A USER'S GUIDE TO G-SPECTRA (UNFINISHED DRAFT)

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Contents

These are expository notes on equivariant stable homotopy theory, or G-spectra for short. We discuss different notions of stable G-equivalence, as well as the elementary tools such as equivariant transfer maps, the tom Dieck splitting, and the Adams and Wirthmuller isomorphisms. The author is aware of many other sources on G-spectra with varying levels of depth and modernity; the idea behind these notes is to give a focused treatment of those aspects of G-spectra that have been most useful for computations in THH and K-theory, with enough exposition to give completely precise definitions. In other words, we aim to pave a short path into the land of G-spectra that has enough precision to allow the reader to start working with them directly. In exchange for being completely explicit about the defintions, we will omit most of the proofs.

Our overall approach is to make, once and for all, a single category of G-spectra. Then we put different model structures on this category, corresponding to the different flavors of equivariant stable homotopy theory, and discuss how they are related. This is the same approach taken in [EM97] except that we look at orthogonal spectra instead of coordinate-free Ω-spectra as in [**?**]. These notes are also meant to complement and connect [**?**] and [**?**] which are excellent resources.

1. The basics of G-spaces

We are going to work in the category of compactly generated weak Hausdorff spaces. All smash products and mapping spaces are taken in that category.

1.1. **Basic definitions.**

Let G be a topological group. Whenever we take a subgroup $H \leq G$ we will always assume that it is closed.

A *(based) G-space* X is a based topological space X with a continuous left G-action that preserves the basepoint. For each closed subgroup $H \leq G$, the subspace of points of X fixed by every element of H will be denoted X^H. Of course X^H always contains the basepoint, so it's also a based space.

An *equivariant map* or *G-map* of based G-spaces is a continuous basepoint-preserving map $f : X \longrightarrow Y$ which commutes with the G-action. In other words,

$$g \circ f = f \circ g \quad \text{for all } g \in G$$

The set of all such maps is denoted $\mathrm{Map}_*^G(X, Y)$. We topologize it as a subspace of the usual space of all not-necessarily-equivariant based maps $\mathrm{Map}_*(X, Y)$.

Now the above condition rearranges to

$$f = g \circ f \circ g^{-1}$$

This motivates us to define a G-action on $\mathrm{Map}_*(X, Y)$ by the rule

$$g(f)(x) := g(f(g^{-1}(x)))$$

We call this action "conjugation" because it looks very similar to conjugation from group theory. Under this action, the subspace of $\mathrm{Map}_*(X, Y)$ fixed by the entire group G is exactly the subspace of G-equivariant maps:

$$\mathrm{Map}_*^G(X, Y) = (\mathrm{Map}_*(X, Y))^G$$

Finally, we can tensor two G-spaces X and Y together by taking the ordinary smash product $X \wedge Y$ and giving it a diagonal G-action:

$$g(x, y) := (gx, gy)$$

It is then a pleasurable exercise to verify that the category of based G-spaces is a closed symmetric monoidal category (as defined in e.g. [Hov07]), with tensor product given by smash product $X \wedge Y$ with diagonal G-action, and internal hom given by based maps $\mathrm{Map}_*(X, Y)$ with the conjugation G-action. In particular, there is a natural homeomorphism of G-spaces

$$\mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)) \cong \mathrm{Map}_*(X \wedge Y, Z)$$

To be mildly consistent with other sources, we will denote this category $\mathbf{Top}_*^G$ or $G\mathbf{Top}_*$.

1.2. **Homotopy theory.** The homotopy theory of G-spaces comes in two flavors. There are two classes of maps we may want to invert:

- A *coarse G-equivalence* (or *naïve G-equivalence*) is an equivariant based map $X \longrightarrow Y$ which is a weak homotopy equivalence when we forget the G-action.
- A *fine G-equivalence* (or *genuine G-equivalence*, or even a *G-equivalence*) is an equivariant based map $X \longrightarrow Y$ which when restricted to X^H induces a weak equivalence $X^H \longrightarrow Y^H$, for every closed subgroup $H \leq G$.

The next step is to define cell complexes:

- A *G-cell* is a map of the form
$$(G/H \times S^{n-1})_+ \hookrightarrow (G/H \times D^n)_+$$
for some integer $n \geq 0$ and closed subgroup $H \leq G$. When $H = \{1\}$ this is called a *free G-cell.*
- A *relative G-cell complex* $A \longrightarrow X$ is a map of G-spaces built up from A as a countably infinite sequence of pushouts along coproducts of G-cells:
$$A \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow \operatorname{colim}_n X_n \cong X$$
$$\begin{array}{ccc} \bigvee_{a \in A_k} (G/(H_a) \times S^{n_a - 1})_+ & \longrightarrow & \bigvee_{a \in A_k} (G/(H_a) \times D^{n_a})_+ \\ \downarrow & \ulcorner & \downarrow \\ X_k & \longrightarrow & X_{k+1} \end{array}$$
When $A = *$ is the one-point space, we say X is a *G-cell complex.* When all the subgroups H_a are trivial we call these cell complexes *free.* Of course, if X is a free G-cell complex, then the G-action on X is free everywhere except for the basepoint, which must be fixed.
- A *relative G-CW complex* $A \longrightarrow X$ is a relative G-cell complex where at level $k+1$ we only attach $(k+1)$-dimensional cells:
$$\begin{array}{ccc} \bigvee_{a \in A_k} (G/(H_a) \times S^{k})_+ & \longrightarrow & \bigvee_{a \in A_k} (G/(H_a) \times D^{k+1})_+ \\ \downarrow & \ulcorner & \downarrow \\ X_k & \longrightarrow & X_{k+1} \end{array}$$
If G is a finite group, this is equivalent to the statement that $A \longrightarrow X$ is a relative CW-complex and G acts by permuting the cells of $X - A$.

One of the most basic homotopy-theoretic facts is that each space is weakly equivalent to some CW complex. In the equivariant setting, there are two versions of this fact:

- For each X there is a free G-CW complex Y and a coarse G-equivalence $Y \longrightarrow X$
- For each X there is a G-CW complex Z and a fine G-equivalence $Y \longrightarrow X$

In other words, each X can be replaced by a free G-CW complex, but that destroys its fixed point information. We could preserve the fixed point information if we allow a (not necessarily free) G-CW complex instead. This can be further formalized into two different model structures on the category of G-spaces. (Reference in [MMSS01])

Proposition 1.1. *There is a* coarse *or* naïve *model structure in which*

- *the cofibrations are the retracts of the relative free G-cell complexes*
- *the weak equivalences are the coarse G-equivalences*
- *the fibrations are G-maps $X \longrightarrow Y$ which are Serre fibrations when we forget the G-action*

This model structure is topological and proper. It is cofibrantly generated by the sets of cofibrations and acyclic cofibrations

$$\mathbf{I} = \{(G \times S^{n-1})_+ \hookrightarrow (G \times D^n)_+ : n \geq 0\}$$
$$\mathbf{J} = \{(G \times D^n)_+ \hookrightarrow (G \times D^n \times I)_+ : n \geq 0\}$$

and it is monoidal if G homeomorphic to a cell complex.

Proposition 1.2. *There is a* fine *or* genuine *model structure in which*

- *the cofibrations are the retracts of the relative G-cell complexes*
- *the weak equivalences are the fine G-equivalences*
- *the fibrations are G-maps $X \longrightarrow Y$ for which each $X^H \longrightarrow Y^H$ is a Serre fibration*

This model structure is topological and proper. It is cofibrantly generated by the sets of cofibrations and acyclic cofibrations

$$\mathbf{I} = \{(G/H \times S^{n-1})_+ \hookrightarrow (G/H \times D^n)_+ : n \geq 0, H \leq G\}$$
$$\mathbf{J} = \{(G/H \times D^n)_+ \hookrightarrow (G/H \times D^n \times I)_+ : n \geq 0, H \leq G\}$$

and it is monoidal if G a compact Lie group.

Remark. These categories are tensored and cotensored over based spaces, and this allows us to define homotopies by $X \wedge I_+$. We then define h-cofibrations by requiring that the inclusion $A \wedge I_+ \cup_{A \wedge 0_+} X \wedge 0_+ \longrightarrow X \wedge I_+$ has a retract, and call an object *compact* if maps out commute with a sequential colimit along a sequence of h-cofibrations. In each of the above model categories, the generating cofibrations and the generating acyclic cofibrations have compact domain and are all h-cofibrations. So the domains are, in Quillen's sense, "small objects" relative to not just the inclusions of cells, but all h-cofibrations. This sort of fact is very common in topological settings, and quite convenient when doing constructions like the realization of a simplicial space (and in particular the bar construction).

Remark. Recall that a *monoidal* model category is a category with both a model structure and a monoidal structure, compatible in the following two ways (cf. [Hov07], Ch. 4):

- Either the unit object I is cofibrant, or for every X the maps $QI \otimes X \longrightarrow X$ and $X \otimes QI \longrightarrow X$ are weak equivalences.
- (Pushout-Product Axiom.) For any cofibrations $f : A \longrightarrow X$ and $g : B \longrightarrow Y$, the pushout-product $f \square g : A \otimes Y \cup_{A \otimes B} X \otimes B \longrightarrow X \otimes Y$ is a cofibration, and if one of f, g is a weak equivalence then $f \square g$ is a weak equivalence.

For us a *topological* model category will have a tensoring and cotensoring over spaces, and the pushout-product axiom holds whenever $A \longrightarrow X$ is a cofibration of spaces and

$B \longrightarrow Y$ is a cofibration in our category. This is easy to check and has many convenient consequences: for instance when Y is fibrant, the space of equivariant maps is a functor

$$\mathrm{Map}_*^G(-, Y)$$

that sends cofibrations to fibrations, acyclic cofibrations to acyclic fibrations, and therefore preserves weak equivalences between cofibrant objects.

The claim that the coarse model structure is monoidal is not that easy to find, but it can be proven easily by rearranging

$$(G \times (S^{n-1} \hookrightarrow D^n))\square(G \times (S^{k-1} \hookrightarrow D^k))$$
$$\cong (G \times (S^{nk-1} \hookrightarrow D^{nk}))\square(\emptyset \hookrightarrow G_{\mathrm{triv}})$$

For the fine model structure we need $G/H \times G/K$ to be an unbased G-cell complex for any pair of subgroups $H, K \leq G$ and this is guaranteed if G is a compact Lie group.

1.3. **Elmendorf's theorem.** Now the coarse model structure is obtained by thinking of G-spaces as topological diagrams over a topological category with one object, whose morphism space is the group G. This is often called the *projective* model structure: the weak equivalences and fibrations are determined by forgetting the G-action.

The fine model structure is also a projective model structure, in some sense. To see this, we define the *orbit category* $\mathbf{O}(G)$ to have one object G/H for every closed subgroup $H \leq G$. The morphisms of $\mathbf{O}(G)$ are the maps of G-sets:

$$\mathbf{O}(G)(G/H, G/K) := \mathrm{Map}^G(G/H, G/K)$$

In summary, $\mathbf{O}(G)$ is a full subcategory of unbased G-spaces $\mathbf{Top}^G$ on the objects $\{G/H : H \leq G\}$.

We define a functor

$$G\mathbf{Top}_* \xrightarrow{\Phi} \mathbf{Top}_*^{\mathbf{O}(G)^{\mathrm{op}}}$$

from G-spaces to diagrams of spaces over the opposite of the orbit category. Φ takes a G-space X to the diagram

$$G/H \rightsquigarrow X^H \cong \mathrm{Map}^G(G/H, X) \cong \mathrm{Map}_*^G(G/H_+, X)$$

So this diagram expresses the fixed points of X for all subgroups H, and all the natural maps that we typically expect between these fixed points. Specifically, the category $\mathbf{O}(G)^{\mathrm{op}}$ acts on the fixed points of X by pre-composition

$$\mathrm{Map}^G(G/H, G/K) \times \mathrm{Map}^G(G/K, X) \longrightarrow \mathrm{Map}^G(G/H, X)$$

giving an action

$$\mathbf{O}(G)(G/H, G/K) \times X^K \longrightarrow X^H$$

(If you prefer you may take the equivalent formulation:)

$$\mathrm{Map}_*^G(G/H_+, G/K_+) \wedge \mathrm{Map}_*^G(G/K_+, X) \longrightarrow \mathrm{Map}_+^G(G/H_+, X)$$

The fixed points functor Φ has a left adjoint Θ, which sends a diagram F indexed by $\mathbf{O}(G)^{\mathrm{op}}$ to the based space $F(G/\{1\})$, with G-action given by

$$G \cong \mathrm{Map}^G(G/\{1\}, G/\{1\}) \text{ acting on } X^{\{1\}} = X$$

So we have an adjunction

$$\mathbf{Top}_*^{\mathbf{O}(G)^{\mathrm{op}}} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} G\mathbf{Top}_*$$

The composite $\Theta \circ \Phi$ is naturally isomorphic to the identity, because if we take a G-space, build a system out of its fixed points, and then only remember the points fixed by $\{1\}$, we get our original G-space back up to isomorphism. However, the composite $\Phi \circ \Theta$ is not isomorphic to the identity. It takes a general diagram indexed by $\mathbf{O}(G)^{\mathrm{op}}$ and replaces $F(G/H)$ by $F(G/\{1\})^H$. We can think of these two functors as expressing the category of G-spaces as a full subcategory of diagrams indexed by $\mathbf{O}(G)^{\mathrm{op}}$.

Though Θ and Φ are not equivalences of categories, they form a Quillen equivalence between $\mathbf{Top}_*^{\mathbf{O}(G)^{\mathrm{op}}}$ with the projective model structure and $G\mathbf{Top}_*$ with the genuine model structure. This means in particular that they have equivalent homotopy categories, a fact often referred to as *Elmendorf's Theorem*.

It may be worth pointing out that despite the simplicity of Θ, it may be reinterpreted as a coend. Recall that a coend takes two diagrams, one indexed by $\mathbf{O}(G)$ and one indexed by $\mathbf{O}(G)^{\mathrm{op}}$, and produces a space. Here the coend combines a given diagram $F : \mathbf{O}(G)^{\mathrm{op}} \longrightarrow \mathbf{Top}_*$ with the tautological diagram $J : \mathbf{O}(G) \longrightarrow G\mathbf{Top}_*$ which sends the object $G/H \in \mathbf{O}(G)$ to the G-space G/H_+. The coend of these two diagrams is then a G-space, and in fact it is isomorphic to $F(G/\{1\})$. Once we have this description, we see that the left-derived functor of Θ is a homotopy coend along J, and can be given by a categorical bar construction. (This is Elmendorf's original argument.)

1.4. **How are the coarse and fine model structures related?** There is a Quillen adjunction

$$\boxed{\begin{array}{c} G\text{-spaces } G\mathbf{Top}_* \\ \text{coarse model structure} \end{array}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} \boxed{\begin{array}{c} G\text{-spaces } G\mathbf{Top}_* \\ \text{fine model structure} \end{array}}$$

with the arrow from left to right being the left adjoint. This is a straightforward check of definitions: the functor from right to left preserves all fibrations and weak equivalences, and therefore it preserves all acyclic fibrations. This Quillen adjunction is, of course, not a Quillen equivalence when $G \neq \{1\}$, since the map

$$\mathbb{L}\mathrm{id}(\mathbb{R}\mathrm{id}X) \longrightarrow X$$

is on fixed points

$$* \longrightarrow X^G$$

and this is usually not a weak equivalence.

1.5. **Other adjunctions.** There are two very elementary ways of relating ordinary spaces to G-spaces. First, we may endow an ordinary space with a trivial G-action. Second, we may take a G-space and forget the action of G to get an ordinary space. Each of these operations has both a left adjoint and a right adjoint! We give them below with the left adjoints pointing from left to right.

$$G\text{-spaces } G\mathbf{Top}_* \underset{\text{trivial } G\text{-action}}{\overset{\text{orbits } -_G}{\rightleftarrows}} \text{spaces } \mathbf{Top}_*$$

$$\text{spaces } \mathbf{Top}_* \underset{\text{fixed points } -^G}{\overset{\text{trivial } G\text{-action}}{\rightleftarrows}} G\text{-spaces } G\mathbf{Top}_*$$

$$\text{spaces } \mathbf{Top}_* \underset{\text{forget } G\text{-action}}{\overset{G_+ \wedge -}{\rightleftarrows}} G\text{-spaces } G\mathbf{Top}_*$$

$$G\text{-spaces } G\mathbf{Top}_* \underset{\mathrm{Map}_*(G_+, -)}{\overset{\text{forget } G\text{-action}}{\rightleftarrows}} \text{spaces } \mathbf{Top}_*$$

1.6. **Homotopy orbits and fixed points.** Let's focus on the adjunction between "orbits" and "trivial G-action." This gives a Quillen adjunction when we give G-spaces the coarse model structure:

$$\begin{array}{c} G\text{-spaces } G\mathbf{Top}_* \\ \text{coarse model structure} \end{array} \underset{\text{trivial } G\text{-action}}{\overset{\text{orbits } -_G}{\rightleftarrows}} \begin{array}{c} \text{spaces } \mathbf{Top}_* \\ \text{Quillen model structure} \end{array}$$

This is great, because it means that if we left-derive the orbits we will arrive at a notion of "orbits" which is invariant under all coarse G-equivalences! This left-derived functor $\mathbb{L}(-)_G$ is denoted *homotopy orbits* X_{hG}.

For the rest of the section we will assume that G is homeomorphic to a cell complex, and the inclusion of the identity element $\{1\} \longrightarrow G$ is a cofibration.

We calculate homotopy orbits by replacing a G-space X with a free based G-CW complex ΓX, and then taking orbits. When X has a non-degenerate basepoint, the homotopy orbits are also equivalent to $EG_+ \wedge_G X$ and $B(S^0, G_+, X)$. This is proven by showing that these constructions preserve all coarse G-equivalences between nondegenerately-based spaces, and that when evaluated on a free G-CW complex they are naturally equivalent to the ordinary orbits. If on the other hand X has a degenerate basepoint, or you want a construction that works for all spaces without fail, simply grow a whisker onto X and then apply the above constructions.

