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## Formula(s)

$$2 = 2 = \frac{e^1}{e^{1/2}} \cdot \frac{e^{1/3}}{e^{1/4}} \cdots, 186$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \arctan\left(\frac{1}{F_{2n+1}}\right), 301$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^z} = \frac{3^z}{3^z+1} \cdot \frac{5^z}{5^z-1} \cdots, 346$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} = (-1)^k \frac{E_{2k}}{2(2k)!} \left(\frac{\pi}{2}\right)^{2k+1}, 346$$

$$e = 2 + \frac{2}{2} + \frac{3}{3} + \frac{4}{4} \cdots, 364$$

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$$\arctan x = \frac{x}{1} + \frac{x^2}{3-x^2} + \frac{3^2 x^2}{5-3x^2} + \cdots, 360$$

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$$\frac{e}{e-1} = \lim_{n \rightarrow \infty} \left\{ \left(\frac{n}{n}\right)^n + \cdots + \left(\frac{1}{n}\right)^n \right\}, 320$$

$$e = \lim_{n \rightarrow \infty} \left\{ \left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \cdots \left(\frac{n+1}{n}\right)^n \right\}^{1/n}, 304$$

$$e = \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{16}{15} \cdots, 314$$

$$\frac{e^{\pi} - e^{-\pi}}{2\pi} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right), 325$$

$$\frac{e^{2/x+1}}{e^{2/x-1}} = x + \frac{1}{3x} + \frac{1}{5x} \cdots, 405$$

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Euler's product ( $\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdots$ ), 312, 347

$$\gamma = 1 - \sum_{n=2}^{\infty} \frac{1}{n} (\zeta(n) - 1), 298$$

$$\gamma = \frac{3}{2} - \log 2 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} (n-1) (\zeta(n) - 1), 298$$

$$\gamma = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n), 229, 298$$

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$$\log(1+x) = \frac{x}{1} + \frac{1^2 x}{(2-1x)} + \frac{2^2 x}{(3-2x)} \cdots, 364$$

$\log 2$

$$\log 2 = \frac{1}{1} + \frac{1^2}{1} + \frac{2^2}{1} + \cdots, 358$$

$\log 2$

$$\log 2 = \sum_{n=2}^{\infty} \frac{1}{2^n} \zeta(n), 269$$

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$$\Phi = \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n}, 384$$

$$\Phi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1}}, 384$$

$$\Phi^{-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{F_n F_{n+2}}, 384$$

$$\pi = 3 + \frac{1^2}{6} + \frac{3^2}{6} + \frac{5^2}{6} \cdots, 362$$

$$\pi \cot \pi x = \frac{1}{x} + \frac{x^2}{1-2x} + \frac{(1-x)^2}{2x} \cdots, 365$$

$$\frac{\pi}{2} = 1 + \frac{1}{1} + \frac{1 \cdot 2}{1} + \frac{2 \cdot 3}{1} \cdots, 361$$

$$\frac{\cos \frac{\pi x}{2}}{\frac{x}{2}} = x + 1 + \frac{(x+1)^2}{-2 \cdot 1} + \frac{(x-1)^2}{-2} \cdots, 365$$

$$\frac{\sin \pi x}{\pi x} = 1 - \frac{x}{1} + \frac{(1-x)^2}{2x} + \frac{(1+x)^2}{1-2x} \cdots, 365$$

$$\frac{\sin \pi x}{\pi x} = 1 - \frac{x}{1} + \frac{1 \cdot (1-x)}{x} + \frac{1 \cdot (1+x)}{1-x} \cdots, 365$$

$$\frac{\tan \pi x}{\pi x} = 1 + \frac{x}{1-2x} + \frac{(1-x)^2}{2x} \cdots, 365$$

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$$\frac{6}{\pi^2} = 0^2 + 1^2 - \frac{1^4}{1^2+2^2} + \frac{-2^4}{2^2+3^2} + \frac{-3^4}{3^2+4^2} \cdots, 361$$

$$\frac{\pi^2}{6} = \frac{1}{0^2+1^2} + \frac{-1^4}{1^2+2^2} + \frac{-2^4}{2^2+3^2} + \frac{-3^4}{3^2+4^2} \cdots, 361$$

$$\frac{6}{\pi^2} = 1 - \frac{1}{2^2} - \frac{1}{3^2} \cdots + \frac{\mu(n)}{n^2} + \cdots, 345$$

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