6.3. Multiplication of power series and infinite series. If we consider two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$, then *formally* multiplying and combining like powers of z, we get

$$\begin{aligned}\big(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\big)\big(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots\big) = \\ a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 \\ + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) z^3 + \cdots .\end{aligned}$$

In particular, taking $z = 1$, we get (again, only formally!)

$$\begin{aligned}\big(a_0 + a_1 + a_2 + a_3 + \cdots\big)\big(b_0 + b_1 + b_2 + b_3 + \cdots\big) = \\ a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) \\ + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0) + \cdots .\end{aligned}$$

These thoughts suggest the following definition. Given two series $\sum_{n=0}^\infty a_n$ and $\sum_{n=0}^\infty b_n$, their **Cauchy product** is the series $\sum_{n=0}^\infty c_n$, where

$$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0 = \sum_{k=0}^n a_kb_{n-k}.$$

A natural question to ask is if $\sum_{n=0}^\infty a_n$ and $\sum_{n=0}^\infty b_n$ converge, then is it true that

$$\Big(\sum_{n=0}^\infty a_n\Big)\Big(\sum_{n=0}^\infty b_n\Big) = \sum_{n=0}^\infty c_n \ ?$$

The answer is, what may be a surprising, "no".

Example 5.37. Let us consider the example $(\sum_{n=1}^\infty \frac{(-1)^{n-1}}{\sqrt{n}})(\sum_{n=1}^\infty \frac{(-1)^{n-1}}{\sqrt{n}})$, which is due to Cauchy. That is, let $a_0 = b_0 = 0$ and

$$a_n = b_n = (-1)^{n-1}\frac{1}{\sqrt{n}}, \quad n = 1, 2, 3, \ldots .$$

We know, by the alternating series test, that $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{\sqrt{n}}$ converges. However, we shall see that the Cauchy product does not converge. Indeed,

$$c_0 = a_0b_0 = 0, \quad c_1 = a_0b_1 + a_1b_0 = 0,$$

and for $n \geq 2$,

$$c_n = \sum_{k=0}^n a_kb_{n-k} = \sum_{k=1}^{n-1} \frac{(-1)^k(-1)^{n-k}}{\sqrt{k}\sqrt{n-k}} = (-1)^n \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}\sqrt{n-k}}.$$

Since for $1 \leq k \leq n-1$, we have

$$k(n-k) \leq (n-1)(n-1) = (n-1)^2 \quad \Longrightarrow \quad \frac{1}{n-1} \leq \frac{1}{\sqrt{k(n-k)}},$$

we see that

$$(-1)^n c_n = \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}} \geq \sum_{k=1}^{n-1} \frac{1}{n-1} = \frac{1}{n-1}\sum_{k=1}^{n-1} 1 = 1.$$

Thus, the terms c_n do not tend to zero as $n \to \infty$, so by the n-th term test, the series $\sum_{n=0}^\infty c_n$ does not converge.

The problem with this example is that the series $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ does not converge absolutely. However, for absolutely convergent series, there is no problem as the following theorem, due to Franz Mertens (1840–1927), shows.

Theorem 5.32 (**Mertens' multiplication theorem**). *If at least one of two convergent series $\sum a_n = A$ and $\sum b_n = B$ converges absolutely, then their Cauchy product converges with sum equal to AB*

PROOF. Consider the partial sums of the Cauchy product:

$$\begin{aligned} C_n &= c_0 + c_1 + \cdots + c_n \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0(b_0 + \cdots + b_n) + a_1(b_0 + \cdots + b_{n-1}) + \cdots + a_n b_0 \end{aligned} \tag{5.37}$$

We need to show that C_n tends to AB as $n \to \infty$. Because our notation is symmetric in A and B, we may assume that the sum $\sum a_n$ is absolutely convergent. If A_n denotes the n-th partial sum of $\sum a_n$ and B_n that of $\sum b_n$, then from (5.37), we have

$$C_n = a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0.$$

If we set $B_k = B + \beta_k$, then $\beta_k \to 0$, and we can write

$$\begin{aligned} C_n &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_n(B + \beta_0) \\ &= A_n B + (a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0). \end{aligned}$$

Since $A_n \to A$, the first part of this sum converges to AB. Thus, we just need to show that the term in parenthesis tends to zero as $n \to \infty$. To see this, let $\varepsilon > 0$ be given. Putting $\alpha = \sum |a_n|$ and using that $\beta_n \to 0$, we can choose a natural number N such that for all $n > N$, we have $|\beta_n| < \varepsilon/(2\alpha)$. Also, since $\beta_n \to 0$, we can choose a constant C such that $|\beta_n| \le C$ for every n. Then for $n > N$,

$$\begin{aligned} |a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0| &= |a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-N+1}\beta_{N+1} \\ &\qquad + a_{n-N}\beta_N + \cdots + a_n\beta_0| \\ &\le |a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-N+1}\beta_{N+1}| + |a_{n-N}\beta_N + \cdots + a_n\beta_0| \\ &< \Big(|a_0| + |a_1| + \cdots + |a_{n-N+1}|\Big) \cdot \frac{\varepsilon}{2\alpha} + \Big(|a_{n-N}| + \cdots + |a_n|\Big) \cdot C \\ &\le \alpha \cdot \frac{\varepsilon}{2\alpha} + C\Big(|a_{n-N}| + \cdots + |a_n|\Big) \\ &= \frac{\varepsilon}{2} + C\Big(|a_{n-N}| + \cdots + |a_n|\Big). \end{aligned}$$

Since $\sum |a_n| < \infty$, by the Cauchy criterion for series, we can choose $N' > N$ such that

$$n > N' \quad \Longrightarrow \quad |a_{n-N}| + \cdots + |a_n| < \frac{\varepsilon}{2C}.$$

Then for $n > N'$, we see that

$$|a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof of the theorem. □

As an easy corollary, we see that if $\sum_{n=0}^\infty a_n z^n$ and $\sum_{n=0}^\infty b_n z^n$ have radii of convergence R_1, R_2, respectively, then since power series converge absolutely within their radii of convergence, for all $z \in \mathbb{C}$ with $|z| < R_1, R_2$, we have

$$\left(\sum_{n=0}^\infty a_n z^n\right)\left(\sum_{n=0}^\infty b_n z^n\right) = \sum_{n=0}^\infty c_n z^n$$

where $c_n = \sum_{k=0}^n a_k b_{n-k}$. In words: *The product of power series is a power series.*

Here's a question: Suppose that $\sum a_n$ and $\sum b_n$ converge and their Cauchy product $\sum c_n$ also converges; is it true that $\sum c_n = \big(\sum a_n\big)\big(\sum b_n\big)$? The answer may seem to be an "obvious" yes. However, it's not so "obvious' because the definition of the Cauchy product was based on a formal argument. Here is a proof of this "obvious" fact.

THEOREM 5.33 (**Abel's multiplication theorem**). *If the Cauchy product of two convergent series $\sum a_n = A$ and $\sum b_n = B$ converges, then the Cauchy product has the value AB.*

PROOF. In my opinion, the slickest proof of this theorem is Abel's original, prove in 1826 using his limit theorem, Theorem 5.20 [**92**, p. 321]. Let

$$f(z) = \sum a_n z^n, \quad g(z) = \sum b_n z^n, \quad h(z) = \sum c_n z^n,$$

where $c_n = a_0 b_n + \cdots + a_n b_0$. These power series converge at $z = 1$, so they must have radii of convergence at least 1. In particular, each series converges absolutely for $|z| < 1$ and for these values of z according to according to Merten's theorem, we have

$$h(z) = f(z) \cdot g(z).$$

Since each of the sums $\sum a_n$, $\sum b_n$, and $\sum c_n$ converges, by Abel's limit theorem, the functions f, g, and h converge to A, B, and $C = \sum c_n$, respectively, as $z = x \to 1$ from the left. Thus,

$$C = \lim_{x \to 1-} h(x) = \lim_{x \to 1-} f(x) \cdot g(x) = A \cdot B.$$

□

Example 5.38. For example, let us square $\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. It turns out that it will be convenient to write $\log 2$ in two ways: $\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (here $a_0 = 0$ and $a_n = \frac{(-1)^{n-1}}{n}$ for $n = 1, 2, \ldots$) and as $\log 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ (here $b_n = \frac{(-1)^n}{n+1}$). Thus, $c_0 = a_0 b_0 = 0$ and for $n = 1, 2, \ldots$, we see that

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=1}^{n} \frac{(-1)^{k-1}(-1)^{n-k}}{k(n+1-k)} = (-1)^{n-1} \alpha_n,$$

where $\alpha_n = \sum_{k=1}^{n} \frac{1}{k(n+1-k)}$. By Abel's multiplication theorem, we have $(\log 2)^2 = \sum_{n=0}^{\infty} c_n = \sum_{n=1}^{\infty} (-1)^{n-1} \alpha_n$ as long as this latter sum converges. By the alternating series test, this sum converges if we can prove that $\{\alpha_n\}$ is nonincreasing and converges to zero. To prove these statements hold, observe that we can write

$$\frac{1}{k(n-k+1)} = \frac{1}{n+1}\Big(\frac{1}{k} + \frac{1}{n-k+1}\Big),$$

therefore

$$\begin{aligned} \alpha_n &= \frac{1}{1} \cdot \frac{1}{n} + \frac{1}{2} \cdot \frac{1}{n-1} + \frac{1}{3} \cdot \frac{1}{n-2} + \cdots + \frac{1}{n} \cdot \frac{1}{1} \\ &= \frac{1}{n+1}\Big[\Big(1 + \frac{1}{n}\Big) + \Big(\frac{1}{2} + \frac{1}{n-1}\Big) + \Big(\frac{1}{3} + \frac{1}{n-2}\Big) + \cdots + \Big(\frac{1}{n} + \frac{1}{1}\Big)\Big]. \end{aligned}$$

In the brackets there are two copies of $1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Thus,

$$\alpha_n = \frac{2}{n+1} H_n, \qquad \text{where } H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

It is common to use the notation H_n for the n-th partial sum of the harmonic series. Now, recall from Section 4.6.5 on the Euler-Mascheroni constant that $\gamma_n := H_n - \log n$ is bounded above by 1, so

$$\begin{aligned}\alpha_n = \frac{2}{n+1}(\gamma_n + \log n) \leq \frac{2}{n+1} + 2\frac{\log n}{n+1} &= \frac{2}{n+1} + 2 \cdot \frac{n}{n+1} \cdot \frac{1}{n} \log n \\ &= \frac{2}{n+1} + 2 \cdot \frac{n}{n+1} \cdot \log(n^{1/n}) \to 0 + 2 \cdot 1 \cdot \log 1 = 0\end{aligned}$$

as $n \to \infty$. Thus, $\alpha_n \to 0$. Moreover,

$$\begin{aligned}\alpha_n - \alpha_{n+1} = \frac{2}{n+1}H_n - \frac{2}{n+2}H_{n+1} &= \frac{2}{n+1}H_n - \frac{2}{n+2}\Big(H_n + \frac{1}{n+1}\Big) \\ &= \Big(\frac{2}{n+1} - \frac{2}{n+2}\Big)H_n - \frac{2}{(n+1)(n+2)} \\ &= \frac{2}{(n+1)(n+2)}H_n - \frac{2}{(n+1)(n+2)} \\ &= \frac{2}{(n+1)(n+2)}(H_n - 1) \geq 0.\end{aligned}$$

Thus, $\alpha_n \geq \alpha_{n+1}$, so $\sum c_n = \sum (-1)^{n-1}\alpha_n$ converges. Hence, we have proved the following pretty formula:

$$\boxed{\begin{aligned}\frac{1}{2}\big(\log 2\big)^2 &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+1} H_n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+1}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{n}\Big)\ .\end{aligned}}$$

Our final theorem, Cauchy's multiplication theorem, basically says that we can multiply absolutely convergent series without worrying about anything. To introduce this theorem, note that if we have *finite* sums $\sum a_n$ and $\sum b_n$, then

$$\Big(\sum a_n\Big) \cdot \Big(\sum b_n\Big) = \sum a_m b_n,$$

where the sum on the right means to add over all such products $a_m b_n$ in any order we wish. One can ask if this holds true in the infinite series realm. The answer is "yes" if both series on the left are absolutely convergent.

THEOREM 5.34 (**Cauchy's multiplication theorem**). *If two series $\sum a_n = A$ and $\sum b_n = B$ converge absolutely, then the double series $\sum a_m b_n$ converges absolutely and has the value AB.*

PROOF. Since

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |a_m b_n| = \sum_{m=0}^{\infty} |a_m| \sum_{n=0}^{\infty} |b_n| = \Big(\sum_{m=0}^{\infty} |a_m|\Big)\Big(\sum_{n=0}^{\infty} |b_n|\Big) < \infty,$$

by Cauchy's double series theorem, the double series $\sum a_m b_n$ converges absolutely, and we can iterate the sums:

$$\sum a_m b_n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m b_n = \sum_{m=0}^{\infty} a_m \sum_{n=0}^{\infty} b_n = \Big(\sum_{m=0}^{\infty} a_m\Big)\Big(\sum_{n=0}^{\infty} b_n\Big) = A \cdot B.$$

$\square$

We remark that Cauchy's multiplication theorem generalizes to a product of more than two absolutely convergent series.

5.6.4. The exponential function (again). Using Mertens' or Cauchy's multiplication theorem, we can give an alternative and quick proof of the formula $\exp(z)\exp(w) = \exp(z+w)$ for $z, w \in \mathbb{C}$, which was originally proved in Theorem 4.29 using a completely different method:

$$\begin{aligned}\exp(z)\exp(w) &= \Big(\sum_{n=0}^{\infty} \frac{z^n}{n!}\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{w^n}{n!}\Big)\\ &= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!}\Big)\\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k}\Big)\\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}\Big) = \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n = \exp(z+w),\end{aligned}$$

where we used the binomial theorem for $(z+w)^n$ in the last line.

