THE TRANSFER ON THE n-FOLD COVER OF THE CIRCLE

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The purpose of this note is to give details about how to calculate the Becker-Gottlieb transfer [BG75] for the n-fold covering $S^1 \xrightarrow{\cdot n} S^1$. It was written in November of 2014 and posted to my webpage in 2018, but there was an incorrect statement about the coordinate systems. In 2023 I corrected this and added a discussion of how this interacts with the circle transfer.

0.1. **Coordinates.** First we have to talk about choices of coordinates, because the answer depends on which coordinates we use for $\Sigma^\infty_+ S^1$.

In the stable homotopy category, any diagram manifesting A as a retract of X gives canonical isomorphisms in the homotopy category

$$A \vee F \xrightarrow{\sim} X \xrightarrow{\sim} A \times C$$

where F is the fiber of $X \longrightarrow A$ and C is the cofiber of $A \longrightarrow X$. We stress that we get not just an abstract isomorphism $A \oplus C \cong X$ in the stable homotopy category, but a *particular* isomorphism induced by the retract diagram for A and X. Any other isomorphism $A \oplus C \cong X$ may be easily compared against our canonical one, by composing them to get a self-map of $A \oplus C$. The two isomorphisms agree iff this self-map is the identity, and this is easily checked in practice by calculating separately the maps

$$A \longrightarrow A, \qquad C \longrightarrow C, \qquad A \longrightarrow C, \qquad C \longrightarrow A,$$

and checking that the first two are the identity and the second two are zero.

Now consider the suspension spectrum of the circle, $\Sigma^\infty_+ S^1$. Choosing a basepoint for the circle gives a retract diagram

$$\Sigma^\infty S^0 \longrightarrow \Sigma^\infty_+ S^1 \longrightarrow \Sigma^\infty S^0.$$

In the stable category, this leads to a canonical splitting of $\Sigma^\infty_+ S^1$ into a wedge of $\Sigma^\infty S^0$ and another summand. The second summand may be canonically identified with the cofiber of the first map above, or the fiber of the second map. The cofiber is simple enough, it is just the suspension spectrum of the based circle $\Sigma^\infty S^1$.

We want to describe a different way of getting the same isomorphism. We will map $\Sigma^\infty S^0 \longrightarrow \Sigma^\infty_+ S^1$ as before, by inclusion of the basepoint of S^1. For the second summand, we truncate $\Sigma^\infty S^1$ at spectrum level 1, and then map in by Pontryagin-Thom collapse map

$$S^2 \longrightarrow \Sigma_+ S^1. \tag{1}$$

This uses the standard embedding of S^1 into $\mathbb{R}^2$, and the identification of its neighborhood with $S^1 \times (-1, 1)$ by spinning around the origin. If we compose this map with the projection to $\Sigma^\infty S^1 \times \Sigma^\infty S^0$, we get a map

$$\Sigma^\infty S^1 \longrightarrow \Sigma^\infty S^1 \times \Sigma^\infty S^0$$

which is (up to homotopy) the identity on the first factor and zero on the second factor. Therefore it defines a splitting $\Sigma^\infty S^0 \vee \Sigma^\infty S^1 \xrightarrow{\sim} \Sigma^\infty_+ S^1$ which is, in the homotopy category, the inverse of the above splitting. This is the splitting we'll use for our calculation.

0.2. **The answer in "retract" coordinates.** Now that we've fixed the coordinates, we can compute the Becker-Gottlieb transfer [BG75] for the covering

$$S^1 \xrightarrow{\cdot n} S^1, \quad n > 0$$

This will be a stable map from $\Sigma^\infty_+ S^1$ to itself. Rewriting this spectrum as $\mathbb{S}^1 \vee \mathbb{S}^0$, we see that it suffices to fill out the matrix

$$\begin{array}{c|cc} & \mathbb{S}^0 & \mathbb{S}^1 \\ \hline \mathbb{S}^0 & ? \in \pi_0^S & ? \in \pi_1^S \\ \mathbb{S}^1 & ? \in \pi_{-1}^S & ? \in \pi_0^S \end{array}$$

The answer is

$$\begin{pmatrix} n & (n-1)\eta \\ 0 & 1 \end{pmatrix}$$

where $\eta \in \pi_1^S$ is the Hopf map. Remember that $2\eta = 0$, so the upper-right entry is 0 or η depending on the parity of n.

0.3. **How to calculate it.** Let's explain how we got these entries. The lower-left entry is easy, since $\pi_{-1}^S = 0$.

To get the rest, we embed $S^1 \times D^2$ into $\mathbb{R}^3$ in the usual way, by taking a D^2 embedded in the xz plane and spinning it around the z-axis. We then embed another copy of S^1 inside of $S^1 \times D^2$ by winding around n times. Quotienting the complement of the big tube $S^1 \times e^2$ to a point gives the domain $S^2 \wedge S^1_+$. Quotienting the complement of a tubular neighborhood of the windier S^1 gives the target $S^2 \wedge S^1_+$. The top-degree map is the composite

$$S^3 \longrightarrow S^2 \wedge S^1_+ \xrightarrow{\text{collapse}} S^2 \wedge S^1_+ \xrightarrow{\text{collapse}} S^3$$

where the last map collapses onto any point inside $S^1 \times D^2$. We just have to count the degree of this composite. But everything is a collapse map, so the degree is 1; this gives the lower-right entry in the matrix.

To get the bottom-degree map we look at the composite

$$S^2 \hookrightarrow S^2 \wedge S^1_+ \xrightarrow{\text{collapse}} S^2 \wedge S^1_+ \xrightarrow{\text{project}} S^2.$$

This includes one disc D^2 into $S^1 \times D^2$, then collapses the result onto n smaller discs normal to the n places where the windy S^1 passes through this larger disc. The result is a degree n map, giving the upper-left entry of our matrix.

The last entry is tricky. We will use the Pontryagin-Thom correspondence between the stable 1-stem and framed 1-manifolds up to framed bordism. We inspect the composite

$$S^3 \longrightarrow S^2 \wedge S^1_+ \xrightarrow{\text{collapse}} S^2 \wedge S^1_+ \xrightarrow{\text{project}} S^2$$

The first map, as in (1), collapses from $\mathbb{R}^3$ onto $S^1 \times D^2$, embedded into $\mathbb{R}^3$ by spinning a disc about the z-axis. The remaining collapses create a map $S^3 \to S^2$ represented by a circle winding around n times, with the "trivial" framing, where the first vector always points away from the z-axis, and the second vector always points in the positive z-direction.

We unwind this circle in $(n-1)$ steps to get the trivial circle. Each step involves removing a crossing in the knot diagram by flipping a loop; this adds 1 to the framing number. Therefore our map is represented by a trivial circle with a framing that twists around