MATH 478: TEST 2

1. Test 2

Problem 1 (30 points)**.** Prove using only the ϵ-δ definition of limit that

$$\lim_{z \to \frac{1}{2}} \frac{1}{z^2} = 4.$$

Recall that z is a *complex* variable!

Problem 2 (20 points)**.** (No proofs are necessary in any part of this problem. Just make sure your example in (c) is correct!)

(a) (5 points) What does it mean to say that f is pointwise discontinuous on the interval $(-1, 1)$? (i.e. give the definition of "pointwise discontinuous").
(b) (5 points) State Volterra's theorem.
(c) (5 points) If we remove the hypothesis of pointwise discontinuity in Volterra's theorem, the result is no longer true. Give an example to show that the conclusion of the theorem need no longer hold.
(d) (5 points) Say what it means (i.e. give the definition) for a set A to be "dense" in an open interval I.

Problem 3 (20 points)**.** Let z be a complex number.

(a) (5 points) Give the definition of $\exp(z)$ as an infinite series.
(b) (15 points) Show that this series converges.

Problem 4 (10 points)**.** Let X be the set of all functions $f\colon \mathbb{N} \to \{0, 1, 2\}$. Let $f_1, f_2, f_3, \ldots$ be any infinite sequence of elements of X. Prove that there is an element $f \in X$ that is not in this list. Use this to show that X is uncountable.

Problem 5 (20 points)**.** Indicate in your bluebook whether each of the following statements is true or false. No reasoning or proof is required for this question. However, you will lose 2 points for each incorrect answer. Note that this means you can get a *negative* score for this section!

(1) Let f and g be pointwise discontinuous functions on the open interval $(0, 1)$ and let C_f and C_g be the sets of points of continuity of f and g. Then the set $C_f \cap C_g$ is nonempty.
(2) Let f and g be pointwise discontinuous functions on the open interval $(0, 1)$ and let C_f and C_g be the sets of points of continuity of f and g. Then the number of elements of the set $C_f \cap C_g$ is at least 14.
(3) Let $\sum a_n$ be a convergent series of positive real numbers. Then $\sum \sqrt{a_n}/n$ is also convergent.

Date: November 15, 2005.

(4) For any convergent series $\sum a_n$ of real numbers we have $na_n \to 0$.

(5) If $\lim_{x\to c} f$ and $\lim_{x\to c} g$ exist and $f(x) < g(x)$ for x sufficiently close to c, then

$$\lim_{x\to c} f < \lim_{x\to c} g.$$

(6) The series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$$

diverges.

(7) The set of real numbers which do *not* have a unique 3-adic expansion is uncountable.

(8) The 2-adic expansion of $\frac{1}{3}$ is $0._2 01010101\ldots$.

(9) For any given $k \in \mathbb{N}$ we have

$$\lim_{n\to\infty}\left[1 - (1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\cdots(1-\frac{k}{n})\right] = 0$$

(10)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$$

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