Some Facts About QX

Let X be a based topological space. As always, we assume that all our spaces are compactly generated weak Hausdorff. Define

$$QX = \Omega^\infty(\Sigma^\infty X) = \mathrm{colim}_n \Omega^n \Sigma^n X$$

Here the colimit has the usual property that any compact $K \longrightarrow QX$ factors through a map into one of the spaces $\Omega^n\Sigma^n X$, and in fact the space of based maps $F(K, QX)$ is homeomorphic to the colimit of the based mapping spaces $F(K, \Omega^n\Sigma^n X)$. See section 5 of Strickland's paper on CGWH spaces for justification of this.

- Q is a functor into based spaces, obtained by applying Σ^∞ to get a (pre)spectrum, followed by Ω^∞ to land back into spaces.

- The homotopy groups $\pi_k(QX)$ are the stable homotopy groups of X, defined by

  $$\pi_k^{\text{stable}}(X) = \mathrm{colim}_n \pi_{n+k}(\Sigma^n X) = \mathrm{colim}_n \pi_k(\Omega^n\Sigma^n X)$$

  In particular, if we set $X = S^0$ then the homotopy groups of QS^0 are the stable homotopy groups of spheres. Remember that the stable homotopy groups of QX are not the same as the homotopy groups of QX!

- Each based homotopy of maps $X \longrightarrow Y$ induces a based homotopy $QX \longrightarrow QY$. Therefore Q preserves based homotopy equivalences, so it descends to a "derived" functor on the homotopy category of based spaces. Formally, this new functor replaces each space with a CW complex and then applies Q to that complex. If $X \longrightarrow Y$ is a weak homotopy equivalence, and X and Y have nondegenerate basepoints, then $\Sigma^n X \longrightarrow \Sigma^n Y$ is a weak homotopy equivalence (check that it's an isomorphism on homology, then use Whitehead's Theorem). Then clearly $\Omega^n\Sigma^n X \longrightarrow \Omega^n\Sigma^n Y$ is a weak homotopy equivalence, so $QX \longrightarrow QY$ is a weak homotopy equivalence. So the "derived" Q on the homotopy category agrees (up to natural isomorphism) with the simpler underived Q when we restrict to the subcategory of all well-based spaces.

- There is a natural "unit" map $X \longrightarrow QX$ given by the identity map from X into the 0th level of the colimit system for QX. Applying π_*, we get the homomorphism from the homotopy groups of X into the stable homotopy groups of X.

- QX is an infinite loop space. Recall that an infinite loop space Y has deloopings $\{Y_n\}_{n=0}^\infty$, with $Y_0 = Y$, and maps $Y_n \xrightarrow{\sim} \Omega Y_{n+1}$. Depending on the context, these maps may be homeomorphisms, or just weak homotopy equivalences. QX is an infinite loop space in the stronger sense. Specifically, there is a homeomorphism $QX \cong \Omega Q(\Sigma X)$. Notice that this immediately implies that QX has infinitely many deloopings, since then $Q(\Sigma^n X) \cong \Omega Q(\Sigma^{n+1} X)$.

How do we construct this homeomorphism? First observe that $\Omega Q(\Sigma X) = \mathrm{colim}_n \Omega(\Omega^n \Sigma^n(\Sigma X))$, so we have two colimit systems

$$X \xrightarrow{\eta_X} \Omega\Sigma X \longrightarrow \Omega^2\Sigma^2 X \longrightarrow \Omega^3\Sigma^3 X \longrightarrow \cdots \longrightarrow QX$$

$$\Omega\Sigma X \xrightarrow{\Omega(\eta_{\Sigma X})} \Omega^2\Sigma^2 X \longrightarrow \Omega^3\Sigma^3 X \longrightarrow \Omega^4\Sigma^4 X \longrightarrow \cdots \longrightarrow \Omega Q \Sigma X$$

To construct this homeomorphism $QX \cong \Omega Q(\Sigma X)$ explicitly, we just have to draw diagonal homeomorphisms that make everything commute:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] & \Omega\Sigma X \arrow[r] \arrow[dl, leftrightarrow] & \Omega^2\Sigma^2 X \arrow[r] \arrow[dl, leftrightarrow] & \Omega^3\Sigma^3 X \arrow[r] \arrow[dl, leftrightarrow] & \cdots \arrow[r] & QX \arrow[d, leftrightarrow] \\
\Omega\Sigma X \arrow[r, "\Omega(\eta_{\Sigma X})"'] & \Omega^2\Sigma^2 X \arrow[r] & \Omega^3\Sigma^3 X \arrow[r] & \Omega^4\Sigma^4 X \arrow[r] & \cdots \arrow[r] & \Omega Q\Sigma X
\end{tikzcd}$$