Exercises 5.6.

1. Here are some alternating series problems:
   (a) Prove that
   $$\frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} + \cdots = \frac{3}{2}\log 2.$$
   that is, we rearrange the alternating harmonic series so that two positive terms are followed by one negative one, otherwise keeping the ordering the same. Suggestion: Observe that
   $$\begin{aligned}\frac{1}{2}\log 2 &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots\\ &= 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots .\end{aligned}$$
   Add this term-by-term to the series for $\log 2$.
   (b) Prove that
   $$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{2} + \cdots + \frac{1}{8k-7} + \frac{1}{8k-5} + \frac{1}{8k-3} + \frac{1}{8k-1} - \frac{1}{2k} + \cdots = \frac{3}{2}\log 2;$$
   that is, we rearrange the alternating harmonic series so that four positive terms are followed by one negative one, otherwise keeping the ordering the same.
   (c) What's wrong with the following argument?
   $$\begin{aligned}1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots &= \Big(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots\Big)\\ &\quad - 2\Big(\frac{1}{2} + 0 + \frac{1}{4} + 0 + \frac{1}{6} + \cdots\Big)\\ &= \Big(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots\Big) - \Big(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots\Big) = 0.\end{aligned}$$
2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be absolutely convergent for $|z| < 1$. Prove that for $|z| < 1$, we have
   $$\frac{f(z)}{1-z} = \sum_{n=0}^{\infty} (a_0 + a_1 + a_2 + \cdots + a_n) z^n.$$

3. Using the previous problem, prove that for $z \in \mathbb{C}$ with $|z| < 1$,

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n; \quad \text{that is,} \ \Big(\sum_{n=0}^{\infty} z^n \Big) \cdot \Big(\sum_{n=0}^{\infty} z^n \Big) = \sum_{n=0}^{\infty} (n+1) z^n.$$

Using this formula, derive the neat looking formula: For $z \in \mathbb{C}$ with $|z| < 1$,

$$\Big(\sum_{n=0}^{\infty} \cos n\theta \, z^n \Big) \cdot \Big(\sum_{n=0}^{\infty} \sin n\theta \, z^n \Big) = \frac{1}{2} \sum_{n=0}^{\infty} (n+1) \sin n\theta \, z^n. \tag{5.38}$$

Suggestion: Put $z = e^{i\theta} x$ with x real into the formula $(\sum_{n=0}^{\infty} z^n) \cdot (\sum_{n=0}^{\infty} z^n) = \sum_{n=0}^{\infty} (n+1) z^n$, then equate imaginary parts of both sides; this proves (5.38) for $z = x$ real and $|x| < 1$. Why does (5.38) hold for $z \in \mathbb{C}$ with $|z| < 1$?

4. Derive the beautiful formula: For $|z| < 1$,

$$\Big(\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} z^n \Big) \cdot \Big(\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} z^n \Big) = \frac{1}{2} \sum_{n=2}^{\infty} \frac{H_n \sin n\theta}{n} z^n.$$

5. In this problem we prove the following fact: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$ and let $\alpha \in \mathbb{C}$ with $|\alpha| < R$. Then we can write

$$f(z) = \sum_{n=0}^{\infty} b_n (z - \alpha)^n,$$

where this series converges absolutely for $|z - \alpha| < R - \alpha$.

(i) Show that

$$f(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \binom{n}{m} \alpha^{n-m} (z-\alpha)^m. \tag{5.39}$$

(ii) Prove that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |a_n| \binom{n}{m} |\alpha|^{n-m} |z - \alpha|^m = \sum_{n=0}^{\infty} |a_n| \big(|z-\alpha| + |\alpha| \big)^m < \infty.$$

(iii) Verifying that you can change the order of summation in (5.39), prove the result.

5.7. ★ Proofs that $\sum 1/p$ diverges

We know that the harmonic series $\sum 1/n$ diverges. However, if we only sum over the squares, then we get the convergent sum $\sum 1/n^2$. Similarly, if we only sum over the cubes, we get the convergent sum $\sum 1/n^3$. One may ask: What if we sum only over all primes:

$$\sum \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \cdots,$$

do we get a convergent sum? We know that there are arbitrarily large gaps between primes (see Problem 1 in Exercises 2.4), so one may conjecture that $\sum 1/p$ converges. However, following [**19**], [**50**], [**125**] (cf. [**126**]), and [**100**] we shall prove that $\sum 1/p$ diverges! Other proofs can be found in the exercises. An expository article giving other proofs (cf. [**119**], [**41**]) on this fascinating divergent sum can be found in [**175**].

5.7.1. Proof I: Proof by multiplication and rearrangement. This is Bellman [**19**] and Dux's [**50**] argument. Suppose, for sake of contradiction, that $\sum 1/p$ converges. Then we can fix a prime number m such that $\sum_{p>m} 1/p < 1$. Let $2 < 3 < \cdots < m$ be the list of all prime numbers up to m. Given $N > m$, let P_N be the set of natural numbers greater than one and less than or equal to N all of whose prime factors are less than or equal to m, and let Q_N be the set of natural numbers greater than one and less than or equal to N all of whose prime factors are greater than m. Explicitly,

$$\begin{aligned} &k \in P_N \iff 1 < k \leq N \text{ and } k = 2^i 3^j \cdots m^k, \quad \text{some } i, j, \ldots, k,\\ &\ell \in Q_N \iff 1 < \ell \leq N \text{ and } \ell = p\, q \cdots r, \quad p, q, \ldots, r > m \text{ are prime.} \end{aligned} \tag{5.40}$$

In the product $p\, q \cdots r$ prime numbers may be repeated. Observe that any integer $1 < n \leq N$ that is not in P_N or Q_N must have prime factors that are both less than or equal to m and greater than m, and hence can be factored in the form $n = k\,\ell$ where $k \in P_N$ and $\ell \in Q_N$. Thus, the finite sum

$$\sum_{k\in P_N} \frac{1}{k} + \sum_{\ell\in Q_N} \frac{1}{\ell} + \Big(\sum_{k\in P_N} \frac{1}{k}\Big)\Big(\sum_{\ell\in Q_N} \frac{1}{\ell}\Big) = \sum_{k\in P_N} \frac{1}{k} + \sum_{\ell\in Q_N} \frac{1}{\ell} + \sum_{k\in P_N, \ell\in Q_N} \frac{1}{k\,\ell},$$

contains every number of the form $1/n$ where $1 < n \leq N$. (Of course, the resulting sum contains other numbers too.) In particular,

$$\sum_{k\in P_N} \frac{1}{k} + \sum_{\ell\in Q_N} \frac{1}{\ell} + \Big(\sum_{k\in P_N} \frac{1}{k}\Big)\Big(\sum_{\ell\in Q_N} \frac{1}{\ell}\Big) \geq \sum_{n=2}^{N} \frac{1}{n},$$

We shall prove that the finite sums on the left remain bounded as $N \to \infty$, which contradicts the fact that the harmonic series diverges.

To see that $\sum_{P_N} 1/k$ converges, note that each geometric series $\sum_{j=1}^\infty 1/p^j$ converges (absolutely since all the $1/p^j$ are positive) to a finite real number. Hence, by Cauchy's multiplication theorem (or rather its generalization to a product of more than two absolutely convergent series), we have

$$\left(\sum_{i=1}^\infty \frac{1}{2^i}\right)\left(\sum_{j=1}^\infty \frac{1}{3^j}\right)\cdots\left(\sum_{k=1}^\infty \frac{1}{m^k}\right) = \sum \frac{1}{2^i 3^j \cdots m^k}$$

is a finite real number, where the sum on the right is over all $i, j, \ldots, k = 1, 2, \ldots$. Using the definition of P_N in (5.40), we see that $\sum_{P_N} 1/k$ is bounded above by this finite real number uniformly in N. Thus, $\lim_{N\to\infty} \sum_{P_N} 1/k$ is finite.

We now prove that $\lim_{N\to\infty} \sum_{Q_N} 1/\ell$ is finite. To do so observe that since $\alpha := \sum_{p>m} 1/p < 1$ and all the $1/p$'s are positive, the sum $\sum_{p>m} 1/p$, in particular, converges absolutely. Hence, by Cauchy's multiplication theorem, we have

$$\alpha^2 = \left(\sum_{p>m} \frac{1}{p}\right)^2 = \sum_{p,q>m} \frac{1}{p\,q},$$

where the sum is over all primes $p, q > m$, and

$$\alpha^3 = \left(\sum_{p>m} \frac{1}{p}\right)^3 = \sum_{p,q,r>m} \frac{1}{p\,q\,r},$$

where the sum is over all primes $p, q, r > m$. We can continue this procedure showing that α^j is the sum $\sum 1/(p\,q \cdots r)$ where the sum is over all primes $p, q, \ldots, r >$

m. In view of the definition of Q_N in (5.40), it follows that the sum $\sum_{Q_N} 1/\ell$ is bounded by the number $\sum_{j=1}^{\infty} \alpha^j$, which is finite because $\alpha < 1$. Hence, the limit $\lim_{N\to\infty} \sum_{Q_N} 1/\ell$ is finite, and we have reached a contradiction.

5.7.2. An elementary number theory fact. Our next proof depends on the idea of square-free integers. A positive integer is said to be **square-free** if no squared prime divides it, that is, if a prime occurs in its prime factorization, then it occurs with multiplicity one. For instance, 1 is square-free because no squared prime divides it, $10 = 2\cdot 5$ is square-free, but $24 = 2^3\cdot 3 = 2^2\cdot 2\cdot 3$ is not square-free.

We claim that any positive integer can be written uniquely as the product of a square and a square-free integer. Indeed, let n be an integer and let k^2 be the largest square that divides n. Then n/k^2 must be square-free, for if n/k^2 is divided by a squared prime p^2, then $(pk)^2 > k^2$ divides n, which is not possible because k^2 was the largest such square that divided n. Thus, any positive integer n can be uniquely written as $n = k^2$ if n is a perfect square, or

$$n = k^2 \cdot p\, q \cdots r, \tag{5.41}$$

where $k \geq 1$ and where $p, q, \ldots, r$ are primes less than or equal to n that occur with multiplicity one. Using the fact that any positive integer can be uniquely written as the product of a square and a square-free integer, we shall prove that $\sum 1/p$ diverges.

5.7.3. Proof II: Proof by comparison. Here is Niven's [**125, 126**] proof. We first prove that the product

$$\prod_{p<N} \left(1 + \frac{1}{p}\right)$$

diverges to ∞ as $N \to \infty$, where the product is over all primes less than N. Let $2 < 3 < \cdots < m$ be all the primes less than N. Consider the product

$$\prod_{p<N} \left(1 + \frac{1}{p}\right) = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{m}\right).$$

For example, if $N = 5$, then

$$\prod_{p<5} \left(1 + \frac{1}{p}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2\cdot 3}.$$

If $N = 6$, then

$$\begin{aligned}\prod_{p<6} \left(1 + \frac{1}{p}\right) &= \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 5} + \frac{1}{2\cdot 3\cdot 5}.\end{aligned}$$

Using induction on N, we can always write

$$\prod_{p<N} \left(1 + \frac{1}{p}\right) = 1 + \sum_{p<N} \frac{1}{p} + \sum_{p,q<N} \frac{1}{p\cdot q} + \cdots + \sum_{p,q,\ldots,r<N} \frac{1}{p\cdot q\cdots r},$$

where the k-th sum on the right is the sum over over all reciprocals of the form $\frac{1}{p_1 \cdot p_2 \cdots p_k}$ with $p_1, \ldots, p_k$ *distinct* primes less than N. Thus,

$$\prod_{p<N}\Big(1+\frac{1}{p}\Big)\cdot\sum_{k<N}\frac{1}{k^2}=\sum_{k<N}\frac{1}{k^2}+\sum_{k<N}\sum_{p<N}\frac{1}{k^2p}$$
$$+\sum_{k<N}\sum_{p,q<N}\frac{1}{k^2\cdot p\cdot q}+\cdots+\sum_{k<N}\sum_{p,q,\ldots,r<N}\frac{1}{k^2\cdot p\cdot q\cdots r}.$$

By our discussion on square-free numbers around (5.41), the right-hand side contains every number of the form $1/n$ where $n<N$ (and many other numbers too). In particular,

$$\prod_{p<N}\Big(1+\frac{1}{p}\Big)\cdot\sum_{k<N}\frac{1}{k^2}\geq\sum_{n<N}\frac{1}{n}. \tag{5.42}$$

From this inequality, we shall prove that $\sum 1/p$ diverges. To this end, we know that $\sum_{k=1}^\infty 1/k^2$ converges while $\sum_{n=1}^\infty 1/n$ diverges, so it follows that

$$\lim_{N\to\infty}\prod_{p<N}\Big(1+\frac{1}{p}\Big)=\infty.$$

To relate this product to the sum $\sum 1/p$, note that

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\geq 1+x$$

for $x\geq 0$ — in fact, this inequality holds for all $x\in\mathbb{R}$ by Theorem 4.30. Hence,

$$\prod_{p<N}\Big(1+\frac{1}{p}\Big)\leq\prod_{p<N}\exp(1/p)=\exp\Big(\sum_{p<N}\frac{1}{p}\Big).$$

Since the left-hand side increases without bound as $N\to\infty$, so must the sum $\sum_{p<N}1/p$. This ends **Proof II**; see Problem 1 for a related proof.

