Higher scissors congruence

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Joint work with:

Bohmann, Gerhardt, Merling, and Zakharevich (BGMMZ),

Kupers, Lemann, Miller, and Sroka (KLMMS).

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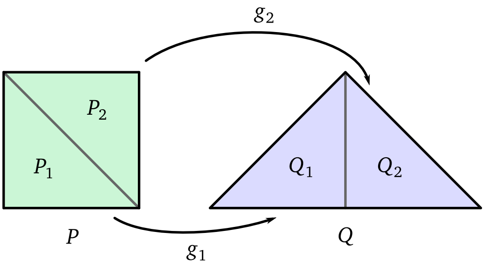

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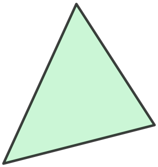
Is this true in dimensions other than 2?

Definition. A **convex polytope** is a convex hull of finitely many points in Euclidean space E^n. (Must be nondegenerate.)



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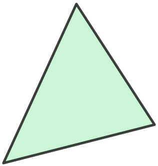
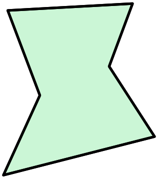
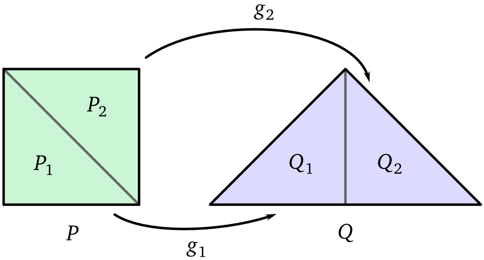


A **polytope** is a finite union of (nondegenerate) convex polytopes.



A **scissors congruence** from P to Q is

$$\begin{cases} P = \cup_{i=1}^k P_i & \text{interiors disjoint,} \\ Q = \cup_{i=1}^k Q_i & \text{interiors disjoint, and} \\ \text{isometries } g_i : P_i \cong Q_i, & i = 1, \dots, k. \end{cases}$$



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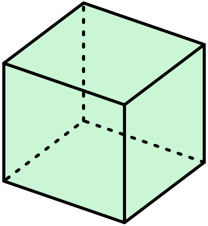
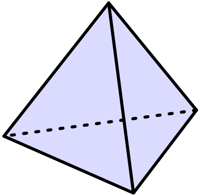
Hilbert's 3rd Problem

Polyhedra in E^3 up to scissors congruence = volume?



P



Q

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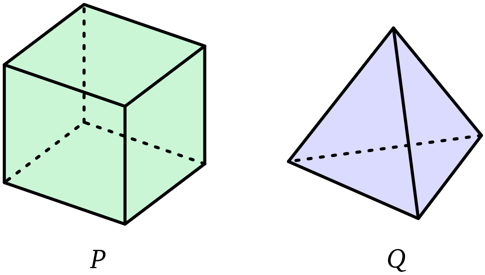

Answer. (Dehn 1901) No! Volume isn't enough.

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Volume $\in \mathbb{R}$ is an example. Are there more?

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Theorem (Dehn 1901)

A cube and a regular tetrahedron are never scissors congruent.

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Every other invariant factors through K-theory: $K_0(E^n) \to A$.

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Theorem (Dehn–Sydler–Jessen)

Volume and Dehn invariant define an injective map

$$K_0(E^3) \longrightarrow \mathbb{R} \times (\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}).$$

So volume and Dehn invariant are everything in dimension 3.

In fact, there is an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow K_0(E^3) \longrightarrow (\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}) \longrightarrow \Omega^1_{\mathbb{R}/\mathbb{Z}} \longrightarrow 0.$$

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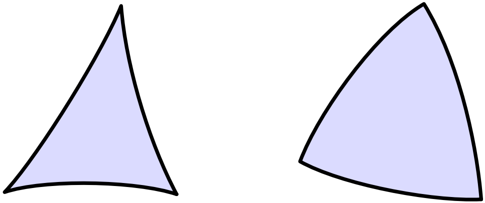
$K_0(E^5)$ has not been computed!

Generalization: consider other geometries!