What are the diagonal maps? They can't be the identity. Instead, we map $\Omega^n\Sigma^n X$ to itself by taking a map $S^n \longrightarrow \Sigma^n X$ and pre-composing it with a self-map $S^n \longrightarrow S^n$ coming from a permutation on n letters. This defines a self-homeomorphism of $\Omega^n\Sigma^n X$. To pick the right permutations to make the diagram commute, we simply color the various sphere coordinates in a way that suggests how the maps of each colimit system act:

$$\begin{tikzcd}
X \arrow[d, "\eta_X"] & \Omega\Sigma X \arrow[d, "\Omega(\eta_{\Sigma X})"] \arrow[dl, leftrightarrow] \\
\Omega\Sigma X \arrow[d] & \Omega\Omega\Sigma\Sigma X \arrow[d] \arrow[dl, leftrightarrow] \\
\Omega\Omega\Sigma\Sigma X \arrow[d] & \Omega\Omega\Omega\Sigma\Sigma\Sigma X \arrow[d] \arrow[dl, leftrightarrow] \\
\Omega\Omega\Omega\Sigma\Sigma\Sigma X \arrow[d] & \vdots \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
QX \arrow[r, leftrightarrow] & \Omega Q(\Sigma X)
\end{tikzcd}$$

Then it becomes clear which permutation to pick to get the colors of the Ω's to line up.

Anyway, this shows that QX is an infinite loop space, in the strongest possible sense. So the infinite little cubes operad acts directly on QX (i.e. QX is an E_∞ space). In fact, all of our constructions were functorial in X, so Q is a functor from based spaces to (grouplike) E_∞ spaces.

- Since QX is an E_∞ space, we can apply the delooping functor B^∞ to get a connective spectrum. Of course, $B^n(QX) \simeq Q(\Sigma^n X)$ because we already know that $Q(\Sigma^n X)$ is an n-fold delooping of QX. The spectrum $\{Q(\Sigma^n X)\}_{n=0}^\infty$ is stably homotopy equivalent to the suspension spectrum $\{\Sigma^n X\}_{n=0}^\infty$. This is because B^∞ and Ω^∞ are inverse equivalences between grouplike E_∞ spaces and connective spectra. Taking homology, we conclude

$$\widetilde{H}_k(X) \cong \mathrm{colim}_n \widetilde{H}_{n+k}(\Sigma^n X) \cong \mathrm{colim}_n \widetilde{H}_{n+k}(Q(\Sigma^n X))$$

- If X is well-based and $* \hookrightarrow X$ is $(n-1)$-connected, then $X \hookrightarrow QX$ is $(2n-1)$-connected. This strengthens the previous statement about the homology of $Q(\Sigma^n X)$. As a special case, the homology of QS^n is $\mathbb{Z}$ in degrees 0 and n, and zero in all other degrees up to $(2n-1)$.

  To prove this, we investigate the maps

$$X \longrightarrow \Omega\Sigma X \longrightarrow \Omega^2\Sigma^2 X \longrightarrow \ldots \longrightarrow QX$$

  The first map on π_k is the suspension map $\pi_k(X) \longrightarrow \pi_{k+1}(\Sigma X)$. By the Freudenthal Suspension Theorem, since X is $(n-1)$-connected, this map is surjective if $k \leq 2n-1$ and an isomorphism if $k \leq 2n-2$. The next map on π_k is $\pi_{k+1}(\Sigma X) \longrightarrow \pi_{k+2}(\Sigma^2 X)$, which is surjective if

$$\begin{aligned}(k+1) &\leq 2(n+1) - 1 \\ k &\leq 2n\end{aligned}$$

  and an isomorphism if $k \leq 2n-1$. Continuing onwards, we see that all maps $X \longrightarrow \Omega\Sigma X \longrightarrow \Omega^2\Sigma^2 X \longrightarrow \ldots$ give an isomorphism on π_k so long as the first one does, that is, if $k \leq 2n-2$. Since π_k commutes with the colimit here, we conclude that $X \longrightarrow QX$ is an isomorphism on π_k if $k \leq 2n-2$ and is surjective on π_k if $k \leq 2n-1$. Therefore $X \longrightarrow QX$ is $(2n-1)$-connected.