5.7.4. Proof III: Another proof by comparison. This is Gilfeather and Meister's argument [**100**]. The first step is to prove that for any natural number $N>1$, we have

$$\prod_{p<N}\frac{p}{p-1}\geq\sum_{n=1}^{N-1}\frac{1}{n}.$$

To prove this we shall prove that $\prod_{p<N}\Big(1-\frac{1}{p}\Big)^{-1}\to\infty$. To see this, observe that

$$\Big(1-\frac{1}{p}\Big)^{-1}=1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\cdots.$$

Let $2<3<\cdots<m$ be all the primes less than N. Then every natural $n<N$ can be written in the form

$$n=2^i\,3^j\,\cdots\,m^k$$

for some nonnegative integers $i, j, \ldots, k$. It follows that the product

$$\begin{aligned}\prod_{p<N}\Big(1-\frac{1}{p}\Big)^{-1} &= \Big(1-\frac{1}{2}\Big)^{-1}\Big(1-\frac{1}{3}\Big)^{-1}\cdots\Big(1-\frac{1}{m}\Big)^{-1}\\ &= \Big(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\cdots\Big)\Big(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\Big)\cdots\\ &\qquad\cdots\Big(1+\frac{1}{m}+\frac{1}{m^2}+\frac{1}{m^3}+\cdots\Big)\end{aligned}$$

after multiplying out using Cauchy's multiplication theorem (or rather its generalization to a product of more than two absolutely convergent series), contains all the numbers $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{N-1}$ (and of course, many more numbers too). Thus,

$$\prod_{p<N}\frac{p}{p-1} = \prod_{p<N}\Big(1-\frac{1}{p}\Big)^{-1} \geq \sum_{n=1}^{N-1}\frac{1}{n}, \tag{5.43}$$

which proves our first step. Now recall from (4.29) that for any natural number n, we have

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}. \tag{5.44}$$

In particular, taking logarithms of both sides of (5.43), we get

$$\begin{aligned}\log\Big(\sum_{n=1}^{N-1}\frac{1}{n}\Big) &\leq \log\Big(\prod_{p<N}\frac{p}{p-1}\Big)\\ &= \sum_{p<N}\big(\log p - \log(p-1)\big) \leq \sum_{p<N}\frac{1}{p-1} \leq \sum_{p<N}\frac{2}{p},\end{aligned}$$

where we used that $p \leq 2(p-1)$ (this is because $n \leq 2(n-1)$ for all natural numbers $n > 1$). Since $\sum_{n=1}^{N-1} 1/n \to \infty$ as $N \to \infty$, $\log\big(\sum_{n=1}^{N-1} 1/n\big) \to \infty$ as $N \to \infty$ as well, so the sum $\sum 1/p$ must diverge.

Exercises 5.7.

1. Niven's proof can be slightly modified to avoid using the square-free fact. Prove that for any prime p, we have
$$\Big(1+\frac{1}{p}\Big)\cdot\sum_{k=0}^{n}\frac{1}{p^{2k}} = \sum_{k=0}^{2n+1}\frac{1}{p^k}.$$
Use this identity to derive the inequality (5.42), which as shown in the main body implies that $\sum 1/p$ diverges.
2. Here is another proof that is similar to Gilfeather and Meister's argument where we replace the inequality (5.44) with the following argument.
   (i) Prove that
$$\frac{1}{1-x/2} \leq e^x \quad \text{for all } 0 \leq x \leq 1. \tag{5.45}$$
   Suggestion: Multiplying $e^{-x}(1-x/2)$ to both sides, it might be slightly easier to prove the inequality $e^{-x} \leq 1 - x/2$ for $0 \leq x \leq 1$. The series expansion for e^{-x} might be helpful.
   (ii) Taking logarithms of (5.45), prove that for any prime number p, we have
$$-\log\Big(1-\frac{1}{p}\Big) = -\log\Big(1-\frac{2/p}{2}\Big) \leq \frac{2}{p}.$$

(iii) Prove that
$$\frac{1}{2}\sum_{p<N}\log\left(\frac{p}{p-1}\right) \le \sum_{p<N}\frac{1}{p}.$$
(iv) Finally, use (5.43) as in the main text to prove that $\sum 1/p$ diverges.

3. Here's Vanden Eynden's proof [**175**]. Assume that $\sum 1/p$ converges. Then we can choose an N such that $\alpha := \sum_{p>N} 1/p < 1/2$ and (since the harmonic series $\sum 1/n$ diverges) $\beta := \sum_{n>N} 1/n > 2$.
   (i) Prove that $\beta - 1 \le \alpha \cdot \beta$.
   (ii) Deduce that $1 - \beta^{-1} \le \alpha$ and use this fact, together with the assumptions that $\alpha < 1/2$ and $\beta > 2$, to derive a contradiction.
4. Here is Paul Erdös' (1913–1996) celebrated proof [**51**]. Assume that $\sum 1/p$ converges. Then we can choose an N such that $\sum_{p>N} 1/p < 1/2$; derive a contradiction as follows.
   (i) For any $x \in \mathbb{N}$, let A_x be the set of all integers $1 \le n \le x$ such that $n = 1$ or all the prime factors of n are $\le N$; that is, $n = p_1 \cdots p_k$ where the p_j's are prime and $p_j \le N$. Given $n \in A_x$, we can write $n = k^2 m$ where m is square free. Prove that $k \le \sqrt{x}$. From this, deduce that
$$\#A_x \le C\sqrt{x},$$
   where $\#A_x$ denotes the number of elements in the set A_x and C is a constant (you can take C to equal the number of square free integers $m \le N$).
   (ii) Given $x \in \mathbb{N}$ and a prime p, prove that the number of integers $1 \le n \le x$ divisible by p is no more than x/p.
   (iii) Given $x \in \mathbb{N}$, prove that $x - \#A_x$ equals the number of integers $1 \le n \le x$ that are divisible by some prime $p > N$. From this fact and Part (b) together with our assumption that $\sum_{p>N} 1/p < 1/2$, prove that
$$x - \#A_x < \frac{x}{2}.$$
   (iv) Using (c) and the inequality $\#A_x \le C\sqrt{x}$ you proved in Part (a), conclude that for any $x \in \mathbb{N}$, we have
$$\sqrt{x} \le 2C.$$
   From this derive a contradiction.

5.8. Composition of power series and Bernoulli and Euler numbers

We've kept you in suspense long enough concerning the extraordinary Bernoulli and Euler numbers, so in this section we finally get to these fascinating numbers.

5.8.1. Composition and division of power series. The Bernoulli and Euler numbers come up when dividing power series, so before we do anything, we need to understand division of power series, and to understand this we first need to consider the composition of power series. The following theorem basically says that the composition of power series is again a power series.

THEOREM 5.35 (**Power series composition theorem**). *If $f(z)$ and $g(z)$ are power series, then the composition $f(g(z))$ can be written as a power series that is valid for all $z \in \mathbb{C}$ such that*
$$\sum_{n=0}^{\infty} |a_n z^n| < \text{the radius of convergence of } f,$$
where $g(z) = \sum_{n=0}^{\infty} a_n z^n$.

PROOF. Let $f(z) = \sum_{n=0}^\infty b_n z^n$ have radius of convergence R and let $g(z) = \sum_{n=0}^\infty a_n z^n$ have radius of convergence r. Then by Cauchy or Mertens' multiplication theorem, for each m, we can write $g(z)^m$ as a power series:

$$g(z)^m = \left(\sum_{n=0}^\infty a_n\, z^n \right)^m = \sum_{n=0}^\infty a_{mn}\, z^n, \qquad |z| < r.$$

Thus,

$$f(g(z)) = \sum_{m=0}^\infty b_m\, g(z)^m = \sum_{m=0}^\infty \sum_{n=0}^\infty b_m a_{mn}\, z^n.$$

If we are allowed to interchange the order of summation in $f(g(z))$, then our result is proved:

$$f(g(z)) = \sum_{n=0}^\infty \sum_{m=0}^\infty b_m a_{mn}\, z^n = \sum_{n=0}^\infty c_n\, z^n, \qquad \text{where } \ c_n = \sum_{m=0}^\infty b_m a_{mn}.$$

Thus, we can focus on interchanging the order of summation in $f(g(z))$. Assume henceforth that

$$\xi := \sum_{n=0}^\infty |a_n z^n| = \sum_{n=0}^\infty |a_n|\,|z|^n < R = \text{ the radius of convergence of } f;$$

in particular, since $f(\xi) = \sum_{m=0}^\infty b_m \xi^m$ is absolutely convergent,

$$\sum_{m=0}^\infty |b_m|\,\xi^m < \infty. \tag{5.46}$$

Now according to Cauchy's double series theorem, we can interchange the order of summation as long as we can show that

$$\sum_{m=0}^\infty \sum_{n=0}^\infty \left| b_m a_{mn} z^n \right| = \sum_{m=0}^\infty \sum_{n=0}^\infty |b_m|\,|a_{mn}|\,|z|^n < \infty.$$

To prove this, we first claim that the inner summation satisfies the inequality

$$\sum_{n=0}^\infty |a_{mn}|\,|z|^n \le \xi^m. \tag{5.47}$$

To see this, consider the case $m = 2$. Recall that the coefficients a_{2n} are defined via the Cauchy product:

$$g(z)^2 = \left(\sum_{n=0}^\infty a_n\, z^n \right)^2 = \sum_{n=0}^\infty a_{2n}\, z^n \quad \text{where} \quad a_{2n} = \sum_{k=0}^n a_k a_{n-k}.$$

Thus, $|a_{2n}| \le \sum_{k=0}^n |a_k|\,|a_{n-k}|$. On the other hand, ξ^2 is defined via the Cauchy product:

$$\xi^2 = \left(\sum_{n=0}^\infty |a_n|\,|z|^n \right)^2 = \sum_{n=0}^\infty \alpha_n\, |z|^n \quad \text{where} \quad \alpha_n = \sum_{k=0}^n |a_k|\,|a_{n-k}|.$$

Hence, $|a_{2n}| \le \sum_{k=0}^n |a_k|\,|a_{n-k}| = \alpha_n$, so

$$\sum_{n=0}^\infty |a_{2n}|\,|z|^n \le \sum_{n=0}^\infty \alpha_n\, |z|^n = \xi^2,$$

which proves (5.47) for $m = 2$. An induction argument shows that (5.47) holds for all m. Finally, using (5.47) and (5.46) we see that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|b_m a_{mn} z^n\right| = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|b_m|\,|a_{mn}|\,|z|^n \le \sum_{m=0}^{\infty}|b_m|\,\xi^m < \infty,$$

which shows that we can interchange the order of summation in $f(g(z))$ and completes our proof. $\square$

We already know (by Mertens' multiplication theorem for instance) that the product of two power series is again a power series. As a consequence of the following theorem, we get the same statement for division.

THEOREM 5.36 (**Power series division theorem**). *If $f(z)$ and $g(z)$ are power series with positive radii of convergence and with $g(0) \neq 0$, then $f(z)/g(z)$ is also a power series with positive radius of convergence.*

PROOF. Since $f(z)/g(z) = f(z) \cdot (1/g(z))$ and we know that the product of two power series is a power series, all we have to do is show that $1/g(z)$ is a power series. To this end, let $g(z) = \sum_{n=0}^{\infty} a_n\, z^n$ and define

$$\tilde{g}(z) := \frac{1}{a_0} g(z) - 1 = \sum_{n=1}^{\infty} \alpha_n\, z^n,$$

where $\alpha_n = \frac{a_n}{a_0}$ and where we recall that $a_0 = g(0) \neq 0$. Then $\tilde{g}$ has a positive radius of convergence and $\tilde{g}(0) = 0$. Now let $h(z) := \frac{1}{a_0(1+z)}$, which can be written as a geometric series with radius of convergence 1. Note that for $|z|$ small, $\sum_{n=1}^{\infty} |\alpha_n|\,|z|^n < 1$ (why?), thus by the previous theorem, for such z,

$$\frac{1}{g(z)} = \frac{1}{a_0(\tilde{g}(z)+1)} = h(\tilde{g}(z))$$

has a power series expansion with a positive radius of convergence. $\square$

5.8.2. Bernoulli numbers. See [**92**], [**43**], [**151**], or [**66**] for more information on Bernoulli numbers. Since

$$\frac{e^z - 1}{z} = \frac{1}{z} \cdot \sum_{n=1}^{\infty} \frac{1}{n!} z^n = \sum_{n=1}^{\infty} \frac{1}{n!} z^{n-1} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^n$$

has a power series expansion and equals 1 at $z = 0$, by our division of power series theorem, the quotient $1/((e^z - 1)/z) = z/(e^z - 1)$ also has a power series expansion near $z = 0$. It is customary to denote its coefficients by $B_n/n!$, in which case we can write

$$\boxed{\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\, z^n} \tag{5.48}$$

where the series has a positive radius of convergence. The numbers B_n are called the **Bernoulli numbers** after Jacob (Jacques) Bernoulli (1654–1705) who discovered them while searching for formulas involving powers of integers; see Problems 3 and 4. We can find a remarkable symbolic equation for these Bernoulli numbers as follows. First, we multiply both sides of (5.48) by $(e^z-1)/z$ and use Mertens' multiplication theorem to get

$$1=\left(\sum_{n=0}^{\infty}\frac{B_n}{n!}\,z^n\right)\cdot\left(\sum_{n=0}^{\infty}\frac{1}{(n+1)!}z^n\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{B_k}{k!}\cdot\frac{1}{(n-k+1)!}\right)z^n.$$

By the identity theorem, the $n=0$ term on the right must equal 1 while all other terms must vanish. The $n=0$ term on the right is just B_0, so $B_0=1$, and for $n>1$, we must have $\sum_{k=0}^{n}\frac{B_k}{k!}\cdot\frac{1}{(n+1-k)!}=0$. Multiplying this by $(n+1)!$ we get

$$0=\sum_{k=0}^{n}\frac{B_k}{k!}\cdot\frac{(n+1)!}{(n+1-k)!}=\sum_{k=0}^{n}\frac{(n+1)!}{k!(n+1-k)!}\cdot B_k=\sum_{k=0}^{n}\binom{n+1}{k}B_k,$$

and adding $B_{n+1}=\binom{n+1}{n+1}B_{n+1}$ to both sides of this equation, we get

$$B_{n+1}=\sum_{k=0}^{n+1}\binom{n+1}{k}B_k.$$

The right-hand side might look familiar from the binomial formula. Recall from the binomial formula that for any complex number a, we have

$$(a+1)^{n+1}=\sum_{k=0}^{n+1}\binom{n+1}{k}a^k\cdot 1^{n-k}=\sum_{k=0}^{n+1}\binom{n+1}{k}a^k.$$