Unfortunately the fixed points $(-)^G$ do not form a right Quillen functor in the coarse model structure, so we can't right-derive them to get a notion of fixed points which is invariant under coarse G-equivalence. However, notice that the fixed points of X are isomorphic to the space to maps from the one-point space to X:

$$X^G \cong \mathrm{Map}^G(*, X) \cong \mathrm{Map}_*^G(S^0, X)$$

Because the coarse model structure is monoidal, and X is always fibrant, if we replace S^0 by a free based G-CW complex, the mapping space will preserve all coarse equivalences in the X input. Therefore

$$X^{hG} := \mathrm{Map}_*^G(EG_+, X) \cong \mathrm{Map}^G(EG, X)$$

preserves all coarse equivalences. We call this construction the *homotopy fixed points*. This construction of homotopy fixed points is dual in some sense to one of our constructions of homotopy orbits:

$$X \xrightarrow{\text{free replacement}} EG_+ \wedge X \xrightarrow{\text{orbits}} EG_+ \wedge_G X = X_{hG}$$

$$X \xrightarrow{\text{cofree replacement}} \mathrm{Map}_*(EG_+, X) \xrightarrow{\text{fixed points}} \mathrm{Map}_*^G(EG_+, X) = X^{hG}$$

One may also build homotopy fixed points as the fixed points of the totalization of a cosimplicial complex, dual to the way one builds X_{hG} by taking orbits of the bar construction $B(G_+, G_+, X)$.

There is a third way of thinking about homotopy orbits and fixed points, and it gives an answer isomorphic to the ones we got from the bar complex and the cobar complex. Namely, we think of a G-space X as a diagram over the category with one object, whose morphisms are G. The homotopy colimit of this diagram, using the based version of the Bousfield-Kan construction of homotopy colimits, is *isomorphic* to $B(S^0, G_+, X)$ and therefore gives a valid model for X_{hG}. Similarly, the homotopy limit of this diagram gives X^{hG}.

Finally, the most natural maps that exist between X and its (homotopy) orbits and fixed points are these:

$$X^G \longrightarrow X^{hG} \longrightarrow X \longrightarrow X_{hG} \longrightarrow X_G$$

The exact definitions depend on your chosen model of X_{hG} and X^{hG}. It is common to define maps out of X_{hG} by simply defining a map out of X_G, and similarly we may define a map into X^{hG} by defining a map into X^G. This is also the way hocolims are related to colims, and holims to lims.

Finally a word about calculating homotopy orbits and homotopy fixed points. We can filter $EG_+ \wedge X$ by the skeleta $EG_+^{(k)} \wedge X$. Taking orbits, we get a tower of cofibrations with X_{hG} as the direct limit (or colimit). Such a tower always leads to a spectral sequence on homology or cohomology.

When G is discrete this homology spectral sequence takes the form

$$E^1_{p,q} = H_{p+q}(\Sigma^p_+ G^{\times(p+1)} \wedge X) \cong \mathbb{Z}[G]^{\otimes(p+1)} \otimes_{\mathbb{Z}} H_q(X)$$
$$E^2_{p,q} = \mathrm{Tor}_q^{\mathbb{Z}[G]}(\mathbb{Z}, H_p(X)) \Rightarrow H_*(X_{hG})$$

Dually, we can co-filter the homotopy fixed points by taking only maps out of the k-skeleton $EG_+^{(k)}$ for each $k \geq 0$. This gives a tower of fibrations with X^{hG} at the top, and every such tower gives a spectral sequence on homotopy groups. (Though I won't try to get the E^2 page right on that one!)

1.7. **Examples.**

- If $*$ is the one-point space then
$$*_{hG} \simeq * \quad \text{and} \quad *^{hG} \simeq *$$
- Give S^0 a trivial $\mathbb{Z}/2$-action. Then
$$S^0_{h\mathbb{Z}/2} \simeq \mathbb{RP}^\infty_+ \quad \text{and} \quad (S^0)^{hZ/2} \simeq S^0$$
or in greater generality
$$S^0_{hG} \simeq BG_+ \quad \text{and} \quad (S^0)^{hG} \simeq S^0$$
If this sounds wrong, remember we are taking *based* homotopy orbits, and these are different from *unbased* homotopy orbits. For the purposes of this example let X_{uG} denote the unbased version of homotopy orbits. Then if X is a based G-space, X_{uG} always contains $*_{uG} \simeq BG$ as a retract and there is a cofiber sequence
$$BG \longrightarrow X_{uG} \longrightarrow X_{hG}$$
Of course, X_{uG} is very computable because we can use the Serre spectral sequence on the fiber bundle
$$G \longrightarrow EG \times X \longrightarrow X_{uG}$$
In general people tend to use the notation X_{hG} for both based and unbased homotopy orbits so be careful!
- If S^n is given the trivial G-action then
$$S^n_{hG} \simeq S^n \wedge BG_+ \cong \Sigma^n_+ BG$$
This example illustrates two principles that the reader is encouraged to verify. First, homotopy orbits commute with the operation $\wedge A$ when A is a based space with a trivial G-action. Second, if X is a based G-space which is $(n-1)$-connected when we forget the G-action, then X_{hG} is also $(n-1)$-connected.
- If K is any finite-dimensional complex with a trivial $\mathbb{Z}/p$-action then
$$K^{h\mathbb{Z}/p} \simeq K$$
This is a consequence of the *Sullivan conjecture*.

- Suppose that G is discrete. Let X be any nondegenerately-based space, and let $\bigvee^G X \cong G_+ \wedge X$ denote a wedge of copies of X, one for each element of G. The G-action permutes the copies of X. Then the folding map $\bigvee^G X \longrightarrow X$, which on each wedge summand is just the identity $X \longrightarrow X$, gives equivalences
$$\left(\bigvee^G X\right)_{hG} \xrightarrow{\sim} \left(\bigvee^G X\right)_G \xrightarrow{\cong} X$$
Depending on your taste, you can prove this with an explicit deformation retract, or with model-theoretic techniques.
- If X is any well-based G-space, and $\widetilde{X}$ is the same space with a trivial G-action, then there is an equivariant homeomorphism $G_+ \wedge X \cong G_+ \wedge \widetilde{X}$, and so $(G_+ \wedge X)_{hG} \simeq \widetilde{X}$.
- The previous example dualizes: if X is any based space, let $\prod^G X \cong \mathrm{Map}_*(G_+, X)$ denote a product of copies of X, one for each element of G. Then the diagonal map $X \longrightarrow \prod^G X$ gives equivalences
$$X \xrightarrow{\cong} \left(\prod^G X\right)^G \xrightarrow{\sim} \left(\prod^G X\right)^{hG}$$
- Let $n \geq 2$ and let $X = M(\mathbb{Z}/3, n)$, the Moore space which is $(n-1)$-connected, has homology $\mathbb{Z}/3$ in dimension n and no other nontrivial homology. Then $M(\mathbb{Z}/3, n)$ can be modeled by a free $\mathbb{Z}/2$-CW complex such that the $\mathbb{Z}/2$-action acts on the $\mathbb{Z}/3$ in homology by negation. One way to do this is to take a free $\mathbb{Z}/2$-cell in dimension n and attach to it a free $\mathbb{Z}/2$-cell in dimension $(n+1)$ along a map with degree
$$x - 2 \in \mathbb{Z}[x]/(x^2 = 1) \cong \mathbb{Z}[\mathbb{Z}/2]$$
The cellular chains are as $\mathbb{Z}[x]/(x^2 = 1)$-modules
$$\mathbb{Z}[x]/(x^2 = 1) \xrightarrow{x-2} \mathbb{Z}[x]/(x^2 = 1)$$
or as $\mathbb{Z}$-modules
$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}} \mathbb{Z}^2$$
taking homology leaves
$$0 \longrightarrow \mathbb{Z}/3$$
so we indeed have a Moore space. Quotienting out the free $\mathbb{Z}/2$-action on $M(\mathbb{Z}/3, n)$ leaves us with a CW complex with one n-cell and one $(n+1)$-cell attached along a map of degree -1. (We got this by substituting $x = 1$ into $x - 2$.) Therefore this quotient CW complex is contractible. In summary, we got the bizarre result
$$M(\mathbb{Z}/3, n)_{h\mathbb{Z}/2} \simeq *$$

In fact, if we compare this to the unbased homotopy orbits we get a curious map of two cofiber sequences

$$\begin{array}{ccccc} \mathbb{RP}^\infty_+ & \longrightarrow & \mathbb{RP}^\infty_+ & \longrightarrow & * \\ \| \wr & & \downarrow & & \downarrow \wr \\ \mathbb{RP}^\infty_+ & \longrightarrow & M(\mathbb{Z}/3,n)_{u\mathbb{Z}/2} & \longrightarrow & M(\mathbb{Z}/3,n)_{h\mathbb{Z}/2} \end{array}$$

which is an equivalence on the left and right vertical maps, but *not* an equivalence in the middle! The middle map is an isomorphism on both π_1 and ordinary homology H_*. However on homology with twisted $\mathbb{Z}[\pi_1]$ coefficients, which is the homology of the universal cover, it is not an isomorphism:

$$H_n(\mathbb{RP}^\infty_+; \mathbb{Z}[\pi_1]) \cong 0, \qquad H_n(M(\mathbb{Z}/3,n)_{u\mathbb{Z}/2}; \mathbb{Z}[\pi_1]) \cong \mathbb{Z}/3$$

This example shows that we can't hope to detect the cells of a G-CW complex, or even a free G-CW complex, from the homotopy orbits.

- A variant of the previous example is

$$\left(\bigvee^\infty M(\mathbb{Z}/3,n) \right)_{h\mathbb{Z}/2} \simeq *$$

which proves that the finiteness of X_{hG} does not imply anything about the finiteness of X.

- We will provide an example of a finite G-CW complex X which is nonequivariantly contractible, but its fixed points are nontrivial. In particular $X \longrightarrow *$ will be an equivariant map of finite G-CW complexes which is a nonequivariant equivalence but not an equivariant equivalence. Of course, if we allow X to be infinite, this is easy, we can just take $X = EG$. The case of finite X is considerably more subtle.

  Set $G = C_2$. We first remark that our example X must have the property that X^{C_2} has homology that is entirely torsion prime to 2, since otherwise $\Sigma(X^{C_2})_{hC_2} \simeq \Sigma X^{C_2} \wedge \mathbb{RP}^\infty_+$ would be infinite, but by the cofiber sequence

$$(X^{C_2})_{hC_2} \longrightarrow X_{hC_2} \simeq * \longrightarrow (X/X^{C_2})_{hC_2} \xrightarrow{\sim} \Sigma(X^{C_2})_{hC_2}$$

  it would have to be finite, which is a contradiction.

  Now for our example. Set $X^{C_2} = M(\mathbb{Z}/3,n)$ with $n \geq 2$. Attach an $n+1$-cell along a map of degree 1, which is possible because $\pi_n(M(\mathbb{Z}/3,n)) \cong \mathbb{Z}/3$, and then extend this to a free C_2-cell. The resulting space $X^{(n+1)}$ has $H_n = 0$ and H_{n+1} equal to the kernel of $(1,1,3) : \mathbb{Z}^3 \longrightarrow \mathbb{Z}$. We check $(-1,-2,1)$ and $(-2,-1,1)$ form both a $\mathbb{Z}/2$-invariant integral basis for this lattice. So if we attach a $n+2$-cell along a map whose image in

$$\pi_{n+1}X^{(n+1)} = H_{n+1}X^{(n+1)} = \ker\{(1,1,3)\}$$

  is $(-1,-2,1)$, and prolong it to a free C_2-cell, the result will kill the remaining homology. Call the final result X. Then X has vanishing homology and so is contractible as a nonequivariant space, but its C_2-fixed points are $M(\mathbb{Z}/3, n)$ which are not contractible!

2. G-spectra on the nose

In the last section we allowed G to be a topological group, but now we will restrict ourselves and only allow G to be a compact Lie group. As before, whenever we take a subgroup $H \leq G$ we assume that it is closed.

Definition 2.1. An orthogonal G-spectrum is a sequence of based spaces $\{X_n\}_{n=0}^{\infty}$ equipped with

- A continuous action of $G \times O(n)$ on X_n for each n
- A G-equivariant structure map $\Sigma X_n \longrightarrow X_{n+1}$ for each n

such that the composite

$$S^p \wedge X_n \longrightarrow \ldots \longrightarrow S^1 \wedge X_{(p-1)+n} \longrightarrow X_{p+n}$$

is $O(p) \times O(n)$-equivariant.

When $G = \{1\}$ is the trivial group, this is just the definition of an *orthogonal spectrum*. Notice that an orthogonal G-spectrum is just an orthogonal spectrum that has been equipped with a G-action commuting with both the structure maps and the $O(n)$-actions.

Now that we have defined our objects, we define our maps. Let X and Y be orthogonal G-spectra. Then a *G-map* from X to Y is a collection of maps $X_n \longrightarrow Y_n$ which commute with the $G \times O(n)$-action and which commute with the structure map:

$$\begin{array}{ccc} \Sigma X_n & \longrightarrow & X_{1+n} \\ \downarrow & & \downarrow \\ \Sigma Y_n & \longrightarrow & Y_{1+n} \end{array}$$

These objects and maps give the category of orthogonal G-spectra, denoted $G\mathbf{Sp}^O$.

The set of all G-maps can be made into a space $\mathrm{Map}^G(X,Y)$. It is topologized as a subset of the product

$$\prod_n \mathrm{Map}_*^G(X_n, Y_n) \subset \prod_n \mathrm{Map}_*(X_n, Y_n)$$

A *(not necessarily equivariant) map* from X to Y is a collection of maps $X_n \longrightarrow Y_n$ which commute with the $O(n)$-action and which commute with the structure maps. These also form a space $\mathrm{Map}(X,Y)$, topologized as a subset of the product

$$\prod_n \mathrm{Map}_*(X_n, Y_n)$$

As before, this space of maps inherits a G-action by conjugation. The G-fixed maps are exactly the equivariant maps:

$$(\mathrm{Map}(X,Y))^G = \mathrm{Map}^G(X,Y)$$

Now the mapping space $\mathrm{Map}(X,Y)$ is naturally the 0th level of a *mapping spectrum*, which we will denote $F(X,Y)$. The nth level of $F(X,Y)$ is defined as a subspace of a product of mapping spaces:

$$F(X,Y)_n \subset \prod_m \mathrm{Map}_*(X_m, Y_{m+n})$$

It consists of all collections of maps $X_m \longrightarrow Y_{m+n}$ which are $O(m)$-equivariant and which commute with the structure maps of X and Y:

$$\begin{array}{ccc} \Sigma X_m & \longrightarrow & X_{1+m} \\ \downarrow & & \downarrow \\ \Sigma Y_{m+n} & \longrightarrow & Y_{1+m+n} \end{array}$$

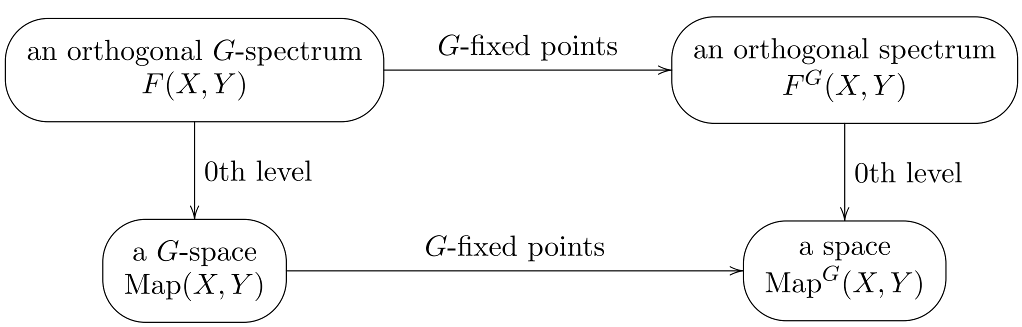
The $O(n)$-action on $F(X,Y)_n$ is through the last n coordinates of the $O(m+n)$-action on Y_{m+n}. The structure map for $F(X,Y)$ takes the extra suspension coordinate and appends it to the Y_{m+n}, applies the structure map of Y, then applies a permutation to Y_{1+m+n} which shuffles the first coordinate past the next m coordinates:

$$\Sigma\mathrm{Map}_*(X_m, Y_{m+n}) \longrightarrow \mathrm{Map}_*(X_m, \Sigma Y_{m+n}) \longrightarrow \mathrm{Map}_*(X_m, Y_{1+m+n}) \longrightarrow \mathrm{Map}_*(X_m, Y_{m+1+n})$$

Of course, when we say that a permutation acts on Y_{1+m+n}, we are thinking about the discrete subgroup of $O(1+m+n)$ consisting of those orthogonal maps which permute the coordinate axes in $\mathbb{R}^{1+m+n}$.

Now that we have constructed the orthogonal spectrum $F(X,Y)$, we simply observe that G acts on the spaces $\mathrm{Map}_*(X_m, Y_{m+n})$ by conjugation, and this makes $F(X,Y)$ into an orthogonal G-spectrum. Taking the G-fixed points of every level gives an ordinary orthogonal spectrum, which we denote $F^G(X,Y)$. Not surprisingly, $F^G(X,Y)$ could also be constructed by following the above recipe, but insisting that all the maps $X_m \longrightarrow Y_{m+n}$ preserve the G-action.

