A function which is infinitely many times differentiable, but is not equal to its McClaurin series

Consider the function

$$f(x) = e^{-1/x^2}.$$

whose domain is all $x \neq 0$, as it is undefined at $x = 0$. Note that

$$\lim_{x\to 0} f(x) = 0,$$

so, we can extend $f(x)$ continuously to the whole real line by taking

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

The derivative of this function for $x \neq 0$ is given by

$$f'(x) = e^{-1/x^2} \cdot 2x^{-3}.$$

At $x = 0$, we have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x}.$$

Introducing the variable $u = 1/x$, allows us to write

$$f(x) = e^{-u^2}, \text{ for } x \neq 0,$$

and the limit as

$$f'(0) = \lim_{u\to\pm\infty} \frac{u}{e^{u^2}} = 0. \tag{1}$$

Moreover, we can write $f'(x)$ in terms of u as

$$f'(x) = \frac{2u^3}{e^{u^2}}, \text{ for } x \neq 0. \tag{2}$$

From here it is easy to see by induction on n that

$$\begin{aligned} &f^{(n)}(x) = \frac{p(u)}{e^{u^2}}, \text{ for } x \neq 0, \quad \text{and} \\ &f^{(n)}(0) = 0, \end{aligned} \tag{3}$$

where $p(u)$ is a polynomial in u (that changes with n).

We have just done the *base case (n=1)* of the induction in (2) and (1) above. Assume now (3) as the inductive hypothesis. Note that we immediately get

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{u\to\pm\infty} \frac{p(u)/e^{u^2}}{1/u} = \lim_{u\to\pm\infty} \frac{p(u)\cdot u}{e^{u^2}} = 0,$$

and for $x \neq 0$, using $\dfrac{du}{dx} = -u^2$

$$\begin{aligned} f^{(n+1)}(x) &= (e^{-u^2}\cdot(-2u)(-u^2)\cdot p(u)) + (e^{-u^2}p'(u)(-u^2)) \\ &= \frac{2u^3p(u) - u^2p'(u)}{e^{u^2}}, \end{aligned}$$

and the last expression has the desired form, a polynomial on u over e^{u^2}, completing the induction argument.

Since every derivative of $f(x)$ at 0 evaluates to 0, we get that the McClaurin series for this function is the 0 function, and is not equal to $f(x)$.