- Q takes finite wedges to finite products:

$$\begin{aligned}
Q(\bigvee_{i=1}^n X_i) &= \mathrm{colim}_n \Omega^n \Sigma^n \left(\bigvee_i X_i\right) \\
&\cong \mathrm{colim}_n \Omega^n \left(\bigvee_i \Sigma^n X_i\right) \\
&\simeq \mathrm{colim}_n \Omega^n \left(\prod_i \Sigma^n X_i\right) \\
&\cong \mathrm{colim}_n \prod_i \Omega^n \Sigma^n X_i \\
&\simeq \prod_i \mathrm{colim}_n \Omega^n \Sigma^n X_i \\
&= \prod_i QX_i
\end{aligned}$$

The two weak equivalences here must be explained. The first comes from the fact that a finite wedge of spectra includes into a finite product, and this inclusion is a stable homotopy equivalence. Therefore it induces a weak homotopy equivalence between the infinite loop spaces of the two spectra. The second equivalence comes from interchanging the finite product with the colimit. There is certainly a map between the two, and we verify that it is a weak homotopy equivalence by choosing any basepoint and using the fact that maps into these spaces from spheres commute with the colimit and the product. It is also worth pointing out that these equivalences are all natural, so their composite $Q(\bigvee_{i=1}^{n} X_i) \simeq \prod_i QX_i$ is natural as well.

- What about infinite wedges? All but the two weak equivalences go through. For the first weak equivalence, we take a CW approximation of the infinite product consisting of all finite products of cells, and use the connectivity of this approximation to show that the map is still a weak equivalence. What about the second weak equivalence? If there is a uniform bound on the dimension of the X_i in the (possibly uncountable) wedge, then $\pi_n(\Sigma^n X_i)$ all stablilize in the same dimension, so the argument goes through. In general, it appears that it does not.

- If $\Omega^n Z$ is an n-fold loop space then n-fold loop maps

$$\Omega^n \Sigma^n X \longrightarrow \Omega^n Z$$

naturally correspond to ordinary continuous maps

$$\Sigma^n X \longrightarrow Z$$
$$X \longrightarrow \Omega^n Z$$

Therefore $\Omega^n \Sigma^n X$ is the free n-fold loop space on X.

- QX is the free infinite loop space on the based space X. As usual, this means that if Y is any other infinite loop space, there is a natural bijection between infinite loop maps $QX \longrightarrow Y$ and ordinary maps $X \longrightarrow Y$. Recall that Y has deloopings $\{Y_n\}_{n=0}^{\infty}$ such that $Y_0 = Y$ and $Y_n \cong \Omega Y_{n+1}$. Composing these homeomorphisms gives a homeomorphism

$$Y_0 \cong \Omega Y_1 \cong \Omega(\Omega Y_2) \cong \Omega^2 Y_2 \cong \Omega^2(\Omega Y_3) \cong \Omega^3 Y_3 \cong \ldots \cong \Omega^n Y_n$$

It turns out that this is the wrong homeomorphism to consider if we want naturally defined diagrams to commute. Since its component maps are of the form $\Omega^k Y_k \longrightarrow \Omega^k(\Omega Y_{k+1})$, the new copies of Ω come in on the right, whereas in the colimit system for QX they come in on the left. Therefore we want to alter this homeomorphism $Y_0 \cong \Omega^n Y_n$ so that the new loops come in on the left. We can accomplish this by changing each component map to $\Omega^k Y_k \longrightarrow \Omega(\Omega^k Y_k)$, or by simply reversing the order of the loops in the final space $\Omega^n Y_n$.

Anyway, to do one part of the adjunction, given an infinite loop map $QX \longrightarrow Y$, we compose $X \longrightarrow QX \longrightarrow Y$ to get an ordinary continuous map $X \longrightarrow Y$. To go the other way, given a map $X \longrightarrow Y$, for each $n > 0$ we take $X \longrightarrow Y \cong \Omega^n Y_n$, which corresponds to a unique map

$\Sigma^n X \longrightarrow Y_n$, which then gives an n-fold loop map $\Omega^n\Sigma^n X \longrightarrow \Omega^n Y_n \cong Y$. Under the above choice of homeomorphism $Y_0 \cong \Omega^n Y_n$, this gives commuting maps from the colimit system for QX into Y, so we get a map $QX \longrightarrow Y$.