In short, orthogonal G-spectra are enriched in four different compatible ways. Between any two orthogonal G-spectra X and Y, we have a mapping object which is:

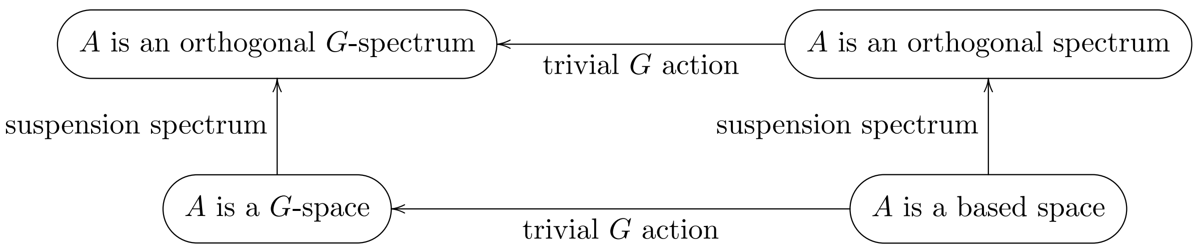


There is also a notion of smash product $X \wedge Y$ of orthogonal spectra. We won't define it explicitly here because there are already many good sources (e.g. [**?**]). The smash product and the mapping spectrum make orthogonal G-spectra $G\mathbf{Sp}^O$ into a closed symmetric monoidal category. In particular we have the adjunction you would expect

$$\mathrm{Map}(X \wedge Y, Z) \cong \mathrm{Map}(X, F(Y, Z))$$

and so the smash product determines and is determined by the mapping spectrum construction $F(Y, Z)$ up to canonical isomorphism. Usually the smash product gets all the attention, but for our purposes here the mapping spectrum will be much more important.

One can also make sense of $A \wedge X$ and $F(A, X)$ when A is an ordinary orthogonal spectrum, or just a G-space, or just a space. Of these four possibilities, if A is in a simpler category then we may promote it to a more sophisticated category by the following rules



and the objects $A \wedge X$ and $F(A, X)$ will be unchanged up to isomorphism. It is not a coincidence that the functors pictured here are the left adjoints of the functors in the previous diagram.

For concreteness, we remark that if A is a G-space then $A \wedge X$ is the orthogonal G-spectrum which at level n is $A \wedge X_n$ with the diagonal G-action, and $F(A, X)$ is the orthogonal G-spectrum which at level n is $\mathrm{Map}_*(A, X_n)$ with the conjugation G-action.

3. Homotopy theory of G-spectra

Now that our point-set category of G-spectra is up and running, our next task is to put a model structure on it. We'll begin by recalling the nonequivariant case $G = \{1\}$. From [MMSS01], we may define homotopy groups π_* of orthogonal spectra in the usual way

$$\pi_k(X) = \operatorname*{colim}_{n \longrightarrow \infty} \pi_{n+k}(X_n), \qquad k \in \mathbb{Z}, \quad n \geq \max(0, -k)$$

and there is a good model structure whose weak equivalences are exactly the maps inducing an isomorphism on π_*. These maps are also called *stable equivalences.* To define this model structure completely, we just need a good notion of "cell" in the stable setting.

First observe that there is a forgetful functor from orthogonal spectra to based spaces

$$Ev_n : X \rightsquigarrow X_n$$

Its left adjoint F_n takes a based space A and constructs a *free orthogonal spectrum on A at level n.* Concretely, $F_n(A)$ is an orthogonal spectrum which at level k is

$$F_n(A)_k = (S^{k-n} \wedge_{O(k-n)} O(k)_+) \wedge A$$

From this formula it is easy to see that $F_n(A) \cong F_n(S^0) \wedge A$. In other words, we may construct the spectrum $F_n(S^0)$ once and for all, and then obtain $F_n(A)$ by smashing in the based space A at every level of $F_n(S^0)$. Despite the disturbing complexity of the above formula for $F_n(S^0)$, its underlying prespectrum is equivalent to a shift of the sphere spectrum:

$$\pi_k(F_n(S^0)) \cong \pi_{k+n}(\mathbb{S})$$

Now a *stable cell* is simply the functor F_n applied to an unstable cell

$$F_n(S^{k-1}_+) \hookrightarrow F_n(D^k_+)$$

We think of F_n as subtracting n from the dimension, so the above map is used to attach a cell whose dimension is $(k - n)$. Finally, a *relative cell complex spectrum* $A \longrightarrow X$ is a countable sequence of pushouts along coproducts of cells:

$$A \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow \operatorname{colim}_n X_n \cong X$$

$$\begin{array}{ccc} \bigvee_{a \in A_k} F_{n_a}(S^{k_a - 1}_+) & \longrightarrow & \bigvee_{a \in A_k} F_{n_a}(D^{k_a}_+) \\ \downarrow & \ulcorner & \downarrow \\ X_k & \longrightarrow & X_{k+1} \end{array}$$

Proposition 3.1. *The category of orthogonal spectra has a* stable model structure *in which*

- *The cofibrations are the retracts of the relative cell complex spectra.*
- *The weak equivalences are the π_*-isomorphisms.*

- *The fibrations are the maps $X \longrightarrow Y$ for which each level $X_n \longrightarrow Y_n$ is a Serre fibration and each square*

$$\begin{array}{ccc} X_n & \longrightarrow & \Omega X_{n+1} \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & \Omega Y_{n+1} \end{array}$$

is a homotopy pullback square.

This model structure is topological, proper, and monoidal. It is compactly generated by the maps

$$\mathbf{I} = \{F_n(S^{k-1}_+) \hookrightarrow F_n(D^k_+) : n, k \geq 0\}$$
$$\mathbf{J} = \{F_n(D^k_+) \hookrightarrow F_n((D^k \times I)_+) : n, k \geq 0\}$$
$$\cup\{(F_n(S^{k-1}_+) \hookrightarrow F_n(D^k_+))\square(F_1(S^1) \hookrightarrow \mathrm{Cyl}(F_1 S^1 \longrightarrow F_0 S^0)))\}$$

where $\square$ denotes the pushout-product.

3.1. **Two "naïve" model structures.** Next we'll make this equivariant following [MM02]. When X is a G-spectrum we still have homotopy groups

$$\pi_k(X) = \operatorname*{colim}_{n \longrightarrow \infty} \pi_{n+k}(X_n), \qquad k \in \mathbb{Z}, \quad n \geq \max(0, -k)$$

but we may also take levelwise fixed points

$$X^H = F^G(G/H_+, X), \qquad (X^H)_n = (X_n)^H$$

and take homotopy groups of the result:

$$\widetilde{\pi}^H_k(X) = \operatorname*{colim}_{n \longrightarrow \infty} \pi_{n+k}(X^H_n), \qquad k \in \mathbb{Z}, \quad n \geq \max(0, -k)$$

We will call these the *naïve equivariant homotopy groups.*

The forgetful functor

$$Ev_n : X \rightsquigarrow X_n$$

now goes from orthogonal G-spectra to based G-spaces. Its left adjoint F_n is still given by the formula

$$F_n(A)_k = (S^{k-n} \wedge_{O(k-n)} O(k)_+) \wedge A \cong F_n(S^0) \wedge A$$

where G acts on A and not on $F_n(S^0)$. A *(naïve) stable G-cell* is simply the functor F_n applied to an unstable G-cell

$$F_n((G/H \times S^{k-1})_+) \hookrightarrow F_n((G/H \times D^k)_+)$$

A *relative (naïve) G-cell complex spectrum* $A \longrightarrow X$ is a countable sequence of pushouts along coproducts of G-cells:

$$A \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow \mathrm{colim}_n X_n \cong X$$

$$\begin{array}{ccc} \bigvee_{a\in A_k} F_{n_a}((G/(H_a)\times S^{k_a-1})_+) & \longrightarrow & \bigvee_{a\in A_k} F_{n_a}((G/(H_a)\times D^{k_a})_+) \\ \downarrow & \ulcorner & \downarrow \\ X_k & \longrightarrow & X_{k+1} \end{array}$$

The following two model structures parallel the ones we discussed for spaces in the first section. (Reference [MM02].)

Proposition 3.2. *The category of orthogonal spectra has a* coarse model structure *in which*

- *The cofibrations are the retracts of the relative (naïve) free G-cell complex spectra.*
- *The weak equivalences are determined by forgetting the G action: they are the maps inducing isomorphisms on π_*. We sometimes call these* coarse stable equivalences*.*
- *The fibrations are determined by forgetting the G action: they are the maps for which each level $X_n \longrightarrow Y_n$ is a Serre fibration and each square*

$$\begin{array}{ccc} X_n & \longrightarrow & \Omega X_{n+1} \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & \Omega Y_{n+1} \end{array}$$

  is a homotopy pullback square.

This model structure is topological, proper, and monoidal. It is compactly generated by the maps

$$\mathbf{I} = \{F_n((G\times S^{k-1})_+) \hookrightarrow F_n((G\times D^k)_+) : n,k\geq 0\}$$
$$\mathbf{J} = \{F_n((G\times D^k)_+) \hookrightarrow F_n((G\times D^k\times I)_+) : n,k\geq 0\}$$
$$\cup\{(F_n((G\times S^{k-1})_+) \hookrightarrow F_n((G\times D^k)_+))\square(F_1(S^1)\hookrightarrow \mathrm{Cyl}(F_1S^1\longrightarrow F_0S^0)))\}$$

where $\square$ denotes the pushout-product.

Proposition 3.3. *The category of orthogonal spectra has a* trivial-universe model structure *in which*

- *The cofibrations are the retracts of the relative (naïve) G-cell complex spectra.*
- *The weak equivalences are the maps inducing isomorphisms on the naïve equivariant homotopy groups $\widetilde{\pi}_*^H$ for all closed subgroups $H\leq G$. We sometimes call these* trivial-universe stable equivalences*.*

- *The fibrations are the maps for which each level fixed point map* $X_n^H \longrightarrow Y_n^H$ *is a Serre fibration and each square*

$$\begin{array}{ccc} X_n^H & \longrightarrow & \Omega X_{n+1}^H \\ \downarrow & & \downarrow \\ Y_n^H & \longrightarrow & \Omega Y_{n+1}^H \end{array}$$

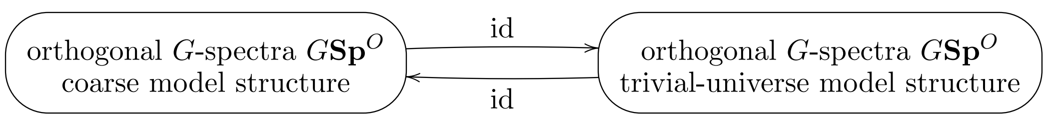
  is a homotopy pullback square.

This model structure is topological, proper, and monoidal. It is compactly generated by the maps

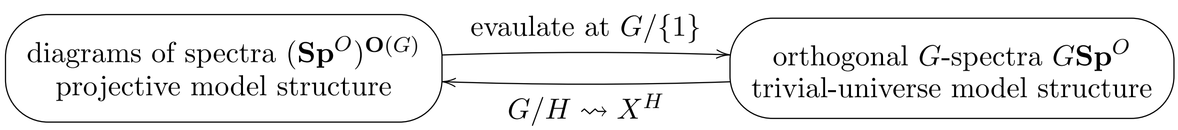
$$\mathbf{I} = \{F_n((G/H \times S^{k-1})_+) \hookrightarrow F_n((G/H \times D^k)_+) : n, k \geq 0, H \leq G\}$$
$$\mathbf{J} = \{F_n((G/H \times D^k)_+) \hookrightarrow F_n((G/H \times D^k \times I)_+) : n, k \geq 0, H \leq G\}$$
$$\cup\{(F_n((G/H \times S^{k-1})_+) \hookrightarrow F_n((G/H \times D^k)_+))\square(F_1(S^1) \hookrightarrow \mathrm{Cyl}(F_1 S^1 \longrightarrow F_0 S^0)))\}$$

where $\square$ *denotes the pushout-product.*

The relationship between the two model structures above is essentially the same as it was for spaces. The identity functors give a Quillen adjunction

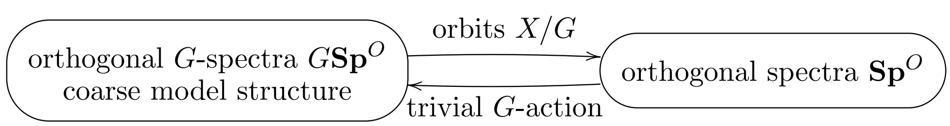


which is not a Quillen equivalence. The fixed point functor is the right leg of a Quillen equivalence



and this is the stable form of Elmendorf's theorem.

3.2. **Homotopy orbits and homotopy fixed points.** If X is an orthogonal G-spectrum then we can take the orbits of X levelwise, giving a spectrum X/G. This gives a left Quillen adjoint



As it was for spaces, so shall it be for spectra. The left derived functor of orbits is called *homotopy orbits* X_{hG}, and it can be calculated by applying the construction $EG_+ \wedge_G -$ or $B(S^0, G_+, -)$ to every level of X. When G is discrete, it does not matter here whether the levels of X are nondegenerately based. (That's probably also true when G is a Lie group.)

We define the *homotopy fixed points* X^{hG} by taking a fibrant replacement RX and applying $\mathrm{Map}_*^G(EG_+, -)$ levelwise. In other words,

$$X^{hG} := F^G(EG_+, RX)$$

Everything from our space-level discussion applies here as well. One should be careful and remember that homotopy orbits commute with suspension spectrum while homotopy fixed points do not:

$$\Sigma^\infty(X_{hG}) \xrightarrow{\cong} (\Sigma^\infty X)_{hG}$$

$$\Sigma^\infty(X^{hG}) \xrightarrow{\not\simeq} (\Sigma^\infty X)^{hG}$$

As a specific example

$$\pi_0(\Sigma_+^\infty((S^0)^{h\mathbb{Z}/p})) \cong \pi_0(\mathbb{S}) \cong \mathbb{Z}, \qquad \pi_0(\mathbb{S}^{h\mathbb{Z}/p}) \cong \mathbb{Z} \oplus \mathbb{Z}_p^\wedge$$

3.3. **A model structure that accomodates Poincaré duality.** Now suppose we want to do an equivariant version of Poincaré duality. It makes sense to suppose that this would come from Atiyah duality, which in the non-equivariant case looks like

$$\Sigma^{-N} M^\nu \simeq F(M_+, \mathbb{S})$$

To make sense of this we need a Euclidean space $\mathbb{R}^N$ big enough that we can embed M inside, and take the normal bundle ν and the Thom space M^ν. To get a spectrum equivalent to $F(M_+, \mathbb{S})$, we finally need to be able to desuspend N times.

How does this work if M is a G-manifold? We can't equivariantly embed M into $\mathbb{R}^n$, so instead we embed it into a representation V. Here V is a G-representation, which for us will always mean a finite-dimensional real inner product space with an orthogonal (= inner-product-preserving) action of G. "Desuspending" by V should mean inverting the operation of smashing with S^V, the one-point compactification of V. It is easy to see that $S^V \wedge -$ has as its right adjoint

$$\Omega^V(-) = \mathrm{Map}^G(S^V, -)$$

but the question is whether these give inverse equivalences on the homotopy category. Finally, since we can't destroy the fixed-point information of the manifold M, we need to keep track of the fixed points of our stable objects.

To summarize, we need a model structure on orthogonal G-spectra which has:

(1) A theory of fixed points.
(2) Smashing with a representation sphere S^V is invertible.

The coarse model structure satisfies (2) but not (1), and the trivial-universe model structure satisfies (1) but not (2). This motivates the search for a third model structure, one where nontrivial G-representations are built in.

Before we get there, we'll show how to evaluate orthogonal spectra at representations. Let U be a complete G-universe. That is, $U \cong \mathbb{R}^\infty$ is a countably-infinite dimensional real inner product space with an orthogonal G-action, which is isomorphic to a direct sum of infinitely many copies of all of the irreducible representations of G. We also suppose that there is a canonical copy of $\mathbb{R}^\infty = \operatorname{colim}_n \mathbb{R}^n$ with the trivial G-action sitting inside U.

Fix one such U and consider the category $\mathbf{J}_G$ whose objects are (finite-dimensional) G-representations $V \subset U$. Between two such objects V and W, we define the mapping space

$$\mathbf{J}_G(V, W) = O(V, W)^{W-V}$$

Here $O(V, W)$ is the space of linear isometric inclusions $i : V \longrightarrow W$, and the above is the Thom space of the bundle over $O(V, W)$ whose fiber over i is the orthogonal complement $i(V)^\perp \subset W$. These mapping spaces are based G-spaces, and in fact $\mathbf{J}_G$ is a category enriched in based G-spaces.

Let $(\mathbf{Top}_*)_G$ denote the category of G-spaces and nonequivariant maps; this is also enriched in based G-spaces. Sometimes categories like $\mathbf{J}_G$ and $(\mathbf{Top}_*)_G$ are called *G-categories*. Anyway, now let

$$X : \mathbf{J}_G \longrightarrow (\mathbf{Top}_*)_G$$

be a functor which is enriched over G-spaces. This means that X sends each representation V to a G-space $X(V)$, and each pair of representations V and W to an equivariant map of G-spaces

$$O(V, W)^{W-V} \longrightarrow \operatorname{Map}_*(X(V), X(W))$$

In particular each linear isometric inclusion $i : V \longrightarrow W$ gives a structure map

$$S^{W-V} \wedge X(V) \longrightarrow X(W)$$

and this structure map is equivariant if $i : V \longrightarrow W$ is equivariant.

Now if we take one such functor X and unwind these definitions, the spaces $X(\mathbb{R}^n)$ actually form the levels of an orthogonal G-spectrum. On the other hand, when $n = \dim V$, the space $X(V)$ is homeomorphic to $X(\mathbb{R}^n)$ and may be canonically recovered via

$$X(V) \cong X(\mathbb{R}^n) \wedge_{O(n)} O(\mathbb{R}^n, V)_+$$

These two operations give an equivalence of categories between orthogonal G-spectra and enriched diagrams $\mathbf{J}_G \longrightarrow (\mathbf{Top}_*)_G$. We can go back and forth without losing *point-set level* information about the spectrum.