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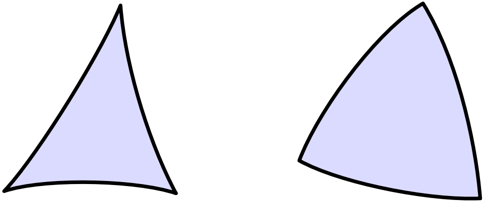

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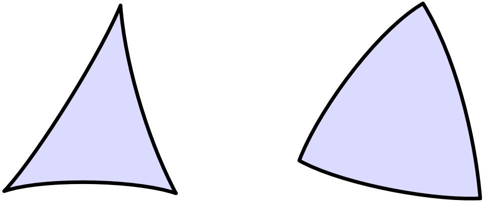

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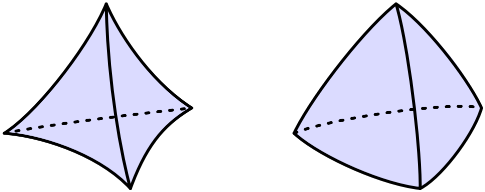
Example: $K_0(S^2) = \mathbb{R}$.
Spherical polygons up to scissors congruence = area.

Next is H^3 and S^3:



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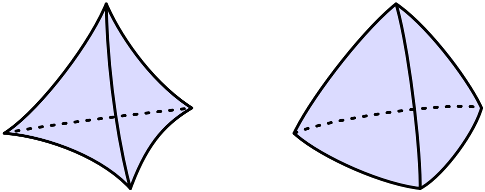

Theorem (Dupont 1982)

There are exact sequences:

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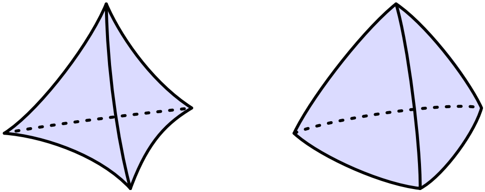

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Still open whether the volume and Dehn invariant are everything!

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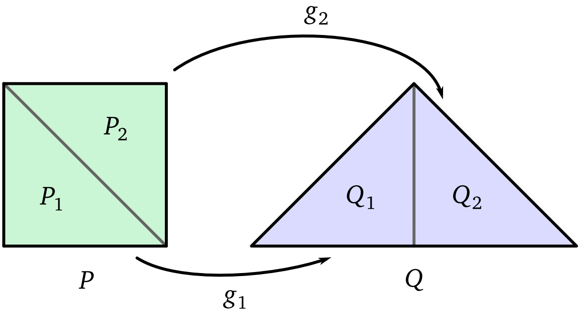
Modern question

How many scissors congruences are there $P \to Q$?

Again, a scissors congruence $P \to Q$ is:



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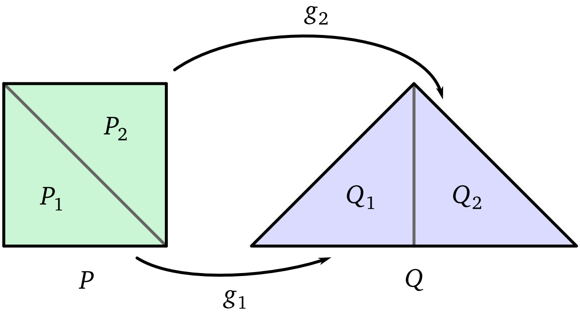

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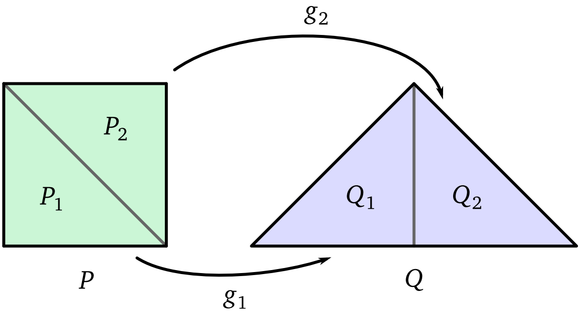

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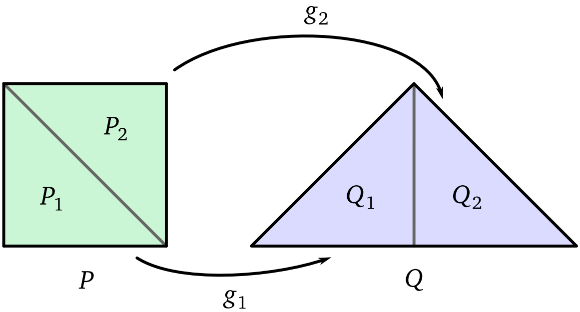