Notice that the right-hand side of this expression is exactly the right-hand side of the previous equation if put $a=B$ and we make the superscript k into a subscript k. Thus, if we use the notation $\doteqdot$ to mean "equals after making superscripts into subscripts", then we can write

$$\boxed{B^{n+1}\doteqdot(B+1)^{n+1}\quad,\quad n=1,2,3,\ldots\quad\text{with } B_0=1.}\tag{5.49}$$

Using the identity (5.49), one can in principle find all the Bernoulli numbers: When $n=1$, we see that

$$B^2\doteqdot(B+1)^2=B^2+2B^1+1\quad\Longrightarrow\quad 0=2B_1+1\quad\Longrightarrow\quad B_1=-\frac{1}{2}.$$

When $n=2$, we see that

$$B^3\doteqdot(B+1)^3=B^3+3B^2+3B^1+1\Longrightarrow 0=3B_2+3B_1+1\Longrightarrow B_2=\frac{1}{6}.$$

Here is a partial list through B_{14}:

$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_3=0,$$
$$B_4=-\frac{1}{30},\quad B_5=B_7=B_9=B_{11}=B_{13}=B_{15}=0,$$
$$B_6=\frac{1}{42},\quad B_8=-\frac{1}{30},\quad B_{10}=\frac{5}{66},\quad B_{12}=-\frac{691}{2730},\quad B_{14}=\frac{7}{6}.$$

These numbers are rational, but besides this fact, there is no known regular pattern these numbers conform to. However, we can easily deduce that all odd Bernoulli numbers greater than one are zero. Indeed, we can rewrite (5.48) as

$$\frac{z}{e^z-1}+\frac{z}{2}=1+\sum_{n=2}^{\infty}\frac{B_n}{n!}\,z^n.\tag{5.50}$$

The fractions on the left-hand side can be combined into one fraction

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z(e^z + 1)}{2(e^z - 1)} = \frac{z(e^{z/2} + e^{-z/2})}{2(e^{z/2} - e^{-z/2})}, \tag{5.51}$$

which an even function of z. Thus,

$$B_{2n+1} = 0, \qquad n = 1, 2, 3, \ldots. \tag{5.52}$$

Other properties are given in the exercises (see e.g. Problem 3).

5.8.3. Trigonometric functions. We already know the power series expansions for $\sin z$ and $\cos z$. It turns out that the power series expansions of the other trigonometric functions involve Bernoulli numbers! For example, to find the expansion for $\cot z$, we replace z with $2iz$ in (5.50) and (5.51) to get

$$\frac{iz(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})} = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!}\,(2iz)^n = 1 + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\,(-1)^n (2z)^{2n},$$

where used that $B_3, B_5, B_7, \ldots$ all vanish in order to sum only over all even Bernoulli numbers. Since $\cot z = \cos z / \sin z$, using the definition of $\cos z$ and $\sin z$ in terms of $e^{\pm iz}$, we see that the left-hand side is exactly $z \cot z$. Thus, we have derived the formula

$$\boxed{z \cot z = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}\, B_{2n}}{(2n)!}\, z^{2n}\ .}$$

From this formula, we can get the expansion for $\tan z$ by using the identity

$$2\cot(2z) = 2\frac{\cos 2z}{\sin 2z} = 2\frac{\cos^2 z - \sin^2 z}{2\sin z\, \cos z} = \cot z - \tan z.$$

Hence,

$$\tan z = \cot z - 2\cot(2z) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}\, B_{2n}}{(2n)!}\, z^{2n} - 2\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}\, B_{2n}}{(2n)!}\, 2^{2n} z^{2n},$$

which, after combining the terms on the right, takes the form

$$\boxed{\tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n} - 1)\, B_{2n}}{(2n)!}\, z^{2n-1}\ .}$$

In Problem 1, we derive the power series expansion of $\csc z$. In conclusion we have power series expansions for $\sin z, \cos z, \tan z, \cot z, \csc z$. What about $\sec z$?

5.8.4. The Euler numbers. It turns out that the expansion for $\sec z$ involves the Euler numbers, which are defined in a similar way as the Bernoulli numbers. By the division of power series theorem, the function $2e^z/(e^{2z} + 1)$ has a power series expansion near zero. It is customary to denote its coefficients by $E_n/n!$, so

$$\boxed{\frac{2e^z}{e^{2z} + 1} = \sum_{n=0}^{\infty} \frac{E_n}{n!}\, z^n} \tag{5.53}$$

where the series has a positive radius of convergence. The numbers E_n are called the **Euler numbers**. We can get the missing expansion for $\sec z$ as follows. First, observe that

$$\sum_{n=0}^{\infty} \frac{E_n}{n!}\, z^n = \frac{2e^z}{e^{2z}+1} = \frac{2}{e^z + e^{-z}} = \frac{1}{\cosh z} = \operatorname{sech} z,$$

where $\operatorname{sech} z := 1/\cosh z$ is the hyperbolic secant. Since $\operatorname{sech} z$ is an even function (that is, $\operatorname{sech}(-z) = \operatorname{sech} z$) it follows that all E_n with n odd vanish. Hence,

$$\boxed{\operatorname{sech} z = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!}\, z^{2n}\ .} \tag{5.54}$$

In particular, putting iz for z in (5.54) and using that $\cosh(iz) = \cos z$, we get the missing expansion for $\sec z$:

$$\boxed{\sec z = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!}\, z^{2n}\ .}$$

Just as with the Bernoulli numbers, we can derive a symbolic equation for the Euler numbers. To do so, we multiply (5.54) by $\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}$ and use Mertens' multiplication theorem to get

$$1 = \left(\sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!}\, z^{2n} \right) \cdot \left(\sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{E_{2k}}{(2k)!} \cdot \frac{1}{(2n-2k)!} \right) z^{2n}$$

By the identity theorem, the $n = 0$ term on the right must equal 1 while all other terms must vanish. The $n = 0$ term on the right is just E_0, so $E_0 = 1$, and for $n > 1$, we must have $\sum_{k=0}^{n} \frac{E_{2k}}{(2k)!} \cdot \frac{1}{(2n-2k)!} = 0$. Multiplying this by $(2n)!$ we get

$$0 = \sum_{k=0}^{n} \frac{E_{2k}}{(2k)!} \cdot \frac{(2n)!}{(2n-2k)!} = \sum_{k=0}^{n} \frac{(2n)!}{(2k)!(2n-2k)!} \cdot E_{2k}. \tag{5.55}$$

Now from the binomial formula, for any complex number a, we have

$$\begin{aligned}
(a+1)^{2n} + (a-1)^{2n} &= \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!} a^k + \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!} a^k (-1)^{2n-k} \\
&= \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!} a^k + \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!} a^k (-1)^k \\
&= \sum_{k=0}^{2n} \frac{(2n)!}{(2k)!(2n-2k)!} a^{2k},
\end{aligned}$$

since all the odd terms cancel. Notice that the right-hand side of this expression is exactly the right-hand side of (5.55) if put $a = E$ and we make the superscript $2k$ into a subscript $2k$. Thus,

$$\boxed{(E+1)^{2n} + (E-1)^{2n} \doteqdot 0\ ,\quad n = 1, 2, \ldots \quad \text{with } E_0 = 1 \text{ and } E_{\text{odd}} = 0.} \tag{5.56}$$

Using the identity (5.56), one can in principle find all the Euler numbers: When $n = 1$, we see that

$$(E^2 + 2E^1 + 1) + (E^2 - 2E^1 + 1) \doteqdot 0 \quad \Longrightarrow \quad 2E_2 + 2 = 0 \quad \Longrightarrow \quad E_2 = -1.$$

Here is a partial list through E_{12}:

$$E_0 = 1, \quad E_1 = E_2 = E_3 = \cdots = 0 \ \ (E_{\rm odd} = 0), \quad E_2 = -1, \quad E_4 = 5$$
$$E_6 = -61, \quad E_8 = 1385, \quad E_{10} = -50,521, \quad E_{12} = 2,702,765, \quad \ldots.$$

These numbers are all integers, but besides this fact, there is no known regular pattern these numbers conform to.

Exercises 5.8.

1. Recall that $\csc z = 1/\sin z$. Prove that $\csc z = \cot z + \tan(z/2)$, and from this identity deduce that
$$\boxed{z \csc z = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(2^{2n} - 2)\, B_{2n}}{(2n)!}\, z^{2n}.}$$
2. (a) Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ with $b_0 \neq 0$ be power series with positive radii of convergence. Show that $f(z)/g(z) = \sum c_n z^n$ where $\{c_n\}$ is the sequence defined recursively as follows:
$$c_0 = \frac{a_0}{b_0} \quad , \quad b_0 c_n = a_n - \sum_{k=1}^{n} b_k\, c_{n-k}.$$
(b) Use Part (a) to find the first few coefficients of the expansion for $\tan z = \sin z/\cos z$.
3. (Cf. [**92**, p. 526] which is reproduced in [**127**]) In this and the next problem we give an elegant application of the theory of Bernoulli numbers to determine the sum of the first k-th powers of integers, Bernoulli's original motivation for his numbers.
   (i) For $n \in \mathbb{N}$, derive the formula
$$1 + e^z + e^{2z} + \cdots + e^{nz} = \frac{z}{e^z - 1} \cdot \frac{e^{(n+1)z} - 1}{z}.$$
   (ii) Writing each side of this identity as a power series (on the right, you need to use the Cauchy product), derive the formula
$$\boxed{1^k + 2^k + \cdots + n^k = \sum_{j=0}^{k} \binom{k}{j} B_j \, \frac{(n+1)^{k+1-j}}{k+1-j}, \quad k = 1, 2, \ldots.} \tag{5.57}$$
   Plug in $k = 1, 2, 3$ to derive some pretty formulas!
4. Here's another proof of (5.57) that is aesthetically appealing.
   (i) Prove that for a complex number a and natural numbers k, n,
$$(n + 1 + a)^{k+1} - (n + a)^{k+1} = \sum_{j=1}^{k+1} \binom{k+1}{j} n^{k+1-j} \Big((a + 1)^j - a^j \Big).$$
   (ii) Replacing a with B, prove that
$$\boxed{1^k + 2^k + \cdots + n^k \doteqdot \frac{1}{k+1} \big\{ (n + 1 + B)^{k+1} - B^{k+1} \big\}.}$$
   Suggestion: Look for a telescoping sum and recall that $(B + 1)^j \doteqdot B^j$ for $j \geq 2$.
5. The n-th Bernoulli polynomial $B_n(t)$ is by definition, $n!$ times the coefficient of z^n in the power series expansion in z of the function $f(z, t) := ze^{zt}/(e^z - 1)$; that is,
$$\boxed{\frac{z\, e^{zt}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!}\, z^n.} \tag{5.58}$$

(a) Prove that $B_n(t) = \sum_{k=0}^n \binom{n}{k} B_k\, t^{n-k}$ where the B_k's are the Bernoulli numbers. Thus, the first few Bernoulli polynomials are

$$B_0(t) = 1, \quad B_1(t) = t - \frac{1}{2}, \quad B_2(t) = t^2 - t + \frac{1}{6}, \quad B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t.$$

(b) Prove that $B_n(0) = B_n$ for $n = 0, 1, \ldots$ and that $B_n(0) = B_n(1) = B_n$ for $n \neq 1$. Suggestion: Show that $f(z,1) = z + f(z,0)$.
(c) Prove that $B_n(t+1) - B_n(t) = nt^{n-1}$ for $n = 0, 1, 2, \ldots$. Suggestion: Show that $f(z, t+1) - f(z,t) = ze^{zt}$.
(d) Prove that $B_{2n+1}(0) = 0$ for $n = 1, 2, \ldots$ and $B_{2n+1}(1/2) = 0$ for $n = 0, 1, \ldots$.

5.9. The logarithmic, binomial, arctangent series, and γ

From elementary calculus, you might have seen the logarithmic, binomial, and arctangent series (discovered by Nicolaus Mercator (1620–1687), Sir Isaac Newton (1643–1727), and Madhava of Sangamagramma (1350–1425), respectively):

$$\log(1+x) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} x^n \ , \ (1+x)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n} x^n \ , \ \arctan x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1}$$

where $\alpha \in \mathbb{R}$. (Below we'll discuss the meaning of $\binom{\alpha}{n}$.) I can bet that you used calculus (derivatives and integrals) to derive these formulæ. In this section we'll derive even more general complex versions of these formulæ without derivatives!

5.9.1. The binomial coefficients. From our familiar binomial theorem, we know that for any $z \in \mathbb{C}$ and $k \in \mathbb{N}$, we have $(1+z)^k = \sum_{n=0}^k \binom{k}{n} z^n$, where $\binom{k}{0} := 1$ and for $n = 1, 2, \ldots, k$,

$$\binom{k}{n} := \frac{k!}{n!(k-n)!} = \frac{1 \cdot 2 \cdots k}{n! \cdot 1 \cdot 2 \cdots (k-n)} = \frac{k(k-1)\cdots(k-n+1)}{n!}. \tag{5.59}$$

Thus,

$$(1+z)^k = 1 + \sum_{n=1}^k \frac{k(k-1)\cdots(k-n+1)}{n!} z^n.$$

With this motivation, given any complex number α, we define the **binomial coefficient** $\binom{\alpha}{n}$ for any nonnegative integer n as follows: $\binom{\alpha}{0} = 1$ and for $n > 0$,

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Note that if $\alpha = 0, 1, 2, \ldots$, then we see that all $\binom{\alpha}{n}$ vanish for $n \geq \alpha + 1$ and $\binom{\alpha}{n}$ is exactly the usual binomial coefficient (5.59). In the following lemma, we derive an identity that will be useful later.