Now that our spectra have levels for every representation, and not just every nonnegative integer, it makes sense to define our homotopy groups using representations as levels in the colimit system:

$$\pi_k^H(X) = \begin{cases} \operatorname*{colim}_{V\subset U} \pi_k((\Omega^V X(V))^H) = \operatorname*{colim}_{V\subset U} \pi_k(\mathrm{Map}_*^H(S^V, X(V))), & k \geq 0 \\ \operatorname*{colim}_{V\subset U} \pi_0((\Omega^{V-\mathbb{R}^{|k|}} X(V))^H) = \operatorname*{colim}_{V\subset U} \pi_0(\mathrm{Map}_*^H(S^{V-\mathbb{R}^{|k|}}, X(V))), & k < 0, \mathbb{R}^k \subset V \end{cases}$$

Here the colimits are over a diagram which has one object for each finite-dimensional $V \subset U$ and one arrow $V \longrightarrow W$ iff $V \subset W$ as subsets of U.

Though this definition of $\pi_k^H(X)$ seems completely unsuitable for computation, these groups actually turn out to have very nice formal properties. As a quick example, they always turn finite products or arbitrary coproducts of G-spectra into direct sums of homotopy groups. We will also see that these homotopy groups play well with cofiber/fiber sequences, pushout/pullback squares, homotopy colimits and finite homotopy limits of spectra. As a result, they are not really much harder to compute with then nonequivariant π_*.

Next, it makes sense to expand our definition of "stable G-cell" to allow desuspension by representations. For any representation V, the forgetful functor

$$Ev_V : X \rightsquigarrow X(V)$$

has a left adjoint F_V, the *free spectrum on a based space A at level V*. This free spectrum can be described concretely as

$$F_V(A)(W) = A \wedge \mathbf{J}_G(V, W) = A \wedge O(V, W)^{W-V}$$

Now we take our G-cells to be maps of the form

$$F_V((G/H \times S^{k-1})_+) \hookrightarrow F_V((G/H \times D^k)_+)$$

and we take our relative G-cell complexes to be maps $A \longrightarrow X$ of the form

$$A \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow \operatorname{colim}_n X_n \cong X$$

$$\begin{array}{ccc} \bigvee_{a\in A_k} F_{V_a}((G/(H_a) \times S^{k_a-1})_+) & \longrightarrow & \bigvee_{a\in A_k} F_{V_a}((G/(H_a) \times D^{k_a})_+) \\ \downarrow & \ulcorner & \downarrow \\ X_k & \longrightarrow & X_{k+1} \end{array}$$

Proposition 3.4. *The category of orthogonal spectra has a* complete-universe model structure *in which*

- *The cofibrations are the retracts of the G-cell complex spectra defined just above.*
- *The weak equivalences are the maps inducing isomorphisms on π_*^H as defined above. We sometimes call these* complete-universe stable equivalences, *or* genuine stable equivalences.

- *The fibrations the maps for which each level fixed point map $X(V)^H \longrightarrow Y(V)^H$ is a Serre fibration and each square*

$$\begin{array}{ccc} X(V)^H & \longrightarrow & (\Omega^W X(V+W))^H \\ \downarrow & & \downarrow \\ Y(V)^H & \longrightarrow & (\Omega^W Y(V+W))^H \end{array}$$

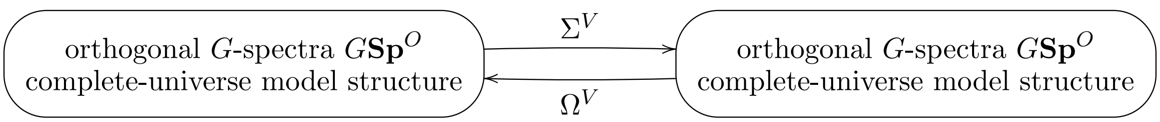
  is a homotopy pullback square.

This model structure is topological, proper, and monoidal. It is compactly generated by the maps

$$\mathbf{I} = \{F_V((G/H \times S^{k-1})_+) \hookrightarrow F_V((G/H \times D^k)_+) : k \geq 0, H \leq G, V \subset U\}$$
$$\mathbf{J} = \{F_V((G/H \times D^k)_+) \hookrightarrow F_V((G/H \times D^k \times I)_+) : k \geq 0, H \leq G, V \subset U\}$$
$$\cup\{(F_V((G/H \times S^{k-1})_+) \hookrightarrow F_V((G/H \times D^k)_+))\square(F_W(S^W) \hookrightarrow \mathrm{Cyl}(F_W S^W \longrightarrow F_0 S^0)))\}$$

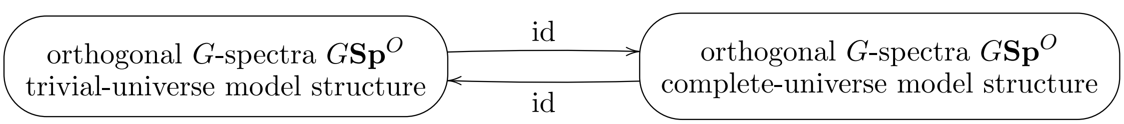
where $\square$ denotes the pushout-product.

Of course, the most important result for this model structure is that Σ^V and Ω^V give inverse Quillen equivalences

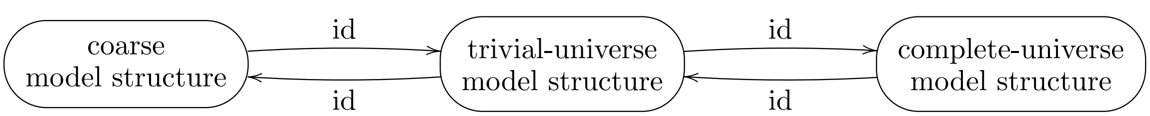


and even before deriving these functors the maps $X \longrightarrow \Omega^V\Sigma^V X$ and $\Sigma^V\Omega^V X \longrightarrow X$ are complete-universe equivalences. So the complete-universe model structure provides a setting for equivariant Atiyah duality. We will develop this to a great extent in section 5 below, proving duality results, defining transfers, and using them to calculate π_*^G in many cases of interest.

3.4. **Relationships between the three model structures.** The identity functors give a Quillen adjunction



which is not a Quillen equivalence. In summary we now have two composable Quillen adjunctions



where the maps from left to right are the left adjoints. This setup is very convenient, but the reader is warned that the identity functor must be derived in order to be homotopically meaningful. The only situations where we can get away with not deriving the identity is in passing down to the coarse model structure from either of the two others. Specifically, we get the inclusions

$$\text{coarse stable equivalences} \supset \text{trivial-universe stable equivalences}$$
$$\text{coarse stable equivalences} \supset \text{complete-universe stable equivalences}$$
$$\text{trivial-universe stable equivalences} \not\subset \text{complete-universe stable equivalences}$$
$$\text{trivial-universe stable equivalences} \not\supset \text{complete-universe stable equivalences}$$

We'll provide counterexamples to show that the trivial and complete universes give incomparable classes of stable equivalences. Consider the map of coordinate-free orthogonal spectra

$$X(V) = S^V \longrightarrow Y(V) = \operatorname*{colim}_{V \subset W \subset U} \Omega^{W-V} S^W$$

Then $X \longrightarrow Y$ is, by construction, a complete-universe equivalence. However it is not a trivial-universe equivalence because when we take naïve homotopy groups $\widetilde{\pi}_0^G$ we get

$$\pi_0(\Omega^\infty S^\infty) \longrightarrow \pi_0(\Omega^{\infty\rho_G} S^{\infty\rho_G})$$

We will see in the section on tom Dieck splitting that this map of groups is

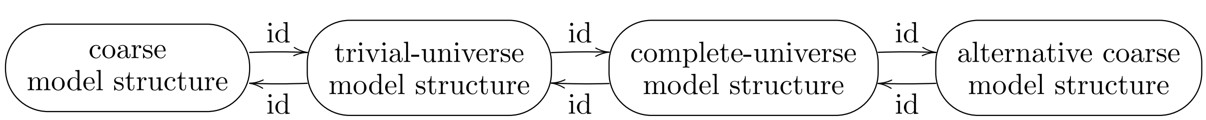
$$\mathbb{Z} \longrightarrow \bigoplus_{(H) \leq G} \mathbb{Z}$$

which is not an isomorphism.

Going the other way, let $G = \mathbb{Z}/2$, let σ denote the sign representation, let $X = F_\sigma S^0$, and let $cX \longrightarrow X$ denote cofibrant replacement in the trivial-universe model structure. Then cX has tom Dieck splitting (see the section on tom Dieck splitting below) but X does not. This implies that the trivial-universe equivalence $cX \longrightarrow X$ is not a complete-universe equivalence.

It can be useful to think of coarse spectra as a full subcategory of complete-universe spectra, and trivial-universe spectra as a subcategory which is not full. To make this precise you have to take cofibrant replacement in the appropriate model structure before passing up to the complete universe. For coarse spectra, this cofibrant replacement makes your spectrum a free G-cell complex at every level. For trivial-universe spectra, they are replaced by G-cell complexes whose cells are never desuspended by nontrivial representations. All complete-universe spectra can be "built" from trivial desuspensions of G/H_+ but only if you allow nontrivial transfer maps that do not show up in the trivial-universe homotopy category.

One can construct a left Bousfield localization of the complete-universe model structure to invert all of the coarse equivalences. This gives a chain of left Quillen functors



whose composite is a Quillen equivalence. See [MM02], IV.6 for more details.

3.5. Homotopy fibers, cofibers, limits, colimits, pullbacks, and pushouts. We have defined coarse equivalences, trivial-universe equivalences, and complete-universe equivalences of orthogonal G-spectra. All three of these notions are extremely well-behaved with respect to standard ways of gluing spectra together.

Let $f : X \longrightarrow Y$ be a map of G-spectra. The *homotopy cofiber* Cf is an orthogonal G-spectrum obtained by applying the mapping cone construction levelwise:

$$(Cf)_n = X_n \wedge I \cup_{X_n \times 1} Y_n$$

Here $I = [0, 1]$ is given the basepoint 0. The *homotopy fiber* Ff is obtained by applying the mapping co-cone levelwise:

$$(Ff)_n = X_n \times_{\mathrm{Map}(1,Y_n)} \mathrm{Map}_*(I, Y_n)$$

All of the notions of "homotopy groups" from the previous section then carry long exact sequences

$$\ldots \longrightarrow \pi_{k+1}(Cf) \longrightarrow \pi_k(X) \longrightarrow \pi_k(Y) \longrightarrow \pi_k(Cf) \longrightarrow \ldots$$
$$\ldots \longrightarrow \pi_k(Ff) \longrightarrow \pi_k(X) \longrightarrow \pi_k(Y) \longrightarrow \pi_{k-1}(Ff) \longrightarrow \ldots$$

and so Cf and Ff are homotopical constructions with respect to all three kinds of weak equivalence. Furthermore the a natural map $Ff \longrightarrow \Omega Cf$ is an equivalence in all three model structures, so the maxim "cofiber and fiber sequences are the same" holds in every case.

From this it follows quickly that the natural map from a finite wedge to a finite product

$$X_1 \vee \ldots \vee X_n \longrightarrow X_1 \times \ldots \times X_n$$

is a stable equivalence in all three senses. It is natural to ask what happens when all the X_i are the same but G acts by permuting the factors of this wedge around; in that case the obvious map

$$G_+ \wedge X \cong \bigvee_G X \longrightarrow \prod_G X \cong F(G_+, X)$$

is a complete-universe equivalence (see the *Wirthmuller isomorphism* below).

Similarly, we may take a square of orthogonal G-spectra

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

We call it a *homotopy pushout square* if the natural map

$$B \cup_{A\times 0} (A \wedge I_+) \cup_{A\times 1} C \longrightarrow D$$

is an equivalence, and a *homotopy pullback square* if the natural map

$$A \longrightarrow B \times_D \mathrm{Map}(I, D) \times_D C$$

is an equivalence. As in the nonequivariant case, these two notions coincide. A square is a coarse homotopy pushout iff it's a coarse homotopy pullback, and similarly for trivial-universe and complete-universe equivalences.

In general, given a small category $\mathbf{I}$ and a diagram $F : \mathbf{I} \longrightarrow G\mathbf{Sp}^O$ we can define its homotopy colimit as the usual Bousfield-Kan construction applied levelwise. This is the same as the "categorical bar contruction" applied to the diagram F:

$$\operatorname*{hocolim}_{\mathbf{I}} F = B(S^0, \mathbf{I}, F)$$

In some sources this is called an *uncorrected homotopy colimit* because it is considered good etiquette to make the spectra in your diagram cofibrant before applying this construction. However the above construction preserves *all* coarse equivalences, trivial-universe equivalences, and complete-universe equivalences, so it is completely unnecessary to make the spectra cofibrant before plugging them in. In other words, the uncorrected hocolim always gives the correct answer: it is always *derived* (or *homotopical*) with respect to any of our notions of stable equivalence. We don't even need hypotheses about nondegenerate basepoints! The usual spectral sequences can be used to calculate π_*^H of the homotopy colimit, using any sense of π_*^H from above.

The same is true for the homotopy limit so long as the category $\mathbf{I}$ has only finitely many distinct strings of composable arrows. So for example based loops $\Omega(-)$ and homotopy pullbacks are always derived if we just apply them levelwise.

However, as soon as $\mathbf{I}$ contains infinitely many distinct strings of composable arrows, it becomes essential to make our spectra fibrant first before taking the homotopy limit. The precise nature of fibrant replacement will vary a lot depending on which model structure we are working in. Even infinite products of spectra need to be derived! (If this comes as a surprise, consider the case $G = \{1\}$ and the fact that infinite products do not commute with the sequential colimits used to define π_* of a spectrum!)

This caveat ties in with our earlier discussion about homotopy fixed points X^{hG}, when we remarked that applying the construction levelwise tends to give the wrong answer when X is not fibrant. (In particular, it does not preserve any of our three notions of stable equivalence.)

3.6. **In summary, complete-universe equivalences are preserved by...**

- homotopy colimits, including
  - homotopy cofibers
  - homotopy pushouts
  - arbitrary coproducts
  - mapping telescopes
  - smashing $(-) \wedge A$ when A is a based nonequivariant cell complex (though equivariant ones work too!)

- finite homotopy limits, including
  - homotopy fibers
  - homotopy pullbacks
  - finite products
  - mapping $F(A, -)$ when A is a finite based nonequivariant cell complex (though equivariant ones work too!)

and the finiteness restriction on homotopy limits can be removed if we first make our spectra fibrant in the complete-universe model structure.

4. Fixed points of various kinds

4.1. **Categorical and genuine fixed points.** Let X be an orthogonal G-spectrum. The *(categorical) H-fixed points* X^H is an orthogonal spectrum whose nth level is just the H-fixed points of the nth level of X:

$$(X^H)_n := (X_n)^H$$

This is isomorphic to the spectrum of maps in from G/H:

$$X^H \cong F^G(G/H_+, X)$$

The group of symmetries of G/H as a left G-set is isomorphic to the *Weyl group*

$$WH = NH/H$$

acting on the right by

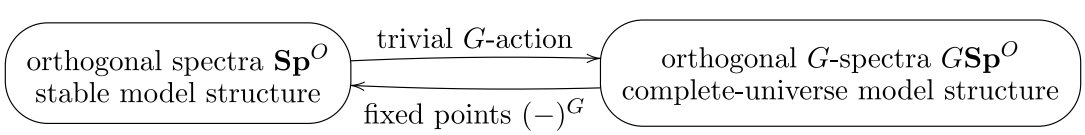
$$(gH)(nH) := gnH$$

When G is abelian, the Weyl group is simply G/H. However, the abelian case is a bit misleading. In the general case, G/H is only a set with a left G-action and a right WH-action that commute. This right action of WH on G/H in turn gives a left WH-action on the fixed point spectrum X^H, making X^H into an orthogonal WH-spectrum.

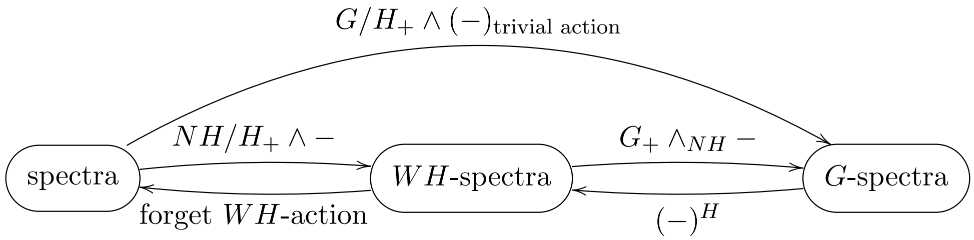
If this exposition makes the WH-action seem mysterious, we can also describe it more concretely. The normalizer $NH \leq G$ is the biggest subgroup of G which preserves the fixed point subspace X^H, and $H \leq NH$ acts trivially, so this gives a left action of WH on X^H which coincides with the action we described above.

The naïve equivariant stable homotopy groups $\widetilde{\pi}^H_*(X)$ are just the stable homotopy groups of X^H. The genuine equivariant stable homotopy groups $\pi^H_*(X)$ are the stable homotopy groups of $(fX)^H$ when $X \longrightarrow fX$ is a fibrant replacement in the complete-universe model structure. For this reason we often call $(fX)^H$ the *genuine H-fixed points of X*.