To check that $QX \longrightarrow Y$ is an infinite loop map, we simply apply the same process to $\Sigma^n X \longrightarrow Y_n$ to get a map $Q(\Sigma^n X) \longrightarrow Y_n$. Then we check that this square commutes

$$\begin{tikzcd}
Q(\Sigma^n X) \arrow[r] \arrow[d] & Y_n \arrow[d] \\
\Omega Q(\Sigma\Sigma^n X) \arrow[r] & \Omega Y_{n+1}
\end{tikzcd}$$

by checking that the individual levels of the colimit system give commuting squares

$$\begin{tikzcd}
\Omega^{k-1}\Omega\Sigma\Sigma^{k-1}\Sigma^n X \arrow[r] \arrow[d, leftrightarrow] & \Omega^k Y_{k+n} \arrow[d, leftrightarrow] & Y_n \arrow[l] \arrow[d] \\
\Omega\Omega^{k-1}\Sigma^{k-1}\Sigma\Sigma^n X \arrow[r] & \Omega\Omega^{k-1} Y_{(k-1)+1+n} & \Omega Y_{n+1} \arrow[l]
\end{tikzcd}$$

As before, the middle vertical maps shuffles one of the sphere coordinates from one side to the other. Both squares commute! So $QX \longrightarrow Y$ is an infinite loop map.

This forms an adjunction because our choices in the second half were forced, so there exists a unique extension of $X \longrightarrow Y$ to an infinite loop map $QX \longrightarrow Y$. (Check this.)

- If Y is a weak infinite loop space, but $Y_n \longrightarrow \Omega Y_{n+1}$ is still a closed inclusion, then let $\widetilde{Y} = \mathrm{colim}_n \Omega^n Y_n$. (As before, we pick the maps in the colimit system so that each new loop comes in "on the left.") Then $\widetilde{Y}$ is a strong infinite loop space with deloopings $\widetilde{Y}_k = \mathrm{colim}_n \Omega^n Y_{n+k}$, and $Y_k \longrightarrow \widetilde{Y}_k$ is a weak homotopy equivalence. In particular, $Y \longrightarrow \widetilde{Y}$ is a weak homotopy equivalence. Tracing through the above constructions, every ordinary map $X \longrightarrow Y$ extends to an n-fold loop map $\Omega^n\Sigma^n X \longrightarrow \Omega^n Y_n \longrightarrow \widetilde{Y}$.

- If X is a monoid in the based sense, then $\Sigma^\infty X$ is a ring spectrum, and $\Omega^\infty\Sigma^\infty X = QX$ gets a "multiplication" that is compatible with its "additive" E_∞ operations. If X is a commutative monoid, or even just an E_∞ monoid, then QX is an E_∞ ring space, in the sense that it has an E_∞ addition and an E_∞ multiplication that distributes over addition in the appropriate sense. So in particular, if X is a monoid then the homotopy groups of QX form a ring, and if X is commutative then they form a commutative ring.

- If G is a discrete monoid, then G_+ is a based monoid and $Q(G_+) \cong \prod_G QS^0$. This follows from the above statement that Q turns wedge sums of bounded-dimension complexes into products. The connected components of this space are in natural bijection with the group ring $\mathbb{Z}[G]$. This bijection preserves addition because a loop in $\prod_G QS^0$ is a product of loops in QS^0, and addition of loops can be carried out in each factor separately. To check that the multiplication agrees, it suffices to check the generators, and this is straightforward. So $\pi_0(QG_+) \cong \mathbb{Z}[G]$ as rings.

- If X is already an infinite loop space, then the identity map $X \longrightarrow X$ extends to a unique infinite loop map $QX \longrightarrow X$, using the above adjunction. (Prove $QX \longrightarrow X$ is a quasifibration.) Let Q' be the fiber over the basepoint of X.

$$\begin{tikzcd}
Q' \arrow[rr, "\mathrm{id}"] \arrow[d] & & Q' \arrow[d] \\
Q' \times X \arrow[r] \arrow[d] & QX \times QX \arrow[r] & QX \arrow[d] \\
X \arrow[rr, "\mathrm{id}"] & & X
\end{tikzcd}$$

  Therefore $QX \simeq Q' \times X$.

- If G is an abelian group, then it has weak deloopings $\{K(G,n)\}_{n=0}^{\infty}$, so G is weakly equivalent to an infinite loop space. Therefore $QG \simeq Q' \times G$. What about QG_+?

- Let G be a topological group. Then $GL_1(\Sigma^\infty G_+) = QG_+|_{\mathbb{Z}[\pi_0 G]^\times}$.

References

[1] J.F. Adams, *Infinite Loop Spaces.*

[2] J.P. May, *The Geometry of Iterated Loop Spaces.*

[3] N.P. Strickland, *The Category of CGWH Spaces.*