Example 3.47. For example, in the base 10 expansion of $\frac{3526}{495} = 7.1232323\ldots = 7.a_1a_2a_3\ldots$, we have $a_n = a_{n+2}$ for all $n \geq \ell = 2$.

We can actually see how the periodic pattern appears by going back to high school long division! Indeed, long dividing 495 into 3526 we get

$$\begin{array}{rl} & 7.123 \\ 495 & \!\!\big)\overline{3526.000} \\ & \underline{3465} \\ & 610 \\ & \underline{495} \\ & 1150 \\ & \underline{990} \\ & 1600 \\ & \underline{1485} \\ & 115 \end{array}$$

At this point, we get another remainder of 115, exactly as we did a few lines before. Thus, by continuing this process of long division, we are going to repeat the pattern $2, 3$. We shall use this long division technique to prove the following theorem.

THEOREM 3.34. *Let b be a positive integer greater than* 1*. A real number is rational if and only if its b-adic expansion is periodic.*

PROOF. We first prove the "only if", then the "if" statement.

Step 1: We prove the "only if": Given integers p, q with $q > 0$, we show that p/q has a periodic b-adic expansion. By the division algorithm (see Theorem 2.15), we can write $p/q = q' + r/q$ where $q' \in \mathbb{Z}$ and $0 \leq r < q$. Thus, we just have to prove that r/q has a periodic b-adic expansion. In particular, we might as well assume from the beginning that $0 < p < q$ so that $p/q < 1$. Proceeding via high school long division, we construct the decimal expansion of p/q.

First, using the division algorithm, we divide bp by q, obtaining a unique integer a_1 such that $bp = a_1q + r_1$ where $0 \leq r_1 < q$. Since

$$\frac{p}{q} - \frac{a_1}{b} = \frac{bp - a_1q}{bq} = \frac{r_1}{bq} \geq 0,$$

we have

$$\frac{a_1}{b} \leq \frac{p}{q} < 1,$$

which implies that $0 \leq a_1 < b$.

Next, using the division algorithm, we divide $b\,r_1$ by q, obtaining a unique integer a_2 such that $br_1 = a_2q + r_2$ where $0 \leq r_2 < q$. Since

$$\frac{p}{q} - \frac{a_1}{b} - \frac{a_2}{b^2} = \frac{r_1}{bq} - \frac{a_2}{b^2} = \frac{br_1 - a_2q}{b^2q} = \frac{r_2}{b^2q} \geq 0,$$

we have

$$\frac{a_2}{b^2} \leq \frac{r_1}{bq} < \frac{q}{bq} = \frac{1}{b},$$

which implies that $0 \leq a_2 < b$.

Once more using the division algorithm, we divide $b\,r_2$ by q, obtaining a unique integer a_3 such that $br_2 = a_3q + r_3$ where $0 \leq r_3 < q$. Since

$$\frac{p}{q} - \frac{a_1}{b} - \frac{a_2}{b^2} - \frac{a_3}{b^3} = \frac{r_2}{b^2q} - \frac{a_3}{b^3} = \frac{br_2 - a_3q}{b^3q} = \frac{r_3}{b^3q} \geq 0,$$