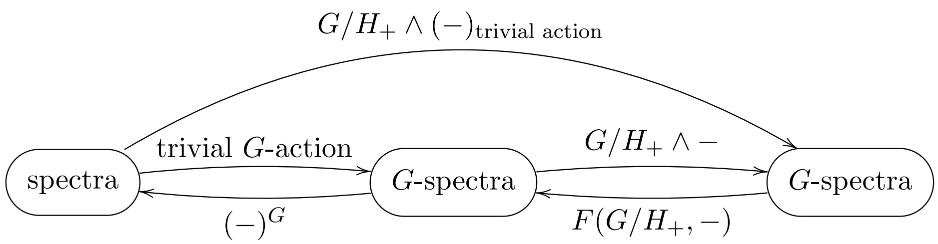
A map $X \longrightarrow Y$ is a complete-universe stable equivalence iff it induces a stable equivalence $(fX)^H \longrightarrow (fY)^H$ for all subgroups H. Even better, the categorical fixed points X^H are a right Quillen functor. In the simplest case $H = G$, we have the Quillen adjunction



In the general case $H \leq G$ the H-fixed points have two different left adjoints, depending on whether we think of it as landing in spectra or WH-spectra. As before, we'll draw them in with the left adjoints pointing from left to right:



We could also factor the H-fixed points functor into two right adjoints in a different way:



Anyway, these are all Quillen adjunctions when we take the complete-universe model structure everywhere, so the genuine fixed points really are a right-derived form of the categorical fixed points.

Even more generally, our mapping spectra $F^G(X, Y)$ and mapping G-spectra $F(X, Y)$ will be derived whenever X is cofibrant and Y is fibrant. (This is true in all three model structures.) More precisely, if Y is fibrant then every equivalence $X \longrightarrow X'$ between cofibrant spectra gives an equivalence of G-spectra

$$F(X', Y) \longrightarrow F(X, Y)$$

and if X is cofibrant than every equivalence $Y \longrightarrow Y'$ of fibrant spectra gives an equivalence of G-spectra

$$F(X, Y) \longrightarrow F(X, Y')$$

This follows from the general theory of closed monoidal model structures, as in [Hov07]. Our discussion of genuine fixed points above is just the special case where $X = \Sigma^\infty_+ G/H$.

4.2. **Genuine fixed points commute with all homotopy limits and homotopy colimits.** It's easy to check that categorical fixed points commute with inverse limits and Bousfield-Kan uncorrected homotopy limits on the nose. Taking fibrant replacements of the spectra in our diagram, we conclude that genuine (derived) fixed points commute with corrected homotopy limits up to equivalence.

Now on the space level, fixed points and colimits don't commute in general. However, fixed points do commute with pushouts along an h-cofibration, and when G is a compact

Lie group, fixed points commute with sequential colimits. Apply these facts to the homotopy pushout squares

$$\begin{array}{ccc} L_n B_\bullet(S^0, \mathbf{I}, F) \wedge \Delta^n \cup_{L_n B_\bullet(S^0,\mathbf{I},F)\times\partial\Delta^n} B_n(S^0, \mathbf{I}, F) \times \partial\Delta^n & \longrightarrow & B_n(S^0, \mathbf{I}, F) \times \Delta^n \\ \downarrow & & \downarrow \\ |\mathrm{Sk}^{n-1} B_\bullet(S^0, \mathbf{I}, F)| & \longrightarrow & |\mathrm{Sk}^n B_\bullet(S^0, \mathbf{I}, F)| \end{array}$$

where the cofiber along either row is equivalent to $\bigvee_{i_n \leftarrow \ldots \leftarrow i_0} F(i_0)$ running over all n-tuples of arrows that contain no identity maps. The categorical fixed points visibly preserve homotopy pushouts/pullbacks, cofiber sequences, and coproducts on each spectrum level, and everything discussed here is defined levelwise, so by induction up the skeleta we conclude that categorical fixed points commute with the Bousfield-Kan uncorrected homotopy colimit on the nose.

Even better, each of the above homotopy pushout squares is a homotopy pullback square, and every cofiber sequence we see is also a fiber sequence. By commutativity of derived right adjoints, the genuine fixed points commute with all of these constructions up to equivalence. Again we induct and observe that fixed points have a compactness condition that causes them to respect sequential colimits along nice inclusions. The surprising conclusion is that genuine fixed points commute with Bousfield-Kan uncorrected homotopy colimits up to equivalence:

$$\left(f \operatorname*{hocolim}_{i \in \mathbf{I}} F(i)\right)^G \xrightarrow{\sim} \operatorname*{hocolim}_{i \in \mathbf{I}} (fF(i))^G$$

Since hocolims of spectra never need to be corrected, genuine (derived) fixed points commute with hocolims up to equivalence.

This result always feels like cheating to the author, since the definition of equivariant homotopy groups and genuine fixed points is so intimidating, but commutativity with homotopy colimits makes them much, much easier to compute and understand. Blumberg has pointed out that this makes sense as a compactness statement, and in fact the above analysis generalizes to something that sounds much more reasonable. If X is a finite G-cell complex spectrum, than the spectrum $F^G(X, f-)$ of derived maps out of X commutes with all hocolims up to equivalence. To get this statement for fixed points we simply take $X = \Sigma^\infty_+ G/H$.

4.3. **Motivation and definition of geometric fixed points.** As Adams remarked in his notes on equivariant stable homotopy theory, one has three intuitions about how fixed points $(-)^G$ of G-spectra should behave:

(1) They should be homotopical (sending complete-universe stable equivalences to stable equivalences).
(2) They should be right adjoint to giving a spectrum the trivial G-action.
(3) They should commute with suspension spectra.

One might dream to find a notion of fixed points that satisfies all three, but this is impossible. However,

- The categorical fixed points X^G satisfy (2) and (3) but not (1).
- The genuine fixed points $(fX)^G$ satisfy (1) and (2) but not (3).
- The *geometric fixed points* $\Phi^G X$ defined below satisfy (1) and (3) but not (2).

Following [**?**] we will define $\Phi^H X$ as the coequalizer in the category of orthogonal spectra

$$\bigvee_{V,W} F_{W^H} S^0 \wedge \mathbf{J}_G^H(V,W) \wedge X(V)^H \rightrightarrows \bigvee_V F_{V^H} S^0 \wedge X(V)^H \longrightarrow \Phi^H X$$

This coequalizer has a natural WH-action, so we have an orthogonal WH-spectrum. The wedges above are taken over all representations V in some complete G-universe U. The action

$$F_{W^H} S^0 \wedge \mathbf{J}_G^H(V,W)$$

is through the projection $\mathbf{J}_G^H(V,W) \longrightarrow \mathbf{J}_{WH}(V^H, W^H)$ that restricts a given linear isometric inclusion $V \hookrightarrow W$ to its fixed points $V^H \hookrightarrow W^H$.

A semi-intuitive way to think about this is

$$(\Phi^H X)_n = \operatorname*{colim}_{V^H \longrightarrow \mathbb{R}^n} X(V)^H$$

so that the geometric fixed points are a colimit of $X(V)^H$ over all representations V whose fixed points V^H are no bigger than n-dimensional. This definition can be made precise by choosing the indexing category carefully and taking an enriched colimit, which results in our definition above.

4.4. **Comparison to other definitions.** Our definition is the one called "monoidal fixed points" by Hill, Hopkins, and Ravenel, and it is isomorphic to the one given by Mandell and May in [MM02]. We walk again through this more-detailed presentation:

Given a G-equivariant diagram

$$\mathbf{J}_G \xrightarrow{X} (\mathbf{Top})_G$$

we first *restrict* all of the mapping spaces in $\mathbf{J}_G$ and $(\mathbf{Top})_G$ to their H-fixed points, giving for each V and W a WH-equivariant map

$$\mathbf{J}_G^H(V,W) \longrightarrow \mathrm{Map}_*^H(X(V), X(W)) \longrightarrow \mathrm{Map}_*(X(V)^H, X(W)^H)$$

This defines a WH-equivariant diagram

$$\mathbf{J}_G^H \xrightarrow{X^H} (\mathbf{Top})_{WH}$$

Second, we observe that $\mathbf{J}_G^H(V,W)$ is the one-point compactification of the space of H-equivariant isometric inclusions $V \longrightarrow W$ and vectors in $W^H - V^H$. We modify this space by restricting the map $V \longrightarrow W$ to fixed points $V^H \longrightarrow W^H$. This gives a span for each $V, W \subset U$

$$\begin{array}{ccc} \mathbf{J}_G^H(V,W) = O^H(V,W)^{W^H - V^H} & \longrightarrow & \mathrm{Map}_*(X(V)^H, X(W)^H) \\ \downarrow & & \\ \mathbf{J}_{WH}(V^H, W^H) = O^H(V^H, W^H)^{W^H - V^H} & & \end{array}$$

The notation $\mathbf{J}_{WH}$ is justified by the fact that the universe U^H is a complete WH-universe:

Lemma 4.1. *If U is a complete G-universe then it is also a complete H-universe and U^H is a complete WH-universe.*

Proof. When G is finite this is easy, because if we take the regular representation ρ_G and forget down from the G-action to an H-action we get

$$\bigoplus_{H\backslash G} \rho_H$$

For compact Lie groups, this is still true but requires more work: every irreducible finite-dimenional H-representation V is a quotient of the infinite-dimensional representation $\mathrm{Ind}_H^G V$ by an adjointness argument, and then since this splits as a completed direct sum of finite-dimensional irreducible G-reps $\hat{\bigoplus} V_i$, we have for some i a nonzero map of H-reps

$$\mathrm{Res}_H^G V_i \longrightarrow V$$

which is therefore a projection and so V is isomorphic to a subrepresentation of $\mathrm{Res}_H^G V_i$.

Now any irreducible WH-rep V may be considered as a NH-rep with H acting trivially; by the above there is an irreducible G-rep V' and an inclusion $V \hookrightarrow \mathrm{Res}_{NH}^G V'$. The G-closure of the image of V under this inclusion is a finite-dimensional G-rep whose H-fixed points contain V. Therefore up to isomorphism V shows up in U^H, so U^H is complete. For finite groups we could also get this by observing that if we restrict ρ_G to its H-fixed points we get

$$\bigoplus_{H\backslash G} \mathbb{R} \cong \bigoplus_{NH\backslash G} \rho_{WH}$$

□

Now we have constructed a span-shaped diagram of WH-equivariant functors

$$\begin{array}{ccc} \mathbf{J}_G^H & \xrightarrow{X^H} & (\mathbf{Top})_{WH} \\ \downarrow{\scriptstyle \phi_H} & & \\ \mathbf{J}_{WH} & & \end{array}$$

Unfortunately that vertical arrow ϕ_H goes the wrong way and we can't simply compose. Our next-best construction is to take a Kan extension, and we choose a left Kan extension because we ultimately want Φ^H to commute with suspension spectra. Specifically, we take a $\mathbf{Top}_*$-*enriched left Kan extension* to get a functor from the category $\mathbf{J}_{WH}(V^H, W^H)$ into spaces:

$$\begin{array}{ccc} \mathbf{J}_G^H & \xrightarrow{X^H} & (\mathbf{Top})_{WH} \\ \downarrow{\scriptstyle \phi_H} & \nearrow_{\mathrm{Lan} X^H} & \\ \mathbf{J}_{WH} & & \end{array}$$

Writing out the local formula for the left Kan extension, we get back the coequalizer we gave in the last section. It has a natural WH-action, and in fact is isomorphic to the $WH\mathbf{Top}_*$-*enriched left Kan extension*, if knowing that fact makes the reader happier. We will see that this definition commutes with hocolims and smash products of cofibrant spectra on the nose.

A second definition appears in Schwede's notes when $H = G$ and G is finite:

$$(\Phi^G X)_n := X(n\rho_G)^G$$

Here $n\rho_G \cong \mathbb{R}^n \otimes \rho_G$ is a direct sum of n copies of the regular representation of G. This visibly includes into our definition, since $V = n\rho_G$ is a perfectly fine G-representation such that $V^G \cong \mathbb{R}^n$. This inclusion map gives an equivalence between the two definitions when X is cofibrant. Amazingly, Schwede's definition is always homotopical. That is, it takes every complete-universe stable equivalence $X \longrightarrow Y$ to a stable equivalence $\Phi^G X \longrightarrow \Phi^G Y$, with no cofibrancy assumptions. Schwede's definition also commutes with hocolims on the nose, but only commutes with smash products up to equivalence.

A third definition of geometric fixed points $\Phi^G X$ is done through genuine fixed points. First, take EP to be a CW complex satisfying

$$EP^H \simeq \begin{cases} \emptyset & H = G \\ * & H < G \end{cases} \qquad \Rightarrow \qquad EP_+^H \simeq \begin{cases} * & H = G \\ S^0 & H < G \end{cases}$$

For instance the unit sphere of the orthogonal complement $U - U^G$ will do. Then let $\widetilde{E}P$ denote the unreduced suspension of EP, or the homotopy cofiber of the collapse map $EP_+ \longrightarrow S^0$. So

$$\widetilde{E}P^H \simeq \begin{cases} S^0 & H = G \\ * & H < G \end{cases}$$

Now this classifying space has a strange and wonderful property: for any G-CW complexes X and Y, the restriction map

$$F^G(X, \widetilde{E}P \wedge Y) \longrightarrow F(X^G, (\widetilde{E}P \wedge Y)^G) \cong F(X^G, Y^G)$$

is always a weak equivalence. One proves this by attaching the nontrivial G-cells to X^G to get X. This allows us to expand the above map into a tower of fibrations. One can check that the fibers of these fibrations are all weakly contractible, so the above map is a weak equivalence!

Now one can show that when X is cofibrant, the construction

$$(f(\widetilde{E}P \wedge X))^G$$

is a third valid definition of geometric fixed points $\Phi^G X$, by giving a zig-zag of stable equivalences

$$(f(\widetilde{E}P \wedge X))^G \xrightarrow{R} \Phi^G(f(\widetilde{E}P \wedge X)) \xleftarrow{\sim} \Phi^G(\widetilde{E}P \wedge X) \xleftarrow{\cong} \Phi^G X$$

We will see below that the second map above is always an acyclic cofibration, and the right-hand map is an isomorphism by an easy induction up the cells of X.

We will also soon find out that Φ^G is homotopical and preserves homotopy colimits, just like genuine fixed points, so the left-hand map is an equivalence for all cofibrant X iff it is an equivalence when $X = \Sigma^\infty_+ G/H$. When H is proper, the based G-space $G/H_+ \wedge \widetilde{E}P$ has all fixed points contractible, so both sides are zero and the left-hand map is an equivalence. On the other hand, when $X = \Sigma^\infty_+ G/G = \mathbb{S}$, the left-hand map becomes

$$(f(\widetilde{E}P))^G \simeq \operatorname*{colim}_V \left(\Omega^V \Sigma^V \widetilde{E}P\right)^G \xrightarrow{R} \operatorname*{colim}_V \left(\Omega^{V^G} \Sigma^{V^G} (\widetilde{E}P)^G\right)$$

which is an equivalence for the strange and wonderful reason given above.

This third definition does not commute with hocolims nor with smash products on the nose. (Of course, it commutes up to equivalence because it's equivalent to the other definitions above!)

4.5. **Properties of geometric fixed points.**

- There is a natural isomorphism of WH-spectra

$$\Phi^H F_V(A) \cong F_{V^H}(A^H)$$

  for any G-rep V and based G-space A. This is because the functor

$$\mathbf{J}^H_G \xrightarrow{(F_V A)^H} (\mathbf{Top})_{WH}$$

  is visibly isomorphic to $F_{V \in \mathbf{J}^H_G} A^H$, and so when we left-Kan extend, the result is two left adjoints $\mathrm{Lan} \circ F_{V \in \mathbf{J}^H_G}$ applied to the WH-space A^H. The corresponding right adjoints compose to Ev_{V^H}, so $\mathrm{Lan} \circ F_{V \in \mathbf{J}^H_G} = F_{V^H}$ and the result is proven.
- Taking $V = 0$ in the last point, we get a natural isomorphism of WH-spectra

$$\Phi^H \Sigma^\infty A \cong \Sigma^\infty(A^H)$$

- Since fixed points of spaces don't commute with all colimits, we are forced to conclude that Φ^H does not commute with all colimits. It cannot be a left adjoint.
- Though Φ^H does not commute with all colimits, it is clear from the definition that Φ^H commutes with any natural construction on spaces that commutes with H-fixed points, smash products, and coequalizers. Therefore Φ^H commutes with

all coproducts (i.e. wedge sums), pushouts along a levelwise closed inclusion, and filtered colimits of levelwise closed inclusions.

- Φ^H commutes with all homotopy colimits on the nose. In other words, the natural map

$$\operatorname*{hocolim}_{i \in \mathbf{I}} \Phi^H F(i) \longrightarrow \Phi^H \left(\operatorname*{hocolim}_{i \in \mathbf{I}} F(i) \right)$$

is an isomorphism of WH-spectra.

- Φ^H commutes with smash products of cofibrant spectra. There is a natural map

$$\Phi^H X \wedge \Phi^H Y \longrightarrow \Phi^H(X \wedge Y)$$

which is an isomorphism when X and Y are cofibrant. (This is proven on the basic cells $F_V(G/K_+ \wedge S^k)$ first, and then it is possible to do induction up to all cofibrant spaces because Φ^H commutes with coproducts, homotopy pushouts, and filtered colimits of levelwise closed inclusions.)

- By a similar inductive argument, Φ^H preserves all of the varieties of "cofibration" and "acyclic cofibration" that we have discussed thus far. Therefore Φ^H takes complete-universe G-equivalences between cofibrant spectra to complete-universe WH-equivalences. (We may as well insist that X is complete-universe cofibrant because the other notions are included in this one.) When Φ^H is evaluated on a cofibrant spectrum we will say that it is *derived* or *homotopical*. Though Φ^H is not a Quillen left adjoint, it acts a lot like one.

- The derived Φ^H is *excisive*. It turns homotopy pushout/pullback squares of G-spectra into homotopy pushout/pullback squares of WH-spectra. Equivalently, it preserves all cofiber/fiber sequences.