Lemma 5.37. *For any $\alpha, \beta \in \mathbb{C}$, we have*

$$\binom{\alpha+\beta}{n} = \sum_{k=0}^n \binom{\alpha}{k}\binom{\beta}{n-k}, \qquad n = 0, 1, 2, \ldots.$$

Proof. Certainly this formula holds when $n = 0$ since in this case both sides equal 1. Assume that this formula holds for n, we shall prove it holds for $n+1$. The following formula will come in handy: For any $\gamma \in \mathbb{C}$ and $k \geq 0$,

$$\binom{\gamma}{k+1} = \frac{\gamma(\gamma-1)\cdots(\gamma-k+1)(\gamma-k)}{(k+1)!} = \binom{\gamma}{k}\frac{(\gamma-k)}{k+1}. \tag{5.60}$$

With $\gamma = \alpha + \beta$ and $k = n$, by induction hypothesis, we can write

$$\binom{\alpha+\beta}{n+1} = \binom{\alpha+\beta}{n}\frac{(\alpha+\beta-n)}{n+1} = \sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}\frac{(\alpha+\beta-n)}{n+1}.$$

Using (5.60) various times, we obtain

$$\begin{aligned}\binom{\alpha}{k}\binom{\beta}{n-k}\frac{(\alpha+\beta-n)}{n+1} &= \binom{\alpha}{k}\binom{\beta}{n-k}\frac{(\alpha-k+\beta-n+k)}{n+1}\\ &= \binom{\alpha}{k}\binom{\beta}{n-k}\frac{(\alpha-k)}{n+1} + \binom{\alpha}{k}\binom{\beta}{n-k}\frac{(\beta-n+k)}{n+1}\\ &= \binom{\alpha}{k+1}\binom{\beta}{n-k}\frac{k+1}{n+1} + \binom{\alpha}{k}\binom{\beta}{n+1-k}\frac{n+1-k}{n+1}\\ &= \binom{\alpha}{k+1}\binom{\beta}{n-k}\frac{k+1}{n+1} + \binom{\alpha}{k}\binom{\beta}{n+1-k} - \binom{\alpha}{k}\binom{\beta}{n+1-k}\frac{k}{n+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\binom{\alpha+\beta}{n+1} = \sum_{k=0}^{n}\binom{\alpha}{k+1}\binom{\beta}{n-k}\frac{k+1}{n+1} &+ \sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n+1-k}\\ &- \sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n+1-k}\frac{k}{n+1}\end{aligned}$$

since the $k = 0$ term in the last expression vanishes. It turns out that the first expression on the right has many terms that cancel with many terms in the last expression, since replacing $k+1$ with k, we get

$$\sum_{k=0}^{n}\binom{\alpha}{k+1}\binom{\beta}{n-k}\frac{k+1}{n+1} = \sum_{k=0}^{n+1}\binom{\alpha}{k}\binom{\beta}{n+1-k}\frac{k}{n+1}.$$

(We really start from $k = 1$ on the right, but the $k = 0$ vanishes anyways so we can start from $k = 0$.) Substituting this expression into the line above it and cancelling appropriate terms from the first and last expressions, we obtain

$$\binom{\alpha+\beta}{n+1} = \sum_{k=0}^{n+1}\binom{\alpha}{k}\binom{\beta}{n+1-k},$$

which completes our proof. □

5.9.2. The complex logarithm and binomial series. In Theorem 5.39 we shall derive (along with a power series for Log) the **binomial series**:

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty}\binom{\alpha}{n}z^n = 1 + \alpha\, z + \frac{\alpha(\alpha-1)}{1!}\, z^2 + \cdots, \quad |z| < 1. \tag{5.61}$$

Let us define $f(\alpha, z) := \sum_{n=0}^{\infty}\binom{\alpha}{n}z^n$. Our goal is to show that $f(\alpha, z) = (1+z)^{\alpha}$ for all $\alpha \in \mathbb{C}$ and $|z| < 1$, where

$$(1+z)^{\alpha} := \exp(\alpha\,\mathrm{Log}(1+z))$$

with Log the principal logarithm of the complex number $1+z$. If $\alpha = k = 0, 1, 2, \ldots$, then we already know that all the $\binom{k}{n}$ vanish for $n \geq k+1$ and these binomial coefficients are the usual ones, so $f(k, z)$ converges with sum $f(k, z) = (1+z)^k$. To see that $f(\alpha, z)$ converges for all other α, assume that $\alpha \in \mathbb{C}$ is not a nonnegative integer. Then setting $a_n = \binom{\alpha}{n}$, we have

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \cdot \frac{(n+1)!}{\alpha(\alpha-1)\cdots(\alpha-n)}\right| = \frac{n+1}{|\alpha - n|},$$

which approaches 1 as $n \to \infty$. Thus, the radius of convergence of $f(\alpha, z)$ is 1 (see (5.12)). In conclusion, $f(\alpha, z)$ is convergent for all $\alpha \in \mathbb{C}$ and $|z| < 1$.

We now prove the real versions of the logarithm series and the binomial series (5.61); see Theorem 5.39 below for the more general complex version. It is worth emphasizing that we do not use the advanced technology of the differential and integral calculus to derive these formulas! (In Sections 9.5 and 11.5 we'll develop this advanced technology to derive these formulas in the way they're usually derived.)

Lemma 5.38. *For all $x \in \mathbb{R}$ with $|x| < 1$, we have*

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, x^n$$

and for all $\alpha \in \mathbb{C}$ and $x \in \mathbb{R}$ with $|x| < 1$, we have

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \alpha\, x + \frac{\alpha(\alpha-1)}{1!}\, x^2 + \cdots.$$

Proof. We prove this lemma in three steps.

Step 1: We first show that $f(r, x) = (1+x)^r$ for all $r = p/q \in \mathbb{Q}$ where $p, q \in \mathbb{N}$ with q *odd* and $x \in \mathbb{R}$ with $|x| < 1$. To see this, observe for any $z \in \mathbb{C}$ with $|z| < 1$, taking the Cauchy product of $f(\alpha, z)$ and $f(\beta, z)$ and using our lemma, we obtain

$$f(\alpha, z) \cdot f(\beta, z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{\alpha}{j} \binom{\beta}{n-j} \right) z^n = \sum_{n=0}^{\infty} \binom{\alpha+\beta}{n} z^n = f(\alpha+\beta, z).$$

By induction it easily follows that

$$f(\alpha_1, z) \cdot f(\alpha_2, z) \cdots f(\alpha_k, z) = f(\alpha_1 + \alpha_2 + \cdots + \alpha_k, z).$$

Using this identity, we obtain

$$f(1/q, z)^q = \underbrace{f(1/q, z) \cdots f(1/q, z)}_{q \text{ times}} = f(\underbrace{1/q + \cdots + 1/q}_{q \text{ times}}, z) = f(1, z) = 1 + z.$$

Now put $z = x \in \mathbb{R}$ with $|x| < 1$ and let $q \in \mathbb{N}$ be odd. Then $f(1/q, x)^q = 1 + x$, so taking q-th roots, we get $f(1/q, x) = (1+x)^{1/q}$. Here we used that every real number has a unique q-th root, which holds because q is odd — for q even we could only conclude that $f(1/q, x) = \pm(1+x)^{1/q}$ (unless we checked that $f(1/q, x)$ is positive, then we would get $f(1/q, x) = (1+x)^{1/q}$). Therefore,

$$\begin{aligned} f(r, x) = f(p/q, x) = f(\underbrace{1/q + \cdots + 1/q}_{p \text{ times}}, x) &= \underbrace{f(1/q, x) \cdots f(1/q, x)}_{p \text{ times}} \\ &= f(1/q, x)^p = (1+x)^{p/q} = (1+x)^r. \end{aligned}$$

Step 2: Second, we prove that for any given $z \in \mathbb{C}$ with $|z| < 1$, $f(\alpha, z)$ can be written as a power series in $\alpha \in \mathbb{C}$:

$$f(\alpha, z) = 1 + \sum_{m=1}^{\infty} a_m(z)\, \alpha^m, \qquad \alpha \in \mathbb{C};$$

in particular, since we know that power series are continuous, $f(\alpha, z)$ is a continuous function of $\alpha \in \mathbb{C}$. Here, the coefficients $a_m(z)$ depend on z (which we'll see are power series in z) and we'll show that

$$a_1(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n \tag{5.62}$$

To prove these statements, note that for $n \geq 1$, $\alpha(\alpha-1)\cdots(\alpha-n+1)$ is a polynomial of degree n in α, so for $n \geq 1$ we can write

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \sum_{m=1}^{n} a_{mn}\, \alpha^m, \tag{5.63}$$

for some coefficients a_{mn}. Defining $a_{mn} = 0$ for $m = n+1, n+2, n+3, \ldots$, we can write $\binom{\alpha}{n} = \sum_{m=0}^{\infty} a_{mn}\, \alpha^m$. Hence,

$$f(z, \alpha) = 1 + \sum_{n=1}^{\infty} \binom{\alpha}{n} z^n = 1 + \sum_{n=1}^{\infty} \bigg(\sum_{m=1}^{\infty} a_{mn}\, \alpha^m \bigg) z^n. \tag{5.64}$$

To make this a power series in α, we need to switch the order of summation, which we can do by Cauchy's double series theorem if we can demonstrate absolute convergence by showing that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \big| a_{mn}\, \alpha^m z^n \big| = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a_{mn}|\, |\alpha|^m\, |z|^n < \infty.$$

To verify this, we first observe that for all $\alpha \in \mathbb{C}$, we have

$$\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{n!} = \sum_{m=1}^{n} b_{mn}\, \alpha^m, \tag{5.65}$$

where the b_{mn}'s are nonnegative real numbers. (This is certainly plausible because the numbers $1, 2, \ldots, n-1$ on the left each come with positive signs; any case, this statement can be verified by induction for instance.) We secondly observe that replacing α with $-\alpha$ in (5.63), we get

$$\begin{aligned} \sum_{m=1}^{n} a_{mn}\, (-1)^m \alpha^m &= \frac{-\alpha(-\alpha-1)\cdots(-\alpha-n+1)}{n!} \\ &= (-1)^n \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} = \sum_{m=1}^{n} (-1)^n b_{mn}\, \alpha^m. \end{aligned}$$

By the identity theorem, we have $a_{mn}(-1)^m = (-1)^n b_{mn}$. In particular, $|a_{mn}| = b_{mn}$ since $b_{mn} > 0$, therefore in view of (5.65), we see that

$$\sum_{m=0}^{\infty} |a_{mn}|\, |\alpha|^m = \sum_{m=0}^{n} |a_{mn}|\, |\alpha|^m = \sum_{m=0}^{n} b_{mn}\, |\alpha|^m = \frac{|\alpha|(|\alpha|+1)\cdots(|\alpha|+n-1)}{n!}.$$

Therefore,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a_{mn}|\, |\alpha|^m\, |z|^n = \sum_{n=1}^{\infty} \frac{|\alpha|(|\alpha|+1)\cdots(|\alpha|+n-1)}{n!} |z|^n.$$

Using the now very familiar ratio test it's easily checked that, since $|z| < 1$, the series on the right converges. Thus, we can iterate sums in (5.64) and conclude that

$$f(\alpha, z) = 1 + \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{mn}\, \alpha^m \right) z^n = 1 + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{mn}\, z^n \right) \alpha^m.$$

Thus, $f(\alpha, z)$ is indeed a power series in α. To prove (5.62), we just need to determine the coefficients of α^1 in (5.63), which we see is just

$$\begin{aligned} a_{1n} &= \text{coefficient of } \alpha \text{ in } \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} \\ &= \frac{(-1)(-2)(-3)\cdots(-n+1)}{n!} = (-1)^{n-1}\frac{(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}. \end{aligned}$$

Therefore,

$$a_1(z) = \sum_{n=1}^{\infty} a_{1n}\, z^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n,$$

just as we stated in (5.62). This completes **Step 2**.

Step 3: We are finally ready to prove our theorem. Let $x \in \mathbb{R}$ with $|x| < 1$. By **Step 2**, we know that for any $\alpha \in \mathbb{C}$,

$$f(\alpha, x) = 1 + \sum_{m=1}^{\infty} a_m(x)\, \alpha^m$$

is a power series in α. However,

$$(1+x)^\alpha = \exp(\alpha \log(1+x)) = \sum_{n=0}^{\infty} \frac{1}{n!} \log(1+x)^n \cdot \alpha^n$$

is also a power series in $\alpha \in \mathbb{C}$. By **Step 1**, $f(\alpha, x) = (1+x)^\alpha$ for all $\alpha \in \mathbb{Q}$ with $\alpha > 0$ having odd denominators. The identity theorem applies to this situation (why?), so we must have $f(\alpha, x) = (1+x)^\alpha$ for all $\alpha \in \mathbb{C}$. Also by the identity theorem, the coefficients of α^n must be identical; in particular, the coefficients of α^1 are identical: $a_1(x) = \log(1+x)$. Now (5.62) implies the series for $\log(1+x)$. $\square$

Using this lemma and the identity theorem, we are ready to generalize these formulas for real x to formulas for complex z.

THEOREM 5.39 (**The complex logarithm and binomial series**). *We have*

$$\boxed{\operatorname{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n, \quad |z| \le 1,\ z \neq -1,}$$

and given any $\alpha \in \mathbb{C}$, we have

$$\boxed{(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n = 1 + \alpha\, z + \frac{\alpha(\alpha-1)}{1!}\, z^2 + \cdots, \quad |z| < 1.}$$

PROOF. We prove this theorem first for $\operatorname{Log}(1+z)$, then for $(1+z)^\alpha$.