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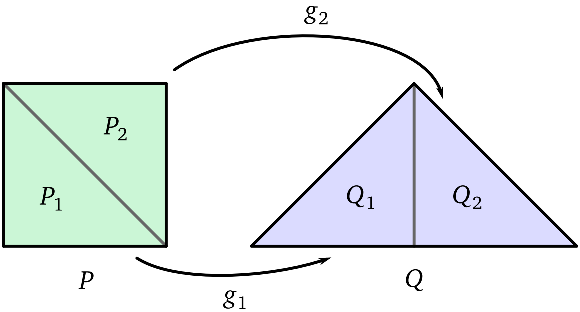

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Cutting a piece P_i into smaller pieces gives the same morphism.

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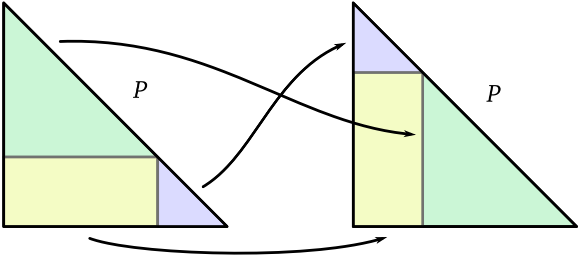
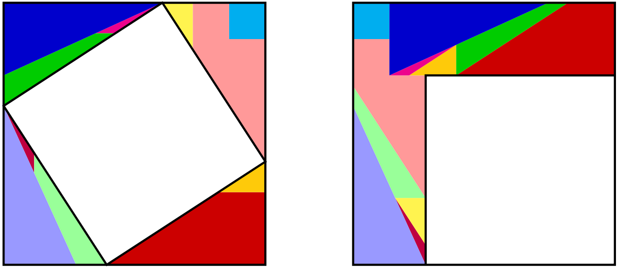
So, scissors congruences from one polytope P to itself form a group, the **scissors automorphism group** $\mathrm{Aut}(P)$.

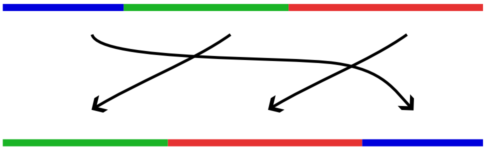


A scissors automorphism of a square:



(image by Inna Zakharevich)

In E^1, if we don't allow reflections, $\mathrm{Aut}(P)$ is the group of **interval exchange transformations**:



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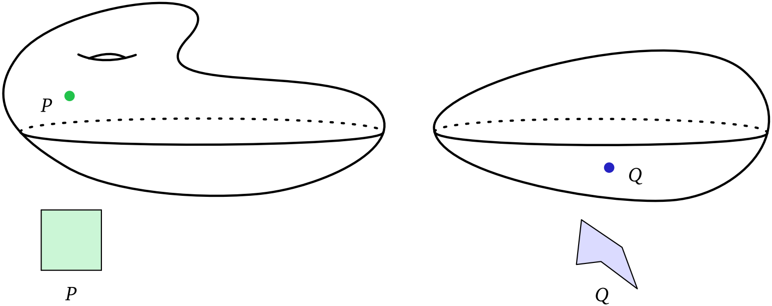

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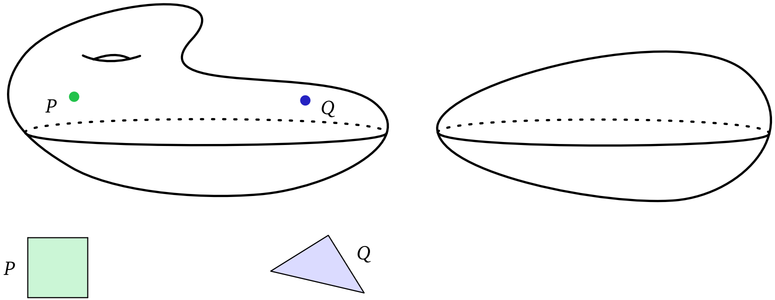