From the definition of $\Phi^H X$ it is clear that there is a natural map of WH-spectra from the categorical fixed points to the geometric fixed points

$$X^H \xrightarrow{R} \Phi^H X$$

which we refer to as the "restriction map." When X is cofibrant and fibrant in the complete-universe model structure, this gives a map between genuine fixed points and geometric fixed points, which we call the derived restriction map. On the derived G-fixed points of the sphere spectrum this restriction map looks like

$$\operatorname*{colim}_V \left(\Omega^V S^V\right)^G \cong \operatorname*{colim}_V \operatorname{Map}^G_*(S^V, S^V) \longrightarrow \operatorname*{colim}_V \operatorname{Map}_*(S^{V^G}, S^{V^G}) \cong \operatorname*{colim}_n \Omega^n S^n$$

$$(f\mathbb{S})^G \xrightarrow{R} f\mathbb{S}$$

In general we get a commuting square

$$\begin{array}{ccc} X^H & \longrightarrow & (fX)^H \\ \downarrow R & & \downarrow R \\ \Phi^H X & \xrightarrow{\sim} & \Phi^H(fX) \end{array}$$

so we can always say there are maps

$$\text{categorical fixed points} \longrightarrow \text{genuine fixed points} \longrightarrow \text{geometric fixed points}$$

Now the derived restriction map usually does not split, but if X comes from the trivial universe then the above map from categorical to geometric fixed points is an equivalence, so the restriction map splits. To prove this, we check that when $X = F_k(S^n \wedge G/H_+)$ the restriction map is an isomorphism. By induction, it is an isomorphism whenever X is cofibrant in the trivial-universe model structure. So in that case the above square becomes

$$\begin{array}{ccc} X^H & \longrightarrow & (fX)^H \\ \cong \downarrow R & & \downarrow R \\ \Phi^H X & \xrightarrow{\sim} & \Phi^H(fX) \end{array}$$

and we conclude that $\Phi^H X$ is a summand of $(fX)^H$ in the stable homotopy category. This splitting is one of the first steps in a general form of tom Dieck splitting originally due to Gaunce Lewis; we will pick it up again in a subsequent section.

4.6. **Iterating fixed points.** Let $H \leq K \leq NH \leq G$. Then given a G-space X, the H-fixed points X^H have a natural K/H action, and we may take the K/H-fixed points of X^H to recover X^K:

$$X^K \cong (X^H)^{K/H}$$

This isomorphism is always as equivariant as possible. In particular, if H and K are both normal, then this is an isomorphism of G/K spaces.

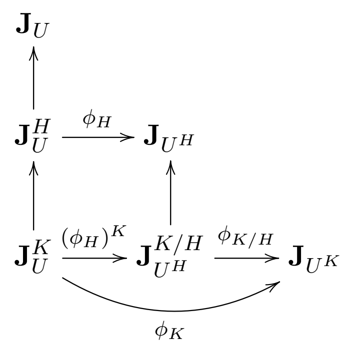
Now iterating the categorical fixed points is easy, because the above analysis applies on each spectrum level. Both $(-)^H$ and $(-)^{K/H}$ are right Quillen, so their composite is equal to the right Quillen functor $(-)^K$. Therefore their derived functors compose in the same way. Composing genuine H-fixed points with genuine K/H-fixed points gives genuine K-fixed points.

For the geometric fixed points the analysis is more subtle, because fixed points and colimits do not often commute. We will construct a natural map

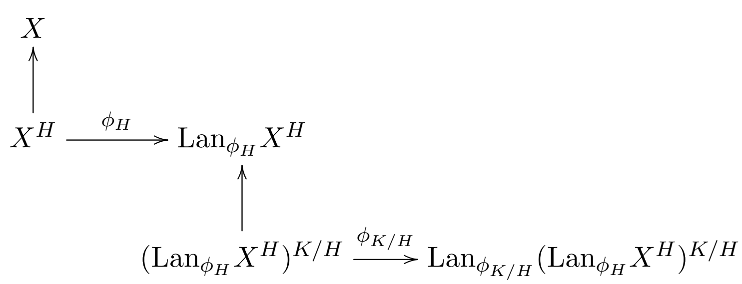
$$\Phi^K X \longrightarrow \Phi^{K/H}(\Phi^H X)$$

and prove that it is an isomorphism when X is cofibrant.

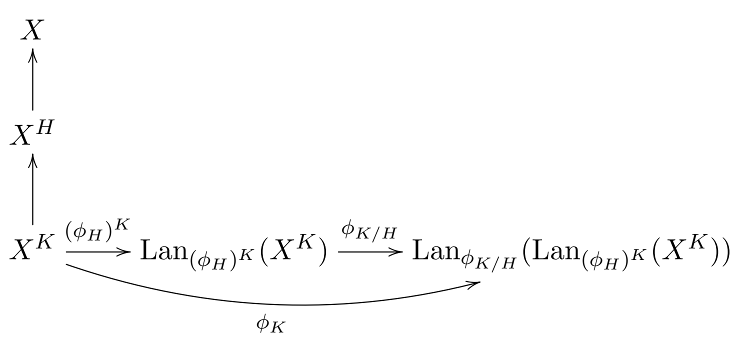
To avoid confusion, we will rename the category $\mathbf{J}_G$ to $\mathbf{J}_U$. As a reminder, it has one object for each finite-dimensional rep $V \subset U$ and the morphism space from V to W is the Thom space $O(V,W)^{W-V}$. Consider the diagram



Now the recipe for constructing the iterated geometric fixed points $\Phi^{K/H}(\Phi^H X)$ can be represented schematically by



and the recipe for constructing $\Phi^K X$ can be represented by



To define a natural map from one to the other, then, it suffices to define a commutation map

$$\mathrm{Lan}_{(\phi_H)^K}(X^K) \longrightarrow (\mathrm{Lan}_{\phi_H} X^H)^{K/H}$$

but we have a straightforward inclusion of coequalizer systems

$$\begin{array}{ccc}
\mathrm{Lan}_{(\phi_H)^K}(X^K) & \longrightarrow & \mathrm{Lan}_{\phi_H}(X^H) \\
\uparrow & & \uparrow \\
\bigvee_V F_{V^H \in \mathbf{J}^K_{U^H}} S^0 \wedge X(V)^K & \longrightarrow & \bigvee_V F_{V^H \in \mathbf{J}_{U^H}} S^0 \wedge X(V)^H \\
\uparrow & & \uparrow \\
\bigvee_{V,W} F_{W^H \in \mathbf{J}^K_{U^H}} S^0 \wedge \mathbf{J}^K_U(V,W) \wedge X(V)^K & \longrightarrow & \bigvee_{V,W} F_{W^H \in \mathbf{J}_{U^H}} S^0 \wedge \mathbf{J}^H_U(V,W) \wedge X(V)^H
\end{array}$$

which we check lands in the K/H-fixed points. Applying $\mathrm{Lan}_{\phi_{K/H}}$, we get our desired map

$$\Phi^K X \longrightarrow \Phi^{K/H}(\Phi^H X)$$

Now when $X = F_V A$, the above map of coequalizer systems becomes

$$F_{V^H \in \mathbf{J}^K_{U^H}} A^K \longrightarrow F_{V^H \in \mathbf{J}_{U^H}} A^H$$

and the inclusion is the most obvious map

$$\mathbf{J}^K_{U^H}(V^H, -^H) \wedge A^K \longrightarrow \mathbf{J}_{U^H}(V^H, -^H) \wedge A^H$$

which is clearly an isomorphism once we apply K/H-fixed points to the right-hand side.

Therefore

$$\Phi^K X \longrightarrow \Phi^{K/H}(\Phi^H X)$$

is an isomorphism when $X = F_V A$. Now since both sides preserve coproducts, pushouts along a closed inclusion, and filtered colimits, it follows that our map is an isomorphism whenever X is cofibrant.

4.7. **Geometric fixed points detect the weak equivalences.** The following are equivalent:

(1) $X \longrightarrow Y$ is a complete-universe G-equivalence.
(2) $(fX)^H \longrightarrow (fY)^H$ is a nonequivariant equivalence for all $H \leq G$.
(3) $(fX)^H \longrightarrow (fY)^H$ is a complete-universe WH-equivalence for all $H \leq G$.
(4) $\Phi^H(cX) \longrightarrow \Phi^H(cY)$ is a nonequivariant equivalence for all $H \leq G$.
(5) $\Phi^H(cX) \longrightarrow \Phi^H(cY)$ is a complete-universe WH-equivalence for all $H \leq G$.

We have already seen (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (4). But (1) $\Rightarrow$ (3) is clear from our discussion of iterating genuine fixed points, and (3) $\Rightarrow$ (2) is obvious, so (1), (2), and (3) are equivalent. Similarly, (1) $\Rightarrow$ (5) by iterating the geometric fixed points, and (5) $\Rightarrow$ (4) is obvious. It remains to prove (4) $\Rightarrow$ (1).

We copy the proof from [**?**] and [**?**], and write $(-)^H$ as shorthand for $(f-)^H$. Since $(-)^H$ and Φ^H preserve cofiber sequences, it suffices to show that if $\Phi^H X$ is nonequivariantly contractible for all $H \leq G$ then X^H is nonequivariantly contractible for all $H \leq G$. So assume that X is cofibrant and $\Phi^H X \simeq *$ for all $H \leq G$. We then prove that $X^H \simeq *$ for all $H \leq G$ by "induction" up the lattice of closed subgroups of G. This sort of induction works precisely because every descending chain of closed subgroups of G stabilizes. By inductive hypothesis, we have already shown that $X^H \simeq *$ for all proper subgroups $H < G$.

Now use the isotropy separation sequence:

$$EP_+ \longrightarrow S^0 \longrightarrow \widetilde{E}P$$

$$(f(EP_+ \wedge X))^G \longrightarrow (fX)^G \longrightarrow \Phi^G X$$

We know $\Phi^G X$ is contractible. The homotopy groups of the spectrum $(f(EP_+ \wedge X))^G$ are

$$\pi_*^G(EP_+ \wedge X) \xrightarrow{\cong} \pi_*^G(EP_+ \wedge fX)$$

but the map $EP_+ \wedge fX \longrightarrow *$ is a level equivalence of G-spectra because $fX(V)^H \simeq *$ for all proper subgroups H and $EP_+^H \simeq *$ when $H = G$. Therefore these homotopy groups are 0, so $(f(EP_+ \wedge X))^G$ is contractible, and by the five-lemma we conclude that $(fX)^G$ is contractible. This finishes (4) ⇒ (1).

4.8. **The HHR norm isomorphism.** When X is an orthogonal spectrum, the smash product $X^{\wedge n}$ has an action of $C_n \cong \mathbb{Z}/n$ which rotates the factors. This makes $X^{\wedge n}$ into an orthogonal C_n-spectrum.

What are its geometric fixed points $\Phi^{C_n} X^{\wedge n}$? A natural guess is X itself. In fact, there is nautral diagonal map

$$X \longrightarrow \Phi^{C_n} X^{\wedge n}$$

and when X is cofibrant this map is an isomorphism. Not just an equivalence, but an *isomorphism.*

More generally, if G is a finite group, $H \leq G$, and X is an orthogonal H-spectrum, we can define a smash product of copies of X indexed by G

$$N_H^G X := \bigwedge_{g_i H \in G/H} (g_i H)_+ \wedge_H X \cong \bigwedge^{|G/H|} X$$

This construction is the *multiplicative norm* defined by Hill, Hopkins, and Ravenel. This can be given a reasonably obvious G-action so long as we fix some choice of representatives $g_i H$ for each left coset of H. Essentially, those representatives tell us to think of the $g_i H$-copy of X as the image of some fixed copy of X under multiplication by g_i.

More explicitly, if x is a given point in the smash product (at some spectrum level), we know that its ith coordinate is equal to $x_i \in X$, and we try to compute the action of

$g \in G$ on x, then we know we can compute the jth coordinate of the output, where j is the unique coordinate such that $gg_i \in g_jH$. There is a unique $h \in H$ such that

$$gg_i = g_jh \quad \Leftrightarrow \quad h = g_j^{-1}gg_i$$

and the jth coordinate of the output is hx_i. The intuition is that there is a "commuting square"

$$\begin{array}{ccc} X_{g_iH} & \xrightarrow{g} & X_{g_jH} \\ \uparrow{\scriptstyle \text{“}g_i\text{”}} & & \uparrow{\scriptstyle \text{“}g_j\text{”}} \\ X & \xrightarrow{h} & X \end{array}$$

(The reader seeking more precision is happily referred to HHR or Bohmann's paper.) Unfortunately, changing our choice of representatives changes this action, but up to natural isomorphism it turns out to be the same. We therefore implicitly assume that such representatives have been chosen.

Theorem 4.2. *There is a natural diagonal of WH-spectra*

$$\Phi^H X \longrightarrow \Phi^G N_H^G X$$

and when X is cofibrant this map is an isomorphism.

This is Thm 2.33 in the six-author paper. It is proven in the HHR paper in the proof of Prop B.96 (section B.6), even though it is not stated in the theorem there. We will reproduce the proof here, since it is surprisingly short.

It is conceptually useful to start by checking that on the space level, the indexed smash product of A over G/H has fixed points A^H:

$$A^H \xrightarrow{\cong} \left(N_H^G A\right)^G \cong \left(\bigwedge^{|G/H|} A \right)^G$$

The map from left to right is the diagonal:

$$a \in A^H \mapsto (a, \dots, a)$$

Now for the spectrum-level argument. We start by taking the coequalizer presentation of the orthogonal H-spectrum X

$$\bigvee_{V,W} F_W S^0 \wedge \mathbf{J}_H(V,W) \wedge X(V) \rightrightarrows \bigvee_V F_V S^0 \wedge X(V) \longrightarrow X$$

and taking $\Phi^G N_H^G$ of everything in sight. Since $\Phi^G N_H^G$ commutes with wedges and smashes up to isomorphism, this gives

$$\bigvee_{V,W} \Phi^G N_H^G F_W S^0 \wedge (N_H^G \mathbf{J}_H(V,W))^G \wedge (N_H^G X(V))^G \rightrightarrows \bigvee_V \Phi^G N_H^G F_V S^0 \wedge (N_H^G X(V))^G \longrightarrow \Phi^G N_H^G X$$

$$\bigvee_{V,W} \Phi^G N_H^G F_W S^0 \wedge \mathbf{J}_H^H(V,W) \wedge X(V)^H \rightrightarrows \bigvee_V \Phi^G N_H^G F_V S^0 \wedge X(V)^H \longrightarrow \Phi^G N_H^G X$$

Note that this is no longer a coequalizer system because $\Phi^G N_H^G$ does not commute with coequalizers! However we can simplify using the string of isomorphisms

$$\begin{aligned}\Phi^G N_H^G F_V A &\cong \Phi^G F_{\mathrm{Ind}_H^G V}(N_H^G A) \\ &\cong F_{(\mathrm{Ind}_H^G V)^G}(N_H^G A)^G \\ &\cong F_{V^H} A^H\end{aligned}$$

for any based H-space A and H-represenation V. This gives

$$\bigvee_{V,W} F_{W^H} S^0 \wedge \mathbf{J}_H^H(V,W) \wedge X(V)^H \rightrightarrows \bigvee_V F_{V^H} S^0 \wedge X(V)^H \longrightarrow \Phi^G N_H^G X$$

and the coequalizer of the first two terms is exactly $\Phi^H X$. This defines the diagonal map

$$\Phi^H X \longrightarrow \Phi^G N_H^G X$$

Now consider the special case when $X = F_V A$. The inclusion of the term

$$F_{V^H} S^0 \wedge A^H$$

into the above coequalizer system maps forward isomorphically to $\Phi^H X$, and so we can evaluate the diagonal map by just examining this term. But back at the top of our proof, the inclusion of the term

$$\Phi^G N_H^G F_V S^0 \wedge (N_H^G A)^G$$

also maps forward isomorphically to $\Phi^G N_H^G X$. Therefore up to isomorphism, the diagonal map becomes the string of maps we used to connect $F_{V^H} S^0 \wedge A^H$ to $\Phi^G N_H^G F_V S^0 \wedge (N_H^G A)^G$, but these maps were all isomorphisms. Therefore the diagonal is an isomorphism when $X = F_V A$. Since both sides preserve coproducts, pushouts along closed inclusions, and sequential colimits along closed inclusions, we get by induction that the diagonal is an isomorphism for all cofibrant X.

5. Duality, transfers, and other magic

This section is where we really lift off the ground, and prove some magical isomorphisms that use "equivariant stability" in an essential way. These isomorphisms are what give the equivariant theory its true power – and they are essential for computations. To develop them, we first need to review duality theory and prove the surprising fact that G-orbits G/H are self-dual when G is finite (and self-dual up to a twist when G is compact Lie).

5.1. **Duality theory.**

Recall that if $\mathcal{C}$ is a closed symmetric monoidal category with unit I, then objects X and Y are *dual* if there are coevaluation and evaluation maps

$$I \xrightarrow{c} Y \otimes X, \quad X \otimes Y \xrightarrow{e} I$$

such that the composites

$$X \cong X \otimes I \xrightarrow{1 \otimes c} X \otimes Y \otimes X \xrightarrow{e \otimes 1} I \otimes X \cong X$$

$$Y \cong I \otimes Y \xrightarrow{c \otimes 1} Y \otimes X \otimes Y \xrightarrow{1 \otimes e} Y \otimes I \cong Y$$

are the identity maps of X and Y, respectively. An object X is said to be *finite* or *dualizable* if there is such collection (Y, c, e). Though dualizability appears to require extra data, that data is canonical when it exists. To prove this you show that dualizability is equivalent to the following map being an isomorphism:

$$F(X, I) \otimes F(I, X) \xrightarrow{f} F(X, X)$$

When it is an isomorphism, we can always take $Y \cong F(X, I)$, the coevaluation c is the lift of the identity $I \longrightarrow F(X, X)$ along f to a map $I \longrightarrow F(X, I) \otimes X$, and the evaluation e is just the obvious evaulation $X \otimes F(X, I) \longrightarrow I$. On the other hand, given any other set (Y, c, e) making X dualizable, Y may be identified with $F(X, I)$, and c and e become the maps we have described here.

