MATH 478: TEST 2

1. Test 2: Solutions

Problem 1 (30 points)**.** Prove using only the ϵ-δ definition of limit that

$$\lim_{z\to\frac{1}{2}} \frac{1}{z^2} = 4.$$

Recall that z is a *complex* variable!

Solution of Problem 1. Let $\epsilon > 0$ be given. We consider the expression

$$\left|\frac{1}{z^2} - 4\right| = \left|\frac{1-4z^2}{z^2}\right| = \left|\frac{4(z-\frac{1}{2})(z+\frac{1}{2})}{z^2}\right| = \frac{4|z+\frac{1}{2}|}{|z^2|} \cdot \left|z - \frac{1}{2}\right|$$

Now let's restrict z so that $|z-\frac{1}{2}| < \frac{1}{4}$. Then we have $|z| = |\frac{1}{2} + (z-\frac{1}{2})| \geq \frac{1}{2} - |(z-\frac{1}{2})| > \frac{1}{4}$, using the triangle inequality. It follows that $\frac{1}{|z^2|} < 16$. In addition, $|(z+\frac{1}{2})| = |1 + (z-\frac{1}{2})| \leq 1 + |(z-\frac{1}{2})| < \frac{5}{4}$, again using the triangle inequality. Thus we have

$$\frac{4|z+\frac{1}{2}|}{|z^2|} \cdot \left|z-\frac{1}{2}\right| < 4 \cdot \frac{5}{4} \cdot 16 \cdot \left|z - \frac{1}{2}\right| = 80\left|z-\frac{1}{2}\right|$$

Now $80|z-\frac{1}{2}| < \epsilon$ iff $|z-\frac{1}{2}| < \frac{\epsilon}{80}$. So we choose $\delta = \min(\frac{1}{4}, \frac{\epsilon}{80})$ and now it follows that for any z with $|z-\frac{1}{2}| < \delta$ that

$$\left|\frac{1}{z^2} - 4\right| < 80\left|z-\frac{1}{2}\right| < 80\frac{\epsilon}{80} = \epsilon,$$

where the first inequality uses $|z-\frac{1}{2}| < \frac{1}{4}$ and the second uses $|z-\frac{1}{2}| < \frac{\epsilon}{80}$.

Thus we have shown that

$$\lim_{z\to\frac{1}{2}} \frac{1}{z^2} = 4.$$

□

Problem 2 (20 points)**.** (No proofs are necessary in any part of this problem. Just make sure your example in (c) is correct!)

(a) (5 points) What does it mean to say that f is pointwise discontinuous on the interval $(-1, 1)$? (i.e. give the definition of "pointwise discontinuous").

(b) (5 points) State Volterra's theorem.

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(c) (5 points) If we remove the hypothesis of pointwise discontinuity in Volterra's theorem, the result is no longer true. Give an example to show that the conclusion of the theorem need no longer hold.

(d) (5 points) Say what it means (i.e. give the definition) for a set A to be "dense" in an open interval I.

Solution of Problem 2. Part (a): "f is pointwise discontinuous" means that the set C_f of points of continuity of f is dense in $(-1,1)$.

Part (b): See p. 162 in the text.

Part (c): Let's define two functions f and g on $(-1,1)$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational}, \\ x & \text{if } x \text{ is irrational}. \end{cases}$$

Then $C_f = (-1,0) \cup (0,1)$ and $C_g = \{0\}$. That is, the set of points of continuity of f is exactly the set of points of discontinuity of g. This example shows that the theorem is not true if we remove the hypothesis of pointwise discontinuity.

Part (d): See p. 162 in the text. □

Problem 3 (20 points)**.** Let z be a complex number.

(a) (5 points) Give the definition of $\exp(z)$ as an infinite series.

(b) (15 points) Show that this series converges.

Solution of Problem 3. See p. 134 in your text. □

Problem 4 (10 points)**.** Let X be the set of all functions $f\colon \mathbb{N} \to \{0,1,2\}$. Let $f_1, f_2, f_3, \ldots$ be any infinite sequence of elements of X. Prove that there is an element $f \in X$ that is not in this list. Use this to show that X is uncountable.

Solution of Problem 4. We define

$$f(n) = \begin{cases} 0 & f_n(n) = 1 \\ 1 & f_n(n) \neq 1 \end{cases}$$

and note that $f \neq f_n$ for all $n \in N$. This is because $f(n) \neq f_n(n)$, i.e. the two functions have different values on n, so they are different functions. Now suppose we have a function $c\colon \mathbb{N} \to X$. Then we can define a sequence $\{f_i\}$ of elements of X by $f_i = c(i)$, and note as above that there is an element f which is not any of the f_i and thus $f \notin \operatorname{Im}(c)$. Thus c is not surjective, hence not bijective, so X is uncountable. □

Problem 5 (20 points)**.** Indicate in your bluebook whether each of the following statements is true or false. No reasoning or proof is required for this question. However, you will lose 2 points for each incorrect answer. Note that this means you can get a *negative* score for this section!

(1) Let f and g be pointwise discontinuous functions on the open interval $(0, 1)$ and let C_f and C_g be the sets of points of continuity of f and g. Then the set $C_f \cap C_g$ is nonempty. *True.*

(2) Let f and g be pointwise discontinuous functions on the open interval $(0, 1)$ and let C_f and C_g be the sets of points of continuity of f and g. Then the number of elements of the set $C_f \cap C_g$ is at least 14. *True.*

(3) Let $\sum a_n$ be a convergent series of positive real numbers. Then $\sum \sqrt{a_n}/n$ is also convergent. *True.*

(4) For any convergent series $\sum a_n$ of real numbers we have $na_n \to 0$. *False.*

(5) If $\lim_{x\to c} f$ and $\lim_{x\to c} g$ exist and $f(x) < g(x)$ for x sufficiently close to c, then

$$\lim_{x\to c} f < \lim_{x\to c} g.$$

False.

(6) The series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$$

diverges. *False.*

(7) The set of real numbers which do *not* have a unique 3-adic expansion is uncountable. *False.*

(8) The 2-adic expansion of $\frac{1}{3}$ is $0._2 01010101\ldots$. *True.*

(9) For any given $k \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \left[1 - (1 - \frac{1}{n})(1 - \frac{2}{n})(1 - \frac{3}{n}) \cdots (1 - \frac{k}{n})\right] = 0$$

True.

(10)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = 0$$

False.

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