Step 1: Let us define $f(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$. Then one can check that the radius of convergence of $f(z)$ is 1, so by our power series composition theorem, we know that $\exp(f(z))$ can be written as a power series:

$$\exp(f(z)) = \sum_{n=0}^{\infty} a_n z^n, \qquad |z| < 1.$$

Restricting to real values of z, by our lemma we know that $f(x) = \log(1+x)$, so

$$\sum_{n=0}^{\infty} a_n x^n = \exp(f(x)) = \exp(\log(1+x)) = 1 + x.$$

By the identity theorem for power series, we must have $a_0 = 1, a_1 = 1$, and all other $a_n = 0$. Thus, $\exp(f(z)) = 1 + z$. Since $\exp(\mathrm{Log}(1+z)) = 1 + z$ as well, we have

$$\exp(f(z)) = \exp(\mathrm{Log}(1+z)),$$

which implies that $f(z) = \mathrm{Log}(1+z) + 2\pi i k$ for some integer k. Setting $z = 0$ shows that $k = 0$ and hence proves that $\mathrm{Log}(1+z) = f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$.

We now prove that $\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$ holds for $|z| = 1$ with $z \neq -1$ (note that for $z = -1$, both sides of this equality are not defined). If $|z| = 1$, then we can write $z = -e^{ix}$ with $x \in (0, 2\pi)$. Recall from Example 5.4 in Section 5.1 that for any $x \in (0, 2\pi)$, the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ and $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converge. Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\cos nx}{n} + i \sum_{n=1}^{\infty} \frac{\sin nx}{n} &= \sum_{n=1}^{\infty} \frac{e^{inx}}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n(-e^{ix})^n}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n, \end{aligned} \tag{5.66}$$

which shows that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$ converges for $|z| = 1$ with $z \neq -1$. Now fix a point z_0 with $|z_0| = 1$ and $z_0 \neq -1$, and let us take $z \to z_0$ through the straight line from $z = 0$ to $z = z_0$ (that is, let $z = tz_0$ where $0 \leq t \leq 1$ and take $t \to 1-$). Since the ratio

$$\frac{|z_0 - z|}{1 - |z|} = \frac{|z_0 - tz_0|}{1 - |tz_0|} = \frac{|z_0 - tz_0|}{1 - t} = \frac{|1 - t|}{1 - t} = 1,$$

which bounded by a fixed constant, by Abel's theorem (Theorem 5.20), it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z_0^n = \lim_{z \to z_0} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n = \lim_{z \to z_0} \mathrm{Log}(1+z) = \mathrm{Log}(1+z_0),$$

where we used that $\mathrm{Log}(1+z)$ is continuous.

Step 2: Let $\alpha \in \mathbb{C}$. To prove the binomial series, we note that by the power series composition theorem, $(1+z)^{\alpha} = \exp(\alpha \, \mathrm{Log}(1+z))$, being the composition of exp and Log, can be written as a power series:

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} b_n z^n, \qquad |z| < 1.$$

Restricting to real $z = x \in \mathbb{R}$ with $|x| < 1$, by our lemma we know that $(1+x)^\alpha = f(\alpha, x)$. Hence, by the identity theorem, we must have $(1+z)^\alpha = f(\alpha, z)$ for all $z \in \mathbb{C}$ with $|z| < 1$. This proves the binomial series. $\square$

For any $z \in \mathbb{C}$ with $|z| < 1$, we have $\operatorname{Log}\big((1+z)/(1-z)\big) = \operatorname{Log}(1+z) - \operatorname{Log}(1-z)$. Therefore, we can use this theorem to prove that (see Problem 1)

$$\boxed{\frac{1}{2}\operatorname{Log}\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}.} \tag{5.67}$$

Here's another consequence of Theorem 5.39.

Example 5.39. In the proof of Theorem 5.39 we used that, for $x \in (0, 2\pi)$, the series $\sum_{n=1}^\infty \frac{\cos nx}{n}$ and $\sum_{n=1}^\infty \frac{\sin nx}{n}$ converge. We shall now determine the sum of these series! In fact, we shall prove that

$$\boxed{\sum_{n=1}^{\infty} \frac{\cos nx}{n} = \log\big(2\sin(x/2)\big) \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{x-\pi}{2}.}$$

To see this, recall from (5.66) that, with $z = -e^{ix}$, we have

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} + i\sum_{n=1}^{\infty} \frac{\sin nx}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n = -\operatorname{Log}(1-z) = -\operatorname{Log}(1-e^{ix}).$$

We can write

$$1 - e^{ix} = e^{ix/2}(e^{-ix/2} - e^{ix/2}) = -2ie^{ix/2}\sin(x/2) = 2\sin(x/2)e^{ix/2 - i\pi/2}.$$

Hence, by definition of Log, we have

$$\operatorname{Log}(1-e^{ix}) = \log\big(2\sin(x/2)\big) + i\frac{x-\pi}{2}.$$

This proves our result.

5.9.3. Gregory-Madhava series and formulæ for γ. Recall from Section 4.9 that

$$\operatorname{Arctan} z = \frac{1}{2i}\operatorname{Log}\frac{1+iz}{1-iz}.$$

Using (5.67), we get the famous formula first discovered by Madhava of Sangamagramma (1350–1425) around 1400 and rediscovered over 200 years later in Europe by James Gregory (1638–1675), who found it in 1671! In fact, the mathematicians of Kerala in southern India not only discovered the arctangent series, they also discoved the infinite series for sine and cosine, but their results were written up in Sanskrit and not brought to Europe until the 1800's. For more history on this fascinating topic, see the articles [**86**], [**145**], and the website [**132**].

Theorem 5.40. *For any complex number z with $|z| < 1$, we have*

$$\boxed{\operatorname{Arctan} z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}, \qquad \textit{Gregory-Madhava's series}.}$$

This series is commonly known as **Gregory's arctangent series**, but we shall call it the **Gregory-Madhava arctangent series** because of Madhava's contribution to this series. Setting $z = x$, a real variable, we obtain the usual formula learned in elementary calculus:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

In Problem 5 we prove the following stunning formulæ for the Euler-Mascheroni constant γ in terms of the Riemann ζ-function $\zeta(z)$:

$$\boxed{\begin{aligned}\gamma &= \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) \\ &= 1 - \sum_{n=2}^{\infty} \frac{1}{n} \big(\zeta(n) - 1\big) \\ &= \frac{3}{2} - \log 2 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} (n-1)\big(\zeta(n) - 1\big).\end{aligned}} \tag{5.68}$$

The first two formulas are due to Euler and the last one to Philippe Flajolet and Ilan Vardi (see [**148**, pp. 4,5], [**58**]). .

Exercises 5.9.

1. Fill in the details in the proof of formula (5.67).
2. Derive the remarkably pretty formulas:

$$\boxed{2(\operatorname{Arctan} z)^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+2} \Big(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}\Big) z^{2n+2},}$$

and the formula

$$\boxed{\frac{1}{2}(\operatorname{Log}(1+z))^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2} \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\Big) z^{n+2},}$$

both valid for $|z| < 1$.

3. Before looking at the next section, prove that

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{and} \quad \log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

are valid for $-1 < x \le 1$. Suggestion: I know you are Abel to do this! From these facts, derive the formulas

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots \quad \text{and} \quad \log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

4. For $\alpha \in \mathbb{R}$, prove that $\sum_{n=0}^{\infty} \binom{\alpha}{n}$ converges if and only if $\alpha \le 0$ or $\alpha \in \mathbb{N}$, in which case,

$$2^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n}.$$

Suggestion: To prove convergence use Gauss' test.

5. Prove the exquisite formulas

$$(a) \ \sum_{n=1}^{\infty} \frac{1}{n} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} \operatorname{Log} \frac{1}{1-z^n}, \qquad |z| < 1,$$

$$(b)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} \mathrm{Log}(1+z^n), \qquad |z|<1.$$

Suggestion: Cauchy's double series theorem.

6. In this problem, we prove the stunning formulæ in (5.68).
   (i) Using the first formula for γ in Problem 7a of Exercises 4.6, prove that $\gamma = \sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$ where $f(z) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} z^n$.
   (ii) Prove that $\gamma = 1 - \log 2 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}(\zeta(n) - 1)$ using (i) and Problem 8 in Exercises 5.5. Show that this formula is equivalent to the first formula in (5.68).
   (iii) Using the second and third formulas in Problem 7a of Exercises 4.6, derive the second and third formulas in (5.68).

5.10. ★ π, Euler, Fibonacci, Leibniz, Madhava, and Machin

In this section, we continue our fascinating study of formulas for π that we initiated in Section 4.10. In particular, we derive Leibniz-Madhava's formula for $\pi/4$, formulas for π discovered by Euler involving the arctangent function and even the Fibonacci numbers, and finally, we look at Machin's formula for π, versions of which has been used to compute trillions of digits of π by Yasumasa Kanada and his coworkers at the University of Tokyo.[5] For other formulas for $\pi/4$ in terms of arctangents, see the articles [**102, 75**]. For more on the history of computations of π, see [**10**], and for interesting historical facets on π in general, see [**16**], [**37, 38**]. The website [**152**] has tons of information.

5.10.1. Leibniz-Madhava's formula for π.

Recall Gregory-Madhava's formula for real values:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}.$$

By the alternating series theorem, we know that $\sum_{n=0}^{\infty}(-1)^{n-1}/(2n-1)$ converges, therefore by Abel's limit theorem (Theorem 5.20) we know that

$$\frac{\pi}{4} = \lim_{x\to 1-} \arctan x = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{1}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots .$$

Therefore,

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots, \qquad \textbf{Gregory-Leibniz-Madhava's series}.}$$

This formula is usually called **Leibniz' series** after Leibniz (1646–1716) because he is usually accredited to be the first to mention this formula in print in 1673, although Madhava of Sangamagramma (1350–1425) discovered this formula over 200 years earlier. Note that the Gregory-Leibniz-Madhava's series is really just a special case of Gregory-Madhava's formula, which was discovered in 1671 by Gregory (and earlier by Madhava), Gregory seems to have not noticed this beautiful expression for $\pi/4$; however, see [**16**, p. 133]. For more history, including Nilakantha Somayaji's (1444–1544) contribution, see [**145, 132**].

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[5] *The value of π has engaged the attention of many mathematicians and calculators from the time of Archimedes to the present day, and has been computed from so many different formulae, that a complete account of its calculation would almost amount to a history of mathematics. James Glaisher (1848–1928)* [**64**].

Example 5.40. Let us say that we want to approximate $\pi/4$ by Gregory-Leibniz-Madhava's series to within, say a reasonable amount of 7 decimal places. Then denoting the n-th partial sum of Gregory-Leibniz-Madhava's series by s_n, according to the alternating series error estimate, we want

$$\left|\frac{\pi}{4} - s_n\right| \leq \frac{1}{2n+1} < 0.00000005 = 5 \times 10^{-8},$$

which implies that $2n + 1 > 10^8/5$, which works for $n \geq 10,000,000$. Thus, we can approximate $\pi/4$ by the n-th partial sum of Gregory-Leibniz-Madhava's series by taking *ten million* terms! Thus, although Gregory-Leibniz-Madhava's series is beautiful, it is quite useless to compute π.

Example 5.41. From Gregory-Leibniz-Madhava's formula, we can easily derive the pretty formula (see Problem 4)

$$\boxed{\pi = \sum_{n=2}^{\infty} \frac{3^n - 1}{4^n}\, \zeta(n+1),} \tag{5.69}$$

due to Philippe Flajolet and Ilan Vardi (see [**149**, p. 1], [**176, 58**]).

5.10.2. Euler's arctangent formula and the Fibonacci numbers. In 1738, Euler derived a very pretty two-angle arctangent expression for π:

$$\boxed{\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}.} \tag{5.70}$$

This formula is very easy to derive. We start off with the addition formula for tangent (see (4.34), but now considering real variables)

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\,\tan\phi}, \tag{5.71}$$

where it is assumed that $1 - \tan\theta\tan\phi \neq 0$. Let $x = \tan\theta$ and $y = \tan\phi$ and assume that $-\pi/2 < \theta + \phi < \pi/2$. Then taking arctangents of both sides of the above equation, we obtain

$$\theta + \phi = \arctan\left(\frac{x+y}{1-xy}\right),$$

or after putting the left-hand in terms of x, y, we get

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right). \tag{5.72}$$

Setting $x = 1/2$ and $y = 1/3$, we see that

$$\frac{x+y}{1-xy} = \frac{5/6}{1-5/6} = 1,$$

which implies that

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan 1.$$

This expression is just (5.70).