For more details and other consequences of duality in a general setting see III.1 of [**?**]. In particular, if X is dualizable then the map

$$F(X, I) \otimes Z \longrightarrow F(X, Z)$$

is an isomorphism for any $Z \in \mathcal{C}$, and the natural map from X into its "double dual"

$$X \longrightarrow F(F(X, I), I)$$

is an isomorphism.

The reader who has never seen duality theory is encouraged to think about the case where $\mathcal{C}$ is vector spaces over a field k. The dualizable vector spaces are exactly the finite-dimensional ones. If V is finite-dimensional, the coevaluation map $k \longrightarrow V \otimes V^*$ sends 1 to the sum $\sum_{i=1}^n v_i \otimes v_i^*$, where $v_1, \dots, v_n$ is any basis and $v_1^*, \dots, v_n^*$ is its dual basis. Strangely, the resulting element of $V \otimes V^*$ does not depend on the choice of basis.

The next important example is the stable homotopy category of ordinary spectra. The finite spectra here are exactly the ones which are bounded below and which have finitely generated total homology. Equivalently, they are the shifts of the finite CW complexes. (To prove this, we show that the sphere spectrum is dualizable and then induct up cofiber and fiber sequences, since dualizable spectra form a thick subcategory.)

5.2. **Atiyah duality.** Let M be a compact manifold, so that $\Sigma^\infty M_+$ is of course a finite spectrum. Then *Atiyah duality* tells us that the dual of $\Sigma^\infty M_+$ is equivalent to the Thom spectrum M^{-TM}. To be more precise, let e be a smooth embeding M into a Euclidean space $\mathbb{R}^N$ and let ν_N be its normal bundle. The *Thom space* M^{ν_N} of this bundle is the one-point compactification of its total space, which is homeomorphic to the unit disc bundle of ν_N quotiented out by its boundary. (If M were not compact then we would have to be more careful.) Our embedding e extends to $\mathbb{R}^{N+k}$ with normal bundle $\nu_N \oplus \mathbb{R}^k$, whose Thom space is naturally homemorphic to $\Sigma^k M^{\nu_N}$. Evidently, these form the levels of a spectrum, which we call M^{-TM}:

$$(M^{-TM})_n := \begin{cases} * & n < N \\ M^{\nu_N \oplus \mathbb{R}^k} & n = N + k \end{cases}$$

It is not even hard to make this into an orthogonal spectrum, since $\mathbb{R}^n$ has an obvious $O(n)$-action. Although our definition of M^{-TM} does depend on a choice of embedding e, this choice is unique up to homotopy, and that homotopy is unique up to homotopy, provided N is sufficiently big. This allows us to regard all constructions of M^{-TM} as canonically isomorphic in the stable homotopy category.

At any rate, to manifest the duality between M_+ and M^{-TM} it suffices to construct maps in the stable category

$$\mathbb{S} \longrightarrow M^{-TM} \wedge M_+, \quad M_+ \wedge M^{-TM} \longrightarrow \mathbb{S}$$

The first map is Pontryagin-Thom collapse followed by the diagonal, and the second is the Alexander map. To define these, we introduce equivalent spectra M_ϵ^{-TM} and $\mathbb{S}_\epsilon$. Let $\epsilon \in (0, 1)$. Let $e(M) \subset \mathbb{R}^N$ denote the image of M under the embedding e, and let $e(M) \times \{0\} \subset \mathbb{R}^{N+k}$ denote the image of this under the inclusion $\mathbb{R}^N \times \{0\} \longrightarrow \mathbb{R}^N \times \mathbb{R}^k$. Define

$$(M_\epsilon^{-TM})_n := \begin{cases} * & n < N \\ N_\epsilon(e(M) \times \{0\})/\partial & n = N + k \end{cases}$$

$$(\mathbb{S}_\epsilon)_n := N_\epsilon(0)/\partial$$

Here the ball of radius ϵ is taken in $\mathbb{R}^n$. The structure maps and $O(n)$-action on these spectra come from thinking of each non-basepoint point as a point in $\mathbb{R}^n$, and using the standard identification $\mathbb{R}^n \times \mathbb{R} \cong \mathbb{R}^{n+1}$ and the standard $O(n)$-action on $\mathbb{R}^n$. The exponential map gives a projection

$$M^{-TM} \longrightarrow M_\epsilon^{-TM}$$

which is an equivalence so long as ϵ is small enough that $N_\epsilon(e(M))$ is actually a tubular neighborhood of $e(M)$. (We will always assume that ϵ is small enough to make this true.) Similarly, there is a projection $\mathbb{S} \longrightarrow \mathbb{S}_\epsilon$ which is always an equivalence. Since we are about to define maps in the stable homotopy category, we will use these equivalences to replace M^{-TM} by M_ϵ^{-TM} and $\mathbb{S}$ by $\mathbb{S}_\epsilon$ whenever necessary.

Now we define our coevaluation map

$$\begin{aligned} \mathbb{S} &\xrightarrow{c} M_\epsilon^{-TM} \wedge M_+ \\ S^n &\xrightarrow{c} (M_\epsilon^{-TM})_n \wedge M_+ \\ x &\mapsto (x, p(x)) \end{aligned}$$

Here $p : N_\epsilon(e(M) \times \{0\}) \longrightarrow M$ is the projection of the tubular neighborhood of $e(M)$ back to M. The evaluation map is defined

$$\begin{aligned} M_+ \wedge M_\epsilon^{-TM} &\xrightarrow{e} \mathbb{S}_\epsilon \\ M_+ \wedge (M_\epsilon^{-TM})_n &\xrightarrow{e} (\mathbb{S}_\epsilon)_n \\ (x, y) &\mapsto e(x) - y \end{aligned}$$

(finish proof by identifying the tensoring with M^{-TM} with taking sections over M, and along that identification, the counit and unit correspond to evaluation and coevaluation. or, prove the triangle identities directly. I thought I remembered this was supposed to be hard, but maybe I was wrong? also it might be conceptually helpful to include a small path in M when you do the evaluation.).

(cf. LMS III.3.5 and III.5.4)

be small enough that the ϵ-neighborhood of $e(M) \subset \mathbb{R}^n$ is a tubular neighborhood.

...

Atiyah duality gives a second proof that M is dualizable, while allowing us to work with the dual in a very geometric way.

5.3. **Duality in the equivariant stable category.** Now we will apply duality theory with $\mathcal{C}$ to be the homotopy category of G-spectra, with respect to the complete-universe equivalences. We start by explaining that for any closed subgroup of H, the object $\Sigma^\infty G/H_+$ is dualizable. This is not too surprising; after all, G/H is a parallelizable manifold, so by Atiyah duality we know that when we forget all G-actions, it is dualizable and its dual is $G/H^{-T(G/H)}$. It turns out that Atiyah duality is true equivariantly as well. The tangent bundle has a G-action...

and its dual is $\Sigma^{-L(H)} G/H_+$, the Thom spectrum of the map

G-spectra (in the complete-universe model structure) form a closed symmetric monoidal model category, so their homotopy category is closed symmetric monoidal. In any such can be done at the level of the homotopy category. Therefore we can rely on LMS to handle our duality theory.

For all X we have a derived equivalence

$$\Sigma^{-L(H)} G/H_+ \wedge X \xrightarrow{\sim} F(G/H_+, \mathbb{S}) \wedge X \xrightarrow{\sim} F(G/H_+, X)$$

The first is easy, we just smashed the duality equivalence above with X. The second map is an equivalence by general duality theory.

Here's an application: we can give a simple description for the mapping spectra between two finite G-sets when G is finite:

$$F(\Sigma^\infty_+ G/H, \Sigma^\infty_+ G/K) \simeq \Sigma^\infty_+ (G/H \times G/K)$$
$$F(\Sigma^\infty_+ S, \Sigma^\infty_+ T) \simeq \Sigma^\infty_+ (S \times T)$$

5.4. **The Wirthmüller isomorphism.** From the previous section, we know that if X is any G spectrum there is a natural G-equivalence

$$\Sigma^{-T(G/H)} G/H_+ \wedge X \xrightarrow{\sim} F(G/H_+, X)$$

There is a closely related statement, not quite implying or implied by this, which is also true. Namely, if X is an orthogonal H-spectrum, there is a natural G-equivalence

$$G_+ \wedge_H \Sigma^{-T(G/H)} X \xrightarrow{\sim} F^H(G_+, X)$$

when everything is derived and $T(G/H)$ is the tangent space to the identity of G/H. This is often called the *Wirthmüller isomorphism*, because of course we can interpret it an isomorphism in the homotopy category, where it was first defined.

We have already proven the Wirthmüller isomorphism in the special case $X = \mathbb{S}$:

$$\Sigma^{-T(G/H)} G/H_+ \xrightarrow{\sim} F(\Sigma^\infty_+ G/H, \mathbb{S})$$

So all we need to do is define a such a map for all X which gives this equivalence when $X = \mathbb{S}$, and use the fact that both sides commute up to equivalence with all homotopy colimits to get the result.

...

5.5. **Geometric fixed points commute with duals.** Let X be dualizable, and let DX denote the mapping spectrum $F(X, \mathbb{S})$. Then there is an equivalence of WH-spectra

$$\Phi^H DX \xrightarrow{\sim} D(\Phi^H X)$$

when everything is derived.

More generally, when X and Y are cofibrant G-spectra, define the "restriction" map of WH-spectra

$$\Phi^H cF(X, Y) \xrightarrow{\rho} F(\Phi^H X, f\Phi^H Y)$$

as the adjoint to

$$\Phi^H cF(X,Y) \wedge \Phi^H X \longrightarrow \Phi^H(cF(X,Y) \wedge X) \longrightarrow \Phi^H(F(X,Y) \wedge X) \longrightarrow \Phi^H Y \longrightarrow f\Phi^H Y$$

Now the reader can check that all these constructions are derived and so ρ may also be considered as a map in the homotopy category

$$\Phi^H F(X,Y) \xrightarrow{\rho} F(\Phi^H X, \Phi^H Y)$$

Using LMS III.1.9, we can immediately conclude that ρ is a nonequivariant equivalence when X is dualizable. Iterating ρ and using the fact that geometric fixed points detect weak equivalences, we conclude that ρ is a WH-equivalence whenever X is dualizable.

5.6. **Definition of the equivariant transfer.** It seems that the word "transfer" really refers to (at least) two separate phenomena. On the one hand, when we have a submersion of smooth manifolds $X \longrightarrow Y$, or more generally a fiber bundle whose fibers are smooth manifolds, a Pontryagin-Thom construction gives us a map

$$\Sigma^\infty_+ Y \longrightarrow \Sigma^{-T(\text{fiber})}_+ X$$

This is sometimes called the dimension-shifting transfer, or the umkehr map. Passing to a more general setting, if $X \longrightarrow Y$ is a fibration with fiber F which is stably finite, there is a Becker-Gottlieb style transfer map

$$\Sigma^\infty_+ Y \longrightarrow \Sigma^\infty_+ X$$

When $X \longrightarrow Y$ is a finite-sheeted covering space, these two transfers are the same. However when the fiber F of $X \longrightarrow Y$ is a manifold of positive dimension, these two transfers must be different. After all, they don't even have the same target! In (LMS) the authors remark that when this second transfer gives zero, the first transfer is often highly nontrivial.

We will define an equivariant version of the first transfer map, which gives for any free G-spectrum X a map in the homotopy category

$$X_{hG} \longrightarrow (\Sigma^{-\mathrm{Ad}(G)} fX)^G$$

Here the fibrant replacement is taken in the category of genuine G-spectra, so that the fixed points are genuine. This map can be rectified to be natural, not just in the homotopy category, but on the nose [**?**]. Our map here will not quite be natural on the nose. Still, it will be enough to prove the Adams isomorphism, and set up the norm cofibration sequence. We have not been able to find our approach here in the literature, but it seems to be known.

The first step is to consider the very special case $X = \Sigma^\infty_+ G$. Embed G into a representation V. Then Pontryagin-Thom collapse gives a G-equivariant map of based spaces

$$S^V \longrightarrow \Sigma^{V-\mathrm{Ad}(G)}_+ G \cong \Sigma^{n-d}_+ G$$

That is, the tangent bundle of G is trivializable but has nontrivial G-action, making it $\mathrm{Ad}(G)$, and the normal bundle of G in V is therefore isomorphic to the complement of $\mathrm{Ad}(G)$ in V. Now apply the desuspension functor F_V to this map and simplify:

$$\mathbb{S} \xleftarrow{\sim} F_V S^V \longrightarrow F_V S^{V-\mathrm{Ad}(G)} \wedge G_+ \xrightarrow{\sim} \Sigma_+^{-\mathrm{Ad}(G)} G$$

We call this "composite" the *pretransfer*. It becomes an actual map, well-defined up to homotopy, if we take fibrant replacement of the last term $\Sigma_+^{-\mathrm{Ad}(G)} G$.

Remark. Because G is free as a G-space, there is a G-equivariant isomorphism

$$\Sigma_+^{V-\mathrm{Ad}(G)} G \cong \Sigma_+^{V-d} G$$

where $d = \dim G$. Therefore $\mathrm{Ad}(G)$ may be replaced by d and the pretransfer is defined just as before, as a map of left G-spaces. However this simplification will not push through the rest of the construction in general, because this isomorphism will not respect both the left and right actions of G.

We need to modify this pretransfer so that it is equivariant with respect to the right G-action on $\Sigma_+^\infty G$ as well. (It clearly isn't as we defined it.) To fix this, we formally define a left $G \times G$-action on G_+, letting the first copy of G act by right-multiplication by the inverse, and the second copy acting by left multiplication:

$$(g, h)k = hkg^{-1}$$

Then we embed G into a $G \times G$-representation V, and as before define the $G \times G$-equivariant map of based spaces

$$S^V \longrightarrow \Sigma_+^{V-\mathrm{Ad}(G)} G$$

giving a zig-zag

$$\mathbb{S} \xleftarrow{\sim} F_V S^V \longrightarrow F_V S^{V-\mathrm{Ad}(G)} \wedge G_+ \xrightarrow{\sim} \Sigma_+^{-\mathrm{Ad}(G)} G$$

of $G \times G$-spectra. This is our powered-up version of the pretransfer.

Now we need some "sleight of hand" in order to blend this map into any other free orthogonal G-spectrum X. Consider the bifunctor $X \wedge_G Y$. We interpret the input X as a genuine $G \times G$-spectrum and the input Y as a genuine G-spectrum. Both actions are on the left, though in practice the second copy of G acting on X is defined via a right action and an inversion. This bifunctor smashes X and Y together, restricts attention to the trivial universe, divides out by the diagonal action of G by the second copy of G on X and the only copy of G on Y, and then induces back up to the complete universe. (The change of universe is essential because G-orbits is very badly behaved on the non-trivial representation levels. If you try to work on nontrivial representations without realizing this it leads to a lot of confusion!) As a result, there is still a remaining G-action coming from the first copy of G on X, and so we interpret the output as a genuine G-spectrum.

We check that this functor preserves weak equivalences when X and Y are cofibrant and Y is free. It is clearly a left adjoint in each slot, so it preserves all colimits in either slot. So it suffices to take the maps

$$X = F_V((G \times G)/K_+ \wedge S^{m-1}) \xrightarrow{i} X' = F_V((G \times G)/K_+ \wedge D^m)$$

$$Y = F_n(G_+ \wedge D^k) \xrightarrow{j} Y' = F_n(G_+ \wedge D^k \times I)$$

and check that $i \Box_G j$ is an acyclic cofibration of genuine G-spectra. The pushout-product simplifies to

$$F_{V+n}((G \times G)/K_+ \wedge_G G_+ \wedge (D^{m+k} \longrightarrow D^{m+k} \times I)_+)$$

$$\simeq F_{V+n}((G \times G)/K_+ \wedge (D^{m+k} \longrightarrow D^{m+k} \times I)_+)$$

Here G acts as expected on the $F_{V+n}((G \times G)/K_+)$, and acts trivially on everything else. This is clearly an acyclic cofibration in the complete-universe model structure. Inducting up the colimits, if X is cofibrant then the functor preserves all free acyclic cofibrations in the Y variable, so it preserves all weak equivalences between free cofibrant Y by Ken Brown's lemma. The argument for weak equivalences in the X slot is almost exactly the same; we just switch the roles of $S^{m-1} \longrightarrow D^m$ and $D^k \longrightarrow D^k \times I$.

Now let X be any G-spectrum which is cofibrant in the coarse model structure. Consider the composite (where the equivalences are complete-universe equivalences)

$$\mathbb{S} \wedge_G X \xleftarrow{\sim} F_V S^V \wedge_G X \longrightarrow F_V S^V \wedge \Sigma^{-\mathrm{Ad}(G)} G_+ \wedge_G X \cong F_V S^V \wedge \Sigma^{-\mathrm{Ad}(G)} X \xrightarrow{\sim} \Sigma^{-\mathrm{Ad}(G)} X$$

It is straightforward to check that all the maps are equivariant, where G acts trivially on the first spectrum (at trivial-representation levels) and G has the usual action on X all the way on the right. (The penultimate term $F_V S^V \wedge \Sigma^{-\mathrm{Ad}(G)} X$ has G acting by the diagonal action on both parts of the smash product.) Therefore we get an equivariant map up to homotopy

$$X_{hG} = \mathbb{S} \wedge_G X \longrightarrow f\Sigma^{-\mathrm{Ad}(G)} X$$

and since the left-hand spectrum has trivial G-action it factors through the fixed points

$$X_{hG} \xrightarrow{\tau} (f\Sigma^{-\mathrm{Ad}(G)} X)^G$$

This is the equivariant transfer.