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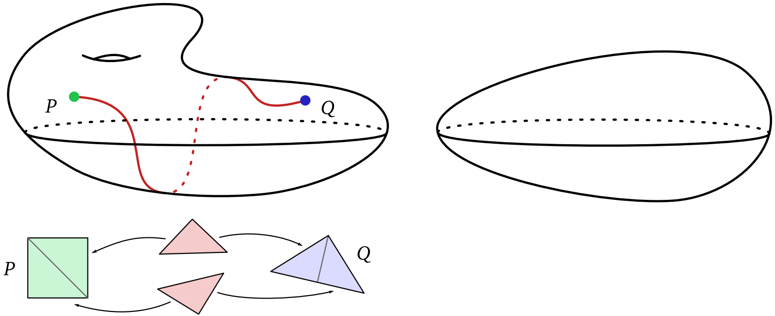

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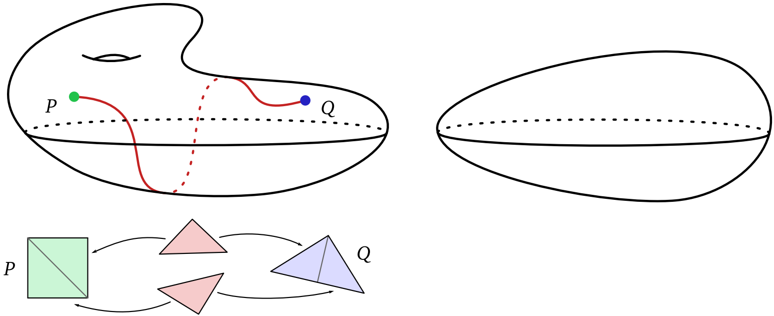

Definition. (Zakharevich) Scissors congruence K-theory is the group completion of this space. (Formally add negatives.)

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No other higher K-groups known! (As of 2022.)

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K_1(E^1_{\mathbb{R}}) & = & \mathbb{R} \wedge \mathbb{R} & K_1(E^1) & = & 0 \\
K_2(E^1_{\mathbb{R}}) & = & \mathbb{R} \wedge \mathbb{R} \wedge \mathbb{R} & K_2(E^1) & = & \mathbb{R} \wedge \mathbb{R} \wedge \mathbb{R} \\
K_3(E^1_{\mathbb{R}}) & = & \Lambda^4(\mathbb{R}) & K_3(E^1) & = & 0 \\
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\vdots & & \vdots & \vdots & & \vdots
\end{array}
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K_0(E^1_{\mathbb{R}}) & = & \mathbb{R} & K_0(E^1) & = & \mathbb{R} \\
K_1(E^1_{\mathbb{R}}) & = & \mathbb{R} \wedge \mathbb{R} & K_1(E^1) & = & 0 \\
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& \vdots & \vdots & & \vdots & \vdots
\end{array}
$$

Theorem (M 2022)

$K_m(E^n)$ is always rational, and

$$K_m(E^n) \cong H_m(\mathrm{Isom}(E^n); St(E^n) \otimes \det).$$

Gives a general method!

Builds on joint work with Bohmann, Gerhardt, Merling, and Zakharevich.

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Ongoing work of Holley, Lemann, and others is drawing conclusions for E^2 and H^2!

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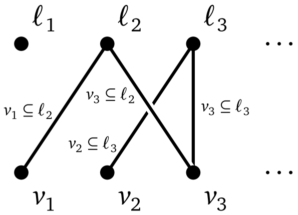

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Polytopes up to subdivision (but no moving around) gives $St(E^n)$.

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Can take $ST(E^n) \simeq \bigvee S^n$ and de-suspend n times to get $\bigvee S^0$!

Theorem (Bohmann–Gerhardt–M–Merling–Zakharevich 2023)

As a spectrum, $K(E^n)$ is homotopy orbits,

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$$\Rightarrow \; K_m(E^n) \cong H_m(\mathrm{Isom}(E^n); St(E^n) \otimes \det).$$

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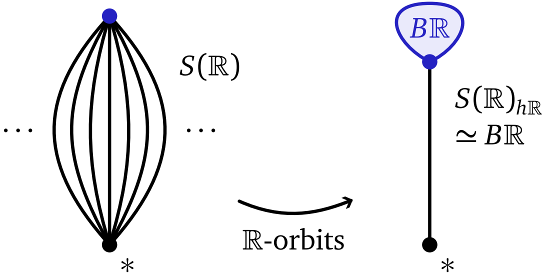