In Problem 7 of Exercises 2.2 we studied the **Fibonacci sequence**, named after Leonardo Fibonacci (1170–1250): $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ and you proved that for every natural number,

$$F_n = \frac{1}{\sqrt{5}}\Big[\Phi^n - (-\Phi)^{-n}\Big], \qquad \Phi = \frac{1+\sqrt{5}}{2}. \tag{5.73}$$

We can use (5.70) and (5.72) to derive the following fascinating formula for $\pi/4$ in terms of the (odd-indexed) Fibonacci numbers due to Lehmer [**101**] (see Problem 2 and [**103**]):

$$\boxed{\frac{\pi}{4} = \sum_{n=0}^{\infty} \arctan\left(\frac{1}{F_{2n+1}}\right).} \tag{5.74}$$

Also, in Problem 3 you will prove the following series for π, due to Castellanos [**37**]:

$$\boxed{\frac{\pi}{\sqrt{5}} = \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1} 2^{2n+3}}{(2n+1)(3+\sqrt{5})^{2n+1}}.} \tag{5.75}$$

5.10.3. Machin's arctangent formula for π. In 1706, John Machin (1680–1752) derived a fairly rapid convergent series for π. To derive this expansion, consider the smallest positive angle α whose tangent is $1/5$:

$$\tan\alpha = \frac{1}{5} \quad \text{(that is, } \alpha := \arctan(1/5)).$$

Now setting $\theta = \phi = \alpha$ in (5.71), we obtain

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2/5}{1-1/25} = \frac{5}{12},$$

so

$$\tan 4\alpha = \frac{2\tan 2\alpha}{1-\tan^2 2\alpha} = \frac{5/6}{1-25/144} = \frac{120}{119},$$

which is just slightly above one. Hence, $4\alpha - \pi/4$ is positive, and moreover,

$$\tan\left(4\alpha - \frac{\pi}{4}\right) = \frac{\tan 4\alpha + \tan\pi/4}{1+\tan 4\alpha\,\tan\pi/4} = \frac{1/119}{1+120/119} = \frac{1}{239}.$$

Taking the inverse tangent of both sides and solving for $\frac{\pi}{4}$, we get

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

Substituting $1/5$ and $1/239$ into the Gregory-Madhava series for the inverse tangent, we arrive at Machin's formula for π:

THEOREM 5.41 (**Machin's formula**). *We have*

$$\boxed{\pi = 16\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n+1}} - 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,239^{2n+1}}.}$$

Example 5.42. Machin's formula gives many decimal places of π without much effort. Let s_n denote the n-th partial sum of $s := 16\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)5^{2n+1}}$ and t_n that of $t := 4\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)\,239^{2n+1}}$. Then $\pi = s - t$ and by the alternating series error estimate,

$$|s - s_3| \le \frac{16}{9\cdot 5^9} \approx 9.102\times 10^{-7}$$

and

$$|t - t_0| \le \frac{4}{3\cdot(239)^3} \approx 10^{-7}.$$

Therefore,

$$|\pi - (s_3 - t_0)| = |(s - t) - (s_3 - t_0)| \le |s - s_3| + |t - t_0| < 5\times 10^{-6}.$$

A manageable computation (even without a calculator!) shows that $s_3 - t_0 = 3.14159\ldots$. Therefore, $\pi = 3.14159$ to five decimal places!

Exercises 5.10.

1. From Gregory-Madhava's series, derive the following pretty series

$$\boxed{\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - + \cdots.}$$

How many terms of this series do you need to approximate $\pi/2\sqrt{3}$ to within seven decimal places? Suggestion: Consider $\arctan(1/\sqrt{3}) = \pi/6$. *History Bite*: Abraham Sharp (1651–1742) used this formula in 1669 to compute π to 72 decimal places, and Thomas Fantet de Lagny (1660–1734) used this formula in 1717 to compute π to 126 decimal places (with a mistake in the 113-th place) [**37**].

2. In this problem we prove (5.74).
   (i) Prove that $\arctan\frac{1}{3} = \arctan\frac{1}{5} + \arctan\frac{1}{8}$, and use this prove that
$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8}.$$
   Prove that $\arctan\frac{1}{8} = \arctan\frac{1}{13} + \arctan\frac{1}{21}$, and use this prove that
$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{13} + \arctan\frac{1}{21}.$$
   From here you can now see the appearance of Fibonacci numbers.
   (ii) To continue this by induction, prove that for every natural number n,
$$F_{2n} = \frac{F_{2n+1}F_{2n+2} - 1}{F_{2n+3}}.$$
   Suggestion: Can you use (5.73)?
   (iii) Using the formula in (b), prove that
$$\arctan\left(\frac{1}{F_{2n}}\right) = \arctan\left(\frac{1}{F_{2n+1}}\right) + \arctan\left(\frac{1}{F_{2n+2}}\right).$$
   Using this formula derive (5.74).

3. In this problem we prove (5.75).
   (i) Using (5.72), prove that
$$\tan^{-1}\frac{\sqrt{5}\,x}{1 - x^2} = \tan^{-1}\left(\frac{1+\sqrt{5}}{2}\right)x - \tan^{-1}\left(\frac{1-\sqrt{5}}{2}\right)x.$$
   (ii) Now prove that
$$\tan^{-1}\frac{\sqrt{5}\,x}{1 - x^2} = \sum_{n=0}^\infty \frac{(-1)^n F_{2n+1}\,x^{2n+1}}{5^n\,(2n+1)}.$$

(iii) Finally, derive the formula (5.75).

4. In this problem, we prove the breath-taking formula (5.69).
   (i) Prove that

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \Big(\frac{1}{4n-3} - \frac{1}{4n-1} \Big) = \sum_{n=1}^{\infty} f\Big(\frac{1}{n}\Big)$$

   where $f(z) = \frac{z}{4-3z} - \frac{z}{4-z}$.

   (ii) Use Theorem 5.29 to derive our breath-taking formula.

5.11. ★ Proofs that $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$ (The Basel problem)

Pietro Mengoli (1625–1686) posed the question: What's the value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \,?$$

This problem was made popular through Jacob (Jacques) Bernoulli (1654–1705) when he wrote about it in 1689 and was solved by Leonhard Euler (1707–1783) in 1734. Bernoulli wrote

> *If somebody should succeed in finding what till now withstood our efforts and communicate it to us, we shall be much obliged to him.* [**37**, p. 73].

Before Euler's solution to this request, known as the *Basel problem* (Bernoulli lived in Basel, Switzerland), this problem eluded many of the great mathematicians of that day. Needless to say, it shocked the mathematical community when Euler found the sum to be $\pi^2/6$. (You'll have to wait until Section 6.6 to see Euler's original proof.) In this section, we present two "elementary" proofs, where "elementary" means that we use nothing involving calculus or beyond. The first proof is of historical interest, but long, and goes back to Nicolaus Bernoulli (1687–1759) (Jacob Bernoulli's nephew) from 1738, and the other proof is more recent and is truly very elementary, and short, which we found in an article by Hofbauer [**79**]. For more on various solutions to the Basel problem, see [**84**] and [**39**] and for more on Euler, see [**8**], [**91**].

5.11.1. Cauchy's arithmetic mean theorem.

5.11.1. Cauchy's arithmetic mean theorem. Before giving our first proof of Euler's sum, we prove the following theorem.

THEOREM 5.42 (**Cauchy's arithmetic mean theorem**). *If a sequence a_1, a_2, a_3, … converges to L, then the sequence of arithmetic means (or averages)*

$$m_n := \frac{1}{n}\Big(a_1 + a_2 + \cdots + a_n\Big)$$

also converges to L. Moreover, if the sequence $\{a_n\}$ is nonincreasing, then so is its sequence of arithmetic means $\{m_n\}$.

PROOF. To show that $m_n \to L$, we need to show that

$$m_n - L = \frac{1}{n}\Big((a_1 - L) + (a_2 - L) + \cdots + (a_n - L)\Big)$$

tends to zero as $n \to \infty$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that for all $n > N$, we have $|a_n| < \varepsilon/2$. Then for $n > N$, we can write

$$\begin{aligned}|m_n| &\le \frac{1}{n}\Big(|(a_1 - L) + \cdots + (a_N - L)|\Big) + \frac{1}{n}\Big(|(a_{N+1} - L) + \cdots + (a_n - L)|\Big)\\ &\le \frac{1}{n}\Big(|(a_1 - L) + \cdots + (a_N - L)|\Big) + \frac{1}{n}\Big(\frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2}\Big)\\ &= \frac{1}{n}\Big(|(a_1 - L) + \cdots + (a_N - L)|\Big) + \frac{n-N}{n}\cdot\frac{\varepsilon}{2}\\ &\le \frac{1}{n}\Big(|a_1 + \cdots + a_N|\Big) + \frac{\varepsilon}{2}.\end{aligned}$$

By choosing n larger, we can make $\frac{1}{n}\Big(|a_1 + \cdots + a_N|\Big)$ also less than $\varepsilon/2$, which shows that $|m_n| < \varepsilon$ for n sufficiently large. This shows that $m_n \to 0$.

Assume now that $\{a_n\}$ is nonincreasing. We shall prove that $\{m_n\}$ is also nonincreasing; that is, for each n,

$$\frac{1}{n+1}\Big(a_1 + \cdots + a_n + a_{n+1}\Big) \le \frac{1}{n}\Big(a_1 + \cdots + a_n\Big),$$

or, after multiplying both sides by $n(n+1)$,

$$n\Big(a_1 + \cdots + a_n\Big) + na_{n+1} \le n\Big(a_1 + \cdots + a_n\Big) + \Big(a_1 + \cdots + a_n\Big).$$

Cancelling, we conclude that the sequence $\{m_n\}$ is nonincreasing if and only if

$$na_{n+1} = \underbrace{a_{n+1} + a_{n+1} + \cdots a_{n+1}}_{n \text{ times}} \le a_1 + a_2 + \cdots + a_n.$$

But this inequality certainly holds since $a_{n+1} \le a_k$ for $k = 1, 2, \ldots, n$. This completes the proof. □

There is a related theorem for geometric means found in Problem 2, which can be used to derive the following neat formula:

$$\boxed{e = \lim_{n\to\infty}\left\{\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n+1}{n}\right)^n\right\}^{1/n}.} \tag{5.76}$$

5.11.2. Proof I of Euler's formula for $\pi^2/6$. Assuming only Gregory-Leibniz-Madhava's series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$$

we give our first of many proofs of the fact that

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.}$$

According to Knopp [**92**, p. 324], the proof we are about to give "may be regarded as the most elementary of all known proofs, since it borrows nothing from the theory of functions except the Leibniz series". Knopp attributes the main ideas of the proof to Nicolaus Bernoulli (1687–1759). First we shall apply Abel's multiplication theorem to Gregory-Leibniz-Madhava's series

$$\left(\frac{\pi}{4}\right)^2 = \left(\sum_{n=0}^\infty (-1)^n \frac{1}{2n+1}\right) \cdot \left(\sum_{n=0}^\infty (-1)^n \frac{1}{2n+1}\right).$$

To do so, we first form the n-th term in the Cauchy product:

$$c_n = \sum_{k=0}^n (-1)^k \frac{1}{2k+1} \cdot (-1)^{n-k} \frac{1}{2n-2k+1} = (-1)^n \sum_{k=0}^n \frac{1}{(2k+1)(2n-2k+1)}.$$

Observe that

$$\frac{1}{(2k+1)(2n-2k+1)} = \frac{1}{2(n+1)} \left(\frac{1}{2k+1} + \frac{1}{2n-2k+1}\right),$$

which implies that

$$c_n = \frac{(-1)^n}{2(n+1)} \left(\sum_{k=0}^n \frac{1}{2k+1} + \sum_{k=0}^n \frac{1}{2n-2k+1}\right) = \frac{(-1)^n}{n+1} \sum_{k=0}^n \frac{1}{2k+1}.$$

Thus, the Cauchy product of Gregory-Leibniz-Madhava's series with itself is

$$\sum_{n=0}^\infty (-1)^n \frac{1}{n+1} \left(1 + \frac{1}{3} + \cdots + \frac{1}{2n+1}\right),$$

provided that this series converges. To see that this series converges, note that the sum

$$\frac{1}{n+1} \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} + \frac{1}{2n+1}\right)$$

is just the arithmetic mean, or average, of the numbers $1, 1/3, \ldots, 1/(2n+1)$. Since $1/(2n+1) \to 0$ monotonically, Cauchy's arithmetic mean theorem shows that these averages also tend to zero monotonically. In particular, by the alternating series theorem, the Cauchy product of Gregory-Leibniz-Madhava's series converges, so by Abel's multiplication theorem, we get the formula

$$\left(\frac{\pi}{4}\right)^2 = \sum_{n=0}^\infty (-1)^n \frac{1}{n+1} \left(1 + \frac{1}{3} + \cdots + \frac{1}{2n+1}\right). \tag{5.77}$$

We evaluate the right-hand side using the following technical theorem.

THEOREM 5.43. *Let $\{a_n\}$ be a nonincreasing sequence of positive numbers such that $\sum a_n^2$ converges. Then both series*

$$s := \sum_{n=0}^\infty (-1)^n a_n \qquad \text{and} \qquad \delta_k := \sum_{n=0}^\infty a_n a_{n+k}, \quad k = 1, 2, 3, \ldots$$

converge. Moreover, $\Delta := \sum_{k=1}^\infty (-1)^{k-1} \delta_k$ also converges, and we have the formula

$$\sum_{n=0}^\infty a_n^2 = s^2 + 2\Delta.$$

PROOF. Since $\sum a_n^2$ converges, we must have $a_n \to 0$, which implies that $\sum (-1)^n a_n$ converges by the alternating series test. By monotonicity, $a_n a_{n+k} \leq$

$a_n \cdot a_n = a_n^2$ and since $\sum a_n^2$ converges, by comparison, so does each series $\delta_k = \sum_{n=0}^\infty a_n a_{n+k}$. Also by monotonicity,

$$\delta_{k+1} = \sum_{n=0}^\infty a_n a_{n+k+1} \le \sum_{n=0}^\infty a_n a_{n+k} = \delta_k,$$

so by the alternating series test, the sum Δ converges if $\delta_k \to 0$. To prove this, let $\varepsilon > 0$ and choose N such that $a_{N+1}^2 + a_{N+2}^2 + \cdots < \varepsilon/2$. Then, since the sequence $\{a_n\}$ is nondecreasing, we can write

$$\begin{aligned}\delta_k &= \sum_{n=0}^\infty a_n a_{n+k}\\ &= \Big(a_0 a_k + \cdots + a_N a_{N+k}\Big) + \Big(a_{N+1} a_{N+1+k} + a_{N+2} a_{N+2+k} + \cdots\Big)\\ &\le \Big(a_0 a_k + \cdots + a_N a_k\Big) + \Big(a_{N+1}^2 + a_{N+2}^2 + a_{N+3}^2 + \cdots\Big)\\ &< a_k \cdot \Big(a_0 + \cdots + a_N\Big) + \frac{\varepsilon}{2}.\end{aligned}$$

As $a_k \to 0$ we can make the first term less than $\varepsilon/2$ for all k large enough. Thus, $\delta_k < \varepsilon$ for all k sufficiently large. This proves that $\Delta = \sum_{k=1}^\infty (-1)^{k-1}\delta_k$ converges. Finally, we need to prove the equality

$$\sum_{n=0}^\infty a_n^2 = s^2 + 2\sum_{k=1}^\infty (-1)^{k-1}\delta_k = s^2 + 2\big[\delta_1 - \delta_2 + \delta_3 - + \cdots\big].$$