5.7. **The Adams isomorphism and classical norm isomorphism.** Recall that, in order to actually define $\tau : X_{hG} \longrightarrow (f\Sigma^{-\mathrm{Ad}(G)} X)^G$, we have assumed that X is cofibrant in the coarse model structure, so that it is a "free" G-spectrum. We know that coarsely equivalent free spectra always have equivalent genuine fixed points, but the next theorem allows us to express those fixed points much more concretely.

Theorem 5.1 (Adams isomorphism.). *τ is an equivalence of spectra.*

Proof. Since both homotopy orbits and genuine fixed points commute with homotopy colimits, to prove that τ is an equivalence on all free spectra, it suffices to check that τ is an equivalence when $X = \Sigma^\infty_+ G$. In other words, it suffices to prove that the pretransfer is an equivalence. When G is finite we easily identify it with the diagonal map

$$\mathbb{S} \longrightarrow \left(\prod^G \operatorname*{colim}_{V \subset U} \Omega^V S^V \right)^G \cong \Omega^\infty S^\infty$$

which is an equivalence. When G is a Lie group we use the duality theory from the previous section more carefully. $\square$

For a coarse consequence, note that there is always a map $(fX)^G \longrightarrow X^{hG}$ and it is an equivalence when $X = G_+ \wedge Y$ or $F(G_+, Y)$ for any G-spectrum Y. Composing this with the transfer, we get a natural transformation

$$X_{hG} \xrightarrow{N} X^{hG}$$

called the *norm map*. A similar induction allows us to prove that the norm map is an isomorphism for a large range of free spectra, but we do not get all free spectra because $-^{hG}$ does not commute with homotopy colimits.

Corollary 5.2 (Classical norm isomorphism.)**.** *N is an equivalence when X is expressible as $Z \wedge Y$ or $F(Z, Y)$ with Z finite free.*

We will follow with a second, more geometric definition of the transfer map τ, in the special case that $X = \Sigma^\infty_+ E$, with E a free G-cell complex.

As before let G be a compact Lie group. Let $E \xrightarrow{p} B$ be a principal G-bundle. We filter B into skeleta $B_0 \subset B_1 \subset B_2 \subset \ldots B$. For each skeleton B_k we equivariantly embed the total space $E_k = p^* B_k$ of the bundle into $B \times V_k$ for some sufficiently big representation V_k. This is possible because the colimit

$$\operatorname{Emb}^G(G, U) = \operatorname*{colim}_{V \subset U} \operatorname{Emb}^G(G, V)$$

is weakly contractible, and it is possible to choose a sequence of representations making the embedding spaces more and more highly connected. (Check this.) Anyway, now we Pontryagin-Thom collapse to get an equivariant map

$$\Sigma^{V_k}_+ B_k \longrightarrow \Sigma^{V_k - \operatorname{Ad}(G)}_+ E_k$$

(Choose ϵ-neighborhoods to make this precise.)

These assemble into a map of spaces

$$B \longrightarrow \left(\operatorname*{colim}_{V \subset U} \Omega^V \Sigma^{V - \operatorname{Ad}(G)} E \right)^G$$

which is adjoint to a map of G-spectra

$$\Sigma^\infty_+ B \longrightarrow f(\Sigma^{-\operatorname{Ad}(G)}_+ E)^G$$

Notice that is in fact defined a map from the homotopy orbits of the G-spectrum $\Sigma^\infty_+ E$ into the genuine fixed points of $\Sigma_+^{-\mathrm{Ad}(G)} E$.

(Prove these are the same.)

5.8. **The norm cofiber sequence.** Now let's use this to make a statement about any G-spectrum X, even if X is not free. Smash X with the cofiber sequence

$$EG_+ \longrightarrow S^0 \longrightarrow \widetilde{E}G$$

to get the cofiber sequence

$$EG_+ \wedge X \longrightarrow X \longrightarrow \widetilde{E}G \wedge X$$

Apply genuine fixed points and the Adams isomorphism above, resulting in

Theorem 5.3 (Norm cofibration sequence)**.** *There is a natural cofiber sequence*

$$X_{hG} \longrightarrow (fX)^G \longrightarrow (f(\widetilde{E}G \wedge X))^G$$

Now when $G = C_p$ the space $\widetilde{E}G$ is the same as $\widetilde{E}P$ defined above, so we get

$$X_{hC_p} \longrightarrow (fX)^{C_p} \longrightarrow \Phi^{C_p} X$$

When $G = C_{p^n}$ we can see that $\widetilde{E}G$ when restricted to a C_p-space is $\widetilde{E}P$, so we get

$$X_{hC_{p^n}} \longrightarrow (fX)^{C_{p^n}} \longrightarrow (f\Phi^{C_p} X)^{C_{p^{n-1}}}$$

These observations form the foundation for calculations of topological cyclic homology. We will also use them to prove tom Dieck splitting below.

5.9. **Thick and localizing subcategories.** If $\mathbf{C}$ is a model category with a notion of cofiber sequence which makes $\mathrm{Ho}\mathbf{C}$ triangulated, we say that a full subcategory of $\mathrm{Ho}\mathbf{C}$ is *thick* if it is closed under cofibers, fibers, and retracts. (In particular this implies that it contains the zero object and is closed under isomorphisms.) Furthermore, a thick subcategory of $\mathrm{Ho}\mathbf{C}$ is *localizing* if it is also closed under arbitrary coproducts.

The entire category $\mathbf{C}$ is trivially both thick and localizing, so it makes sense to take a collection of objects $A \subset \mathrm{ob}\mathbf{C}$ and speak of the thick or localizing subcategory *generated* by A: it's just the intersection of all thick (localizing) subcategories containing A. This notion is very handy for proving natural equivalences: if you have a natural map which preserves retracts and cofibers, and it's an equivalence on every object in A, then it's automatically an equivalence on the thick subcategory generated by A.

By abuse of notation, we will also apply the terms "thick" and "localizing" to any subcategory of $\mathbf{C}$ itself by looking at its image in the homotopy category.

The standard example is that in spectra, the thick subcategory generated by $\mathbb{S}$ is the subcategory of *finite* or *dualizable* spectra. The localizing subcategory generated by $\mathbb{S}$ is the entire category.

Let $\mathbf{C}$ denote orthogonal G-spectra with the complete-universe model structure. Then the localizing subcategory generated by just $F_0G_+ = \Sigma^\infty_+ G$ is the subcategory of *free G-spectra*. The following characterizations of free spectra are all equivalent:

- X is in the localizing subcategory generated by $\Sigma^\infty_+ G$.
- X is equivalent to a spectrum in the image of the left-derived identity functor from coarse spectra.
- After X is made cofibrant, $EG_+ \wedge X \longrightarrow X$ is an equivalence.

These properties are dual to X being *cofree*, meaning that after X is made fibrant, $X \longrightarrow F(EG_+, X)$ is an equivalence.

We will see later that free G-spectra X have property that there is a natural equivalence between their derived homotopy orbits and derived (genuine) fixed points

$$X_{hG} \xrightarrow{\sim} X^G$$

On the other hand, it is not too hard to check directly that cofree spectra X enjoy an equivalence between genuine and homotopy fixed points

$$X^G \xrightarrow{\sim} X^{hG}$$

This proves that the conditions of being free and cofree do not often overlap, for when they do we can conclude $X_{hG} \simeq X^{hG}$.

Now specialize to the case when G is a finite group. Then the localizing subcategory generated by $\{F_0G/H_+ = \Sigma^\infty_+ G/H : H \leq G\}$ is the entire category! The thick subcategory generated by $\{F_0G/H_+ : H \leq G\}$ is the subcategory of *finite* or *dualizable* spectra. This category has many equivalent characterizations:

- X is in the thick subcategory generated by $\Sigma^\infty_+ G/H$ for $H \leq G$.
- X is a retract in the homotopy category of a complex built from finitely many stable cells.
- The natural map $F(X, \mathbb{S}) \wedge X \longrightarrow F(X, X)$ is a (complete-universe) stable equivalence.
- $F(X, -)$ commutes with all sums (even uncountable ones) up to stable equivalence.

The author is not sure to what extent this comparison breaks down when G is a compact Lie group.

Finally, the thick subcategory generated by $\Sigma^\infty_+ G$ is the subcategory of *finite free* spectra. It's just the intersection of the previous two sets of conditions.

Let's discuss how finiteness interacts with being free and cofree. First of all, by the Wirthmuller isomorphism below, $\Sigma^\infty_+ G \simeq F(G_+, \mathbb{S})$ as G-spectra and so $\Sigma^\infty_+ G$ is a free and cofree G-spectrum. It follows that all finite free G-spectra are both free and cofree, so their homotopy orbits, genuine fixed points, and homotopy fixed points are all equivalent.

Even better, if X is finite free and Y is any G-spectrum, both the smash $X \wedge Y$ and the function spectrum $F(X,Y)$ are both free and cofree (by induction on the cells of X). This gives

$$(X \wedge Y)_{hG} \xrightarrow{\sim} (X \wedge Y)^G \xrightarrow{\sim} (X \wedge Y)^{hG}$$
$$F(X,Y)_{hG} \xrightarrow{\sim} F(X,Y)^G \xrightarrow{\sim} F(X,Y)^{hG}$$

This may seem like a rather large class of G-spectra, since Y is allowed to be absolutely anything. However this class is smaller than it appears, since for most spectra the homotopy orbits and fixed points are not equivalent. As a consequence, "most" G-spectra cannot be written as $X \wedge Y$ or $F(X,Y)$ with X finite free. In particular, the unit of the smash product $\mathbb{S}$ is not finite free.

The conditions of "free" and "cofree" do not imply each other, assuming of course that $G \neq 1$. Indeed, using tom Dieck splitting and the Segal conjecture, one can prove that $\mathbb{S}_{hG} \not\simeq \mathbb{S}^{hG}$ when G is any nontrivial finite group. As a consequence, $\Sigma^\infty_+ EG$ is free but not cofree, and $F(EG_+, \mathbb{S})$ is cofree but not free!

5.10. **tom Dieck splitting.** Corollary: $\pi_0^G(\mathbb{S}) = A(G)$. This is an equivariant analogue of the classical theorem that the operation which takes a map $S^n \longrightarrow S^n$ to its degree gives an isomorphism $\pi_n(S^n) \cong \mathbb{Z}$.

5.11. **A spectrum which does not satisfy tom Dieck splitting.** We want to give a counterexample to show tom Dieck splitting cannot hold in general. We will give a G-spectrum X for which the derived restriction map $X^{C_2} \xrightarrow{R} \Phi^{C_2} X$ does not split, even in the homotopy category, even after dropping all naturality requirements.

Let $G = C_2 = \mathbb{Z}/2$ and let σ denote the sign representation. Let A be any based C_2-CW complex for which the inclusion of the fixed points $A^{C_2} \longrightarrow A$ is nonzero in the homotopy category, even after applying Σ^∞. Any stably non-contractible complex A with the trivial action will do. Then we will show that the desuspension of A by the sign representation

$$X = F_\sigma A \simeq \Omega^\sigma \Sigma^\infty A = F(S^\sigma, \Sigma^\infty A)$$

does not satisfy tom Dieck splitting.

The proof is a satisfying combination of a few different ideas we have seen so far. Start with the cofiber sequence of C_2-spaces

$$(C_2)_+ \longrightarrow S^0 \longrightarrow S^\sigma$$

and apply derived $F(-, \Sigma^\infty A)$ to get the cofiber sequence of C_2-spectra

$$F(S^\sigma, \Sigma^\infty A) \longrightarrow F(S^0, \Sigma^\infty A) \longrightarrow F((C_2)_+, \Sigma^\infty A)$$

This expresses $X = F(S^\sigma, \Sigma^\infty A)$ as a fiber of a map of C_2-spectra which we understand well. Now apply the derived restriction map $(-)^{C_2} \longrightarrow \Phi^{C_2}(-)$ to this entire row and pass to the stable homotopy category. This gives a map of cofiber sequences of ordinary spectra

$$\begin{array}{ccccc}
F^{C_2}(S^\sigma, \Sigma^\infty A) & \longrightarrow & (\Sigma^\infty A)^{C_2} & \longrightarrow & F^{C_2}((C_2)_+, \Sigma^\infty A) \\
\downarrow R & & \downarrow R & & \downarrow R \\
\Phi^{C_2} F(S^\sigma, \Sigma^\infty A) & \longrightarrow & \Phi^{C_2}(\Sigma^\infty A) & \longrightarrow & \Phi^{C_2}(F((C_2)_+, \Sigma^\infty A))
\end{array}$$

which simplifies to

$$\begin{array}{ccccc}
(F_\sigma A)^{C_2} & \longrightarrow & \Sigma^\infty A_{hC_2} \vee \Sigma^\infty(A^{C_2}) & \xrightarrow{F} & \Sigma^\infty A \\
\downarrow R & & \downarrow R & & \downarrow R \\
\Sigma^\infty(A^{C_2}) & \xrightarrow{\sim} & \Sigma^\infty(A^{C_2}) & \longrightarrow & *
\end{array}$$

The "F" on the upper map stands for *Frobenius* as its behavior is similar to the Frobenius map on THH, discussed in a later section. It is easy to express F up to homotopy on these simplified spectra: on the first summand it is the C_2-transfer, and on the second summand it is the inclusion of fixed points $A^{C_2} \hookrightarrow A$, which by assumption is nontrivial. Also, the middle vertical map in the above diagram is projection onto the second factor, by the definition of the tom Dieck splitting.

Now assume for the sake of contradiction that the left-vertical map splits in the homotopy category. Then it gives a splitting of the middle-vertical which when projected onto the second term must give an isomorphism in the stable homotopy category

$$\Sigma^\infty(A^{C_2}) \xrightarrow{\sim} \Sigma^\infty(A^{C_2})$$

This would define a natural map on the first two terms of a cofiber sequence, so it would extend to a map of cofiber sequences. Therefore the composite

$$\begin{array}{ccccc}
(F_\sigma A)^{C_2} & & \Sigma^\infty A_{hC_2} \vee \Sigma^\infty(A^{C_2}) & \xrightarrow{F} & \Sigma^\infty A \\
 & & \uparrow \text{split} & & \\
\Sigma^\infty(A^{C_2}) & & \Sigma^\infty(A^{C_2}) & & *
\end{array}$$

would be zero in the homotopy category. On the level of π_0, we see that this composite must have degree $\pm 1 + 2n$, which cannot be equal to zero, contradiction. Therefore no splitting of this map exists:

$$(F_\sigma A)^{C_2} \xrightarrow{R} \Phi^{C_2}(F_\sigma A)$$

So $X = F_\sigma A$ cannot have tom Dieck splitting. Even better, this shows that

$$(F_\sigma S^0)^{C_2} \simeq \Sigma^\infty_+ \mathbb{RP}^\infty$$

mapping into $\mathbb{S} \vee \Sigma^\infty_+ \mathbb{RP}^\infty$ by the wedge of the transfer and the negative of the identity. Under this equivalence, the restriction map to the geometric fixed points $\mathbb{S}$ is the transfer, which clearly does not split.

Taking $A = S^0$ we can conclude that the class of finite G-spectra and the class of duals of suspension spectra both do not enjoy tom Dieck splitting in general.

One might observe that our X above is the homotopy fiber of a map of two spectra, and each of those spectra is induced up from the trivial universe and therefore has tom Dieck splitting. So in a sense X is "built" from spectra that have tom Dieck splitting. However, the chosen *map* between these two spectra does not commute with the splittings. This map could not have been induced up from any map in the trivial universe. This is an important organizing principle that we will see again: one may think of all G-spectra as being built up from spectra like $\Sigma^\infty_+ G/H$ whose behavior is nice and one might even say trivial, but the *maps* between these building blocks do not behave trivially and so general G-spectra have much more interesting behavior.

5.12. **Segal's Burnside ring conjecture.** Let G be a finite p-group. Then Segal's Burnside ring conjecture, which is not a conjecture but a theorem, states that the natural map of nonequivariant spectra

$$(f\mathbb{S})^G = F^G(S^0, f\mathbb{S}) \longrightarrow F^G(EG_+, f\mathbb{S}) = \mathbb{S}^{hG}$$

is an equivalence after p-completion. This implies that the 0th co-homotopy group of BG is isomorphic to the Burnside ring of G.

Though we won't go into p-completion much here, a p-complete equivalence is a map that becomes a stable equivalence when it is smashed with the cofiber of $\mathbb{S} \xrightarrow{\cdot p} \mathbb{S}$. For connective spectra, this is equivalent to the map being an isomorphism on homology with $\mathbb{Z}/p$ coefficients. See [Bou79] for more details.

Once we have this for $\mathbb{S}$, it is straightfoward to show that the same map

$$(fX)^G \longrightarrow X^{hG}$$

is an equivalence after p-completion when $X = \Sigma^\infty_+ G/H$:

$$\begin{array}{ccc} F^G(S^0, f\Sigma^\infty_+ G/H) & \longrightarrow & F^G(EG_+, f\Sigma^\infty_+ G/H) \\ \downarrow \sim & & \downarrow \sim \\ F^G(S^0, F(G/H_+, f\mathbb{S})) & \longrightarrow & F^G(EG_+, F(G/H_+, f\mathbb{S})) \\ \downarrow \cong & & \downarrow \cong \\ F^H(S^0, f\mathbb{S}) & \xrightarrow{\sim^\wedge_p} & F^H(EH_+, f\mathbb{S}) \end{array}$$

It is then automatically true for the thick subcategory generated by $\Sigma^\infty_+ G/H$ for all subgroups H. As we have seen in a previous section, this is also the subcategory of all dualizable spectra, or the subcategory of all compact spectra (in the sense of triangulated categories).

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