MATH 478: TEST 1

1. Test 1: Solutions

Problem 1 (25 points)**.** Prove using only the definition of limit that

$$\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n+1}-1} = 0.$$

As part of your solution you should state the definition of limit, i.e. say what it means for a sequence $\{a_n\}$ to converge to a limit L.

Solution of Problem 1. The definition of limit is boxed on p. 91 in the text.

Let $\epsilon > 0$. We claim that if $N = (\frac{1}{\epsilon}+1)^2$, for any $n > N$, we have

$$\left|\frac{(-1)^n}{\sqrt{n+1}-1} - 0\right| < \epsilon.$$

To see this, let $n > N$, so

$$n > (\frac{1}{\epsilon}+1)^2$$

it follows that

$$n+1 > (\frac{1}{\epsilon}+1)^2$$

so that

$$\sqrt{n+1} > \frac{1}{\epsilon}+1$$

and thus we have

$$\sqrt{n+1}-1 > \frac{1}{\epsilon}.$$

Inverting both sides gives:

$$\frac{1}{\sqrt{n+1}-1} < \epsilon$$

Now note that

$$\left|\frac{(-1)^n}{\sqrt{n+1}-1} - 0\right| = \frac{1}{\sqrt{n+1}-1} < \epsilon,$$

since $N > 0$ and so $n \geq 1$ making the denominator positive. We have proved the claim above. Thus, by definition of limit,

$$\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n+1}-1} = 0.$$

□

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Problem 2 (25 points). Let $\{a_n\}$ be a monotone increasing sequence of real numbers which is bounded above and has least upper bound L. Show that

$$\lim_{n\to\infty} a_n = L.$$

Solution of Problem 2. See the proof of theorem 3.12 in your text. □

Problem 3 (25 points). State the definition of a contractive sequence and prove that the sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{9 - 2a_n}$ is contractive. (Hint: First show that $0 \leq a_n \leq 3$ for all n.)

Solution of Problem 3. See example 3.26 in your text. (This is very similar to problem 3.4.3 (c) from the homework.) □

Problem 4 (25 points). Indicate in your bluebook whether each of the following statements is true or false. No reasoning or proof is required for this question. However, you will lose 2.5 points for each incorrect answer.

(1) Every subsequence of a convergent sequence converges. *True.*
(2) There exists a convergent sequence which is unbounded. *False.*
(3) Every sequence of real numbers has a monotone subseqence. *True.*
(4) Every Cauchy sequence is a contractive sequence. *False.*
(5) If a sequence of real numbers $\{a_n\}$ has the property that $a_{n+1} > a_n$ for all n (note the strict inequality), then $\{a_n\}$ diverges to $+\infty$. *False.*
(6) A monotone decreasing sequence of real numbers is bounded above. *True.*
(7) If $\{a_n\}$ is a sequence of positive real numbers and $a_n \to 0$, then there exist real numbers C, r with $C > 0$ and $0 < r < 1$ such that $a_n \leq Cr^n$ for all sufficiently large n. *False.*
(8) If a is a positive real number and $a < 1$, then for any positive integer k, $lim_{n\to\infty} n^k a^n = 0$. *True.*
(9) Every bounded sequence has a convergent subsequence. *True.*
(10) There exists a convergent sequence which is not a Cauchy seqence. *False.*

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