To prove this, let s_n denote the n-th partial sum of the series $s = \sum_{n=0}^\infty (-1)^n a_n$. We have

$$s_n^2 = \left(\sum_{k=0}^n (-1)^k a_k\right)^2 = \sum_{k=0}^n \sum_{\ell=0}^n (-1)^{k+\ell} a_k\, a_\ell.$$

Consider the following array (called a **matrix**) of numbers:

$$\begin{bmatrix} +a_0^2 & -a_0a_1 & +a_0a_2 & -a_0a_3 & \cdots & (-1)^n a_0 a_n \\ -a_1a_0 & +a_1^2 & -a_1a_2 & +a_1a_3 & \cdots & -(-1)^n a_1 a_n \\ +a_2a_0 & -a_2a_1 & +a_2^2 & -a_2a_3 & \cdots & (-1)^n a_2 a_n \\ & & & \vdots & & \\ (-1)^n a_n a_0 & -(-1)^n a_n a_1 & (-1)^n a_n a_2 & -(-1)^n a_n a_3 & \cdots & a_n^2 \end{bmatrix}.$$

Then s_n^2 is the sum of all the terms in this array. We can sum diagonally along every diagonal stretching from the north-west to south-east, obtaining

$$\begin{aligned}s_n^2 = \sum_{k=0}^n a_k^2 &- 2\big[a_0a_1 + a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n\big]\\ &+ 2\big[a_0a_2 + a_1a_3 + \cdots + a_{n-2}a_n\big]\\ &\qquad - 2\big[a_0a_3 + a_1a_4 + \cdots + a_{n-3}a_n\big] + - \cdots + 2(-1)^n a_0 a_n,\end{aligned}$$

where we used that the matrix is symmetric about the main diagonal. Let

$$d_n := 2\big[\delta_1 - \delta_2 + \delta_3 - + \cdots + (-1)^{n+1}\delta_n\big].$$

We need to show that $s_n^2 + d_n \to \sum_{k=0}^\infty a_k^2$ as $n \to \infty$. To prove this, we add the expressions for s_n^2 and d_n given in the previous two displayed equations, recalling that $\delta_k = a_0 a_k + a_1 a_{k+1} + a_2 a_{k+2} + \cdots$, to obtain

$$\begin{aligned} s_n^2 + d_n = \sum_{k=0}^{n} a_k^2 &+ 2\big[a_n a_{n+1} + a_{n+1} a_{n+2} + a_{n+2} a_{n+3} + \cdots\big] \\ &- 2\big[a_{n-1} a_{n+1} + a_n a_{n+2} + \cdots\big] \\ &+ 2\big[a_{n-2} a_{n+1} + a_{n-1} a_{n+2} + \cdots\big] \\ &- + \cdots + 2(-1)^{n+1}\big[a_1 a_{n+1} + a_2 a_{n+2} + \cdots\big]. \end{aligned}$$

With $\alpha_k := a_k a_{n+1} + a_{k+1} a_{n+2} + a_{k+2} a_{n+3} + \cdots$, this expression takes the form

$$s_n^2 + d_n - \sum_{k=0}^{n} a_k^2 = 2(-1)^{n+1}\big[\alpha_1 - \alpha_2 + \alpha_3 - + \cdots + (-1)^{n+1}\alpha_n\big].$$

Since the sequence $\{a_n\}$ is nonincreasing, it follows that the sequence $\{\alpha_k\}$ is also nonincreasing:

$$\alpha_k = a_k a_{n+1} + a_{k+1} a_{n+2} + \cdots \geq a_{k+1} a_{n+1} + a_{k+2} a_{n+2} + \cdots = \alpha_{k+1}.$$

Now assuming n is even, we have

$$\begin{aligned} \left| s_n^2 + d_n - \sum_{k=0}^{n} a_k^2 \right| &= 2\big|(\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4) + \cdots + (\alpha_{n-1} - \alpha_n)\big| \\ &= (\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4) + \cdots + (\alpha_{n-1} - \alpha_n) \\ &= \alpha_1 - (\alpha_2 - \alpha_3) - (\alpha_4 - \alpha_5) - \cdots - (\alpha_{n-2} - \alpha_{n-1}) - \alpha_n \\ &\leq \alpha_1 - \alpha_n \leq \alpha_1 = a_1 a_{n+1} + a_2 a_{n+2} + \cdots = \delta_n - a_0 a_n, \end{aligned}$$

where we used the fact that the terms in the parentheses are all nonnegative because the α_k's are nonincreasing. Using a very similar argument, we get

$$\left| s_n^2 + d_n - \sum_{k=0}^{n} a_k^2 \right| \leq \delta_n - a_0 a_n \tag{5.78}$$

for n odd. Therefore, (5.78) holds for all n. We already know that $\delta_n \to 0$ and $a_n \to 0$, so (5.78) shows that the left-hand side tends to zero as $n \to \infty$. This completes the proof of the theorem. □

Finally, we are ready to prove Euler's formula for $\pi^2/6$. To do so, we apply the preceding theorem to the sequence $a_n = 1/(2n+1)$. In this case,

$$\delta_k = \sum_{n=0}^{\infty} a_n a_{n+k} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2k+1)}.$$

Observe that

$$\frac{1}{(2n+1)(2n+2k+1)} = \frac{1}{2k}\left\{\frac{1}{2n+1} - \frac{1}{2n+2k+1}\right\},$$

so

$$\delta_k = \frac{1}{2k}\sum_{n=0}^{\infty}\left\{\frac{1}{2n+1} - \frac{1}{2n+2k+1}\right\} = \frac{1}{2k}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}\right),$$

since the sum telescoped. Hence, the equality $\sum_{n=0}^{\infty} a_n^2 = s^2 + 2\Delta$ takes the form

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \left(\frac{\pi}{4}\right)^2 + \sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{k}\left(1 + \frac{1}{3} + \cdots \frac{1}{2k-1}\right).$$

However, see (5.77), we already proved that the Cauchy product of Gregory-Leibniz-Madhava's series with itself is given by the sum on the right. Thus,

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{4}\right)^2 = \frac{\pi^2}{8}. \tag{5.79}$$

Finally, summing over the even and odd numbers, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2},$$

and solving for $\sum_{n=1}^{\infty} 1/n^2$, we obtain Euler's formula:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

5.11.3. Proof II. Follow Hofbauer [**79**] we give another proof of Euler's sum, which is short and sweet! For this proof, we shall use many properties of trigonometric functions. We begin with the identity

$$\frac{1}{\sin^2 x} = \frac{1}{4\sin^2 \frac{x}{2} \cos^2 \frac{x}{2}} = \frac{1}{4}\left[\frac{1}{\sin^2 \frac{x}{2}} + \frac{1}{\cos^2 \frac{x}{2}}\right] = \frac{1}{4}\left[\frac{1}{\sin^2 \frac{x}{2}} + \frac{1}{\sin^2\left(\frac{\pi+x}{2}\right)}\right],$$

or, since $\sin(x) = \sin(\pi - x)$,

$$\frac{1}{\sin^2 x} = \frac{1}{4}\left[\frac{1}{\sin^2 \frac{x}{2}} + \frac{1}{\sin^2\left(\frac{\pi-x}{2}\right)}\right]. \tag{5.80}$$

In particular, setting $x = \pi/2$, we obtain

$$1 = \frac{1}{4}\left[\frac{1}{\sin^2 \frac{\pi}{2^2}} + \frac{1}{\sin^2 \frac{\pi}{2^2}}\right] = \frac{2}{4} \cdot \frac{1}{\sin^2 \frac{\pi}{2^2}}.$$

Applying (5.80) (with $x = \pi/2^2$) to the right-hand side of this equation gives

$$1 = \frac{2}{4^2}\left[\frac{1}{\sin^2 \frac{\pi}{2^3}} + \frac{1}{\sin^2 \frac{3\pi}{2^3}}\right] = \frac{2}{4^2}\sum_{k=0}^{1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^3}}.$$

Repeatedly applying (5.80), we arrive at the interesting formula

$$1 = \frac{2}{4^n} \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}}. \tag{5.81}$$

To establish Euler's formula, we need the following lemma.

LEMMA 5.44. *For $0 < x < \pi/2$, we have*

$$\frac{1}{\sin^2 x} > \frac{1}{x^2} > -1 + \frac{1}{\sin^2 x}.$$

PROOF. Taking reciprocals in the formula from Lemma 4.56: For $0 < x < \pi/2$,

$$\sin x < x < \tan x,$$

we get $\cot^2 x < x^{-2} < \sin^{-2} x$. Since $\cot^2 x = \cos^2 x/\sin^2 x = -1 + \sin^{-2} x$, we conclude that

$$\frac{1}{\sin^2 x} > \frac{1}{x^2} > -1 + \frac{1}{\sin^2 x}, \quad 0 < x < \frac{\pi}{2},$$

which proves the lemma. □

Now using the inequality

$$\frac{1}{\sin^2 x} > \frac{1}{x^2} > -1 + \frac{1}{\sin^2 x}, \quad 0 < x < \frac{\pi}{2}$$

and the fact that $0 < (2k+1)\pi/2^{n+1} < \pi/2$ for $k = 0, \ldots, 2^{n-1} - 1$, we see that

$$\sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}} > \sum_{k=0}^{2^{n-1}-1} \frac{1}{\left(\frac{(2k+1)\pi}{2^{n+1}}\right)^2} > -2^{n-1} + \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}}.$$

Multiplying both sides by $2/4^n = 2/2^{2n}$ and using (5.81), we get

$$1 > \frac{8}{\pi^2} \sum_{k=0}^{2^{n-1}-1} \frac{1}{(2k+1)^2} > -\frac{1}{2^n} + 1.$$

Taking $n \to \infty$, we obtain

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

This equality, as we showed just after (5.79), easily implies Euler's formula. See Problem 4 for a classic proof.

EXERCISES 5.11.

1. Prove that

$$(a)\ \lim \frac{1 + 2^{1/2} + 3^{1/3} + \cdots + n^{1/n}}{n} = 1,$$

$$(b)\ \lim \left[\frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n}\right] = 1.$$

2. If a sequence $a_1, a_2, a_3, \ldots$ of positive numbers converges to $L > 0$, prove that the sequence of geometric means $(a_1 a_2 \cdots a_n)^{1/n}$ also converges to L. Suggestion: Take logs of the geometric means. Using this result, prove (5.76). Using (5.76), prove that

$$e = \lim \frac{n}{(n!)^{1/n}}.$$

3. (**Partial fraction expansion of** $1/\sin^2 x$) Here's Hofbauer derivation [**79**] of a partial fraction expansion of $1/\sin^2 x$.
   (i) Prove that

$$\frac{1}{\sin^2 x} = \frac{1}{2^{2n}} \sum_{k=0}^{2^n-1} \frac{1}{\sin^2 \frac{x+\pi k}{2^n}}.$$

   (ii) Show that

$$\frac{1}{\sin^2 x} = \frac{1}{2^{2n}} \sum_{k=-2^{n-1}}^{2^{n-1}-1} \frac{1}{\sin^2 \frac{x+\pi k}{2^n}}.$$

   (iii) Finally, prove that $\frac{1}{\sin^2 x} = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{(x+\pi k)^2}$. We usually write this as

$$\boxed{\frac{1}{\sin^2 x} = \sum_{k\in\mathbb{Z}} \frac{1}{(x+\pi k)^2}.} \tag{5.82}$$

4. (**Euler's sum for** $\pi^2/6$**, Proof III**) In this problem we derive Euler's sum via an old argument found in Thomas John l'Anson Bromwich's (1875–1929) book [**31**, p. 218-19] (cf. [**5**], [**134**], [**94**]).

(ii) Recall from Problem 4 in Exercises 4.7 that for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\sin nx = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1} x \, \sin^{2k+1} x.$$

Using this formula, prove that if $\sin x \neq 0$, then

$$\sin(2n+1)x = \sin^{2n+1} x \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} (\cot^2 x)^{n-k}.$$

(iiii) Prove that if $n \in \mathbb{N}$, then the roots of $\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} t^{n-k} = 0$ are the n numbers $t = \cot^2 \frac{m\pi}{2n+1}$ where $m = 1, 2, \ldots, n$.

(iiiii) Prove that if $n \in \mathbb{N}$, then

$$\sum_{k=1}^{n} \cot^2 \frac{k\pi}{2n+1} = \frac{n(2n-1)}{3}.$$

Suggestion: Recall that if $p(t)$ is a polynomial of degree n with roots $r_1, \ldots, r_n$, then $p(t) = a(t - r_1)(t - r_2) \cdots (t - r_n)$ for a constant a. What's the coefficient of t^1 if you multiply out $a(t - r_1) \cdots (t - r_n)$?

(iviv) Prove that if $n \in \mathbb{N}$, then

$$\sum_{k=1}^{n} \frac{1}{\sin^2 \frac{\pi k}{2n+1}} = \frac{2n(n+1)}{3}. \tag{5.83}$$

From this identity, derive Euler's sum.

5. (**Partial fraction expansion of $1/\sin^2 x$, Proof II**) As noted by Hofbauer [**79**], the identity (5.83), which is a key ingredient to the previous proof, can be derived from the partial fraction expansion (5.82).

(i) Use (5.82) to prove that for any $n \in \mathbb{N}$,

$$\frac{1}{\sin^2 x} = \frac{1}{n^2} \sum_{m=0}^{n-1} \frac{1}{\sin^2 \frac{x + \pi m}{n}}. \tag{5.84}$$

Suggestion: Replace x with $\frac{x+\pi m}{n}$ in (5.82) and sum from $m = 0$ to $n - 1$.

(ii) Take the $m = 0$ term in (5.84) to the left, replace n by $2n + 1$, and then take $x \to 0$ to derive (5.83).