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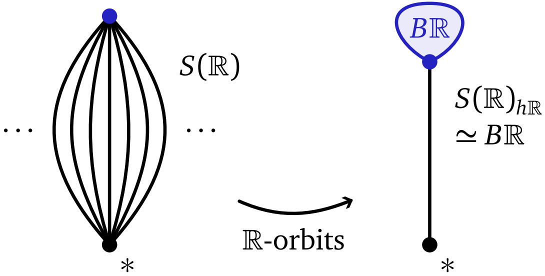

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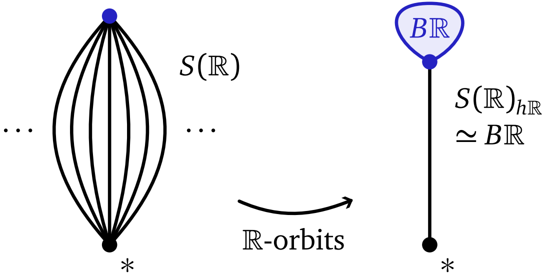

So $K(E^1_{\mathbb{R}}) \simeq \Sigma^{-1} B\mathbb{R}$, and $K_n = \Lambda^{n+1}(\mathbb{R})$. (Including $K_0 = \mathbb{R}$!)

By a similar method, we get

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Also get exact sequences for $K_*(E^3)$, higher Dehn-Sydler-Jessen theorem!

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So *stably*, $H_*(\mathrm{Aut}(P);\mathbb{Q})$ becomes free and the K-groups are the generators.

Theorem (Kupers, Lemann, M, Miller, Sroka 2024)

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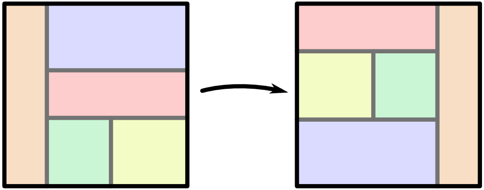
$\vdots$

Example. "Rectangle exchange transformations" (Cornulier–Lacourte 2022)



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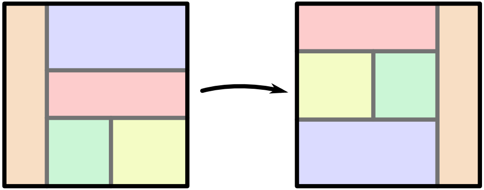

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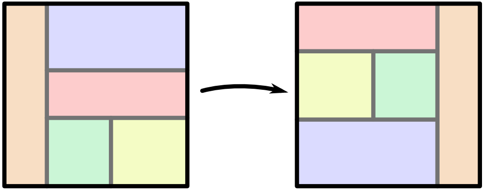

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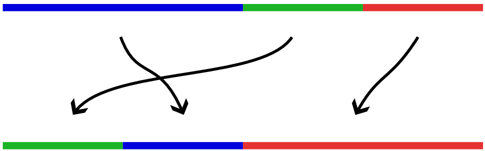
Corollary. $K_1 = H_1 = (\Lambda^2\mathbb{R} \otimes \mathbb{R}^{\otimes(n-1)})^{\oplus n}$ (Cornulier–Lacourte 2022)

Example. Thompson's group V: cut $[0,1]$ at points of the form $\frac{a}{2^k}$, allow translations and scalings by powers of 2.



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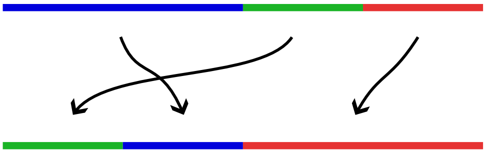

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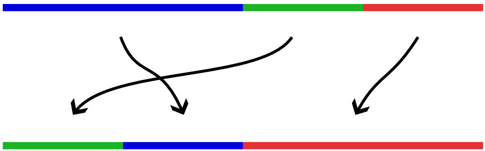

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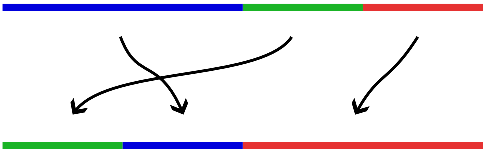

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Can also do variants where the homology was not known before, e.g. the "irrational slope Thompson's group" (Burillo–Nucinkis–Reeves 2022).

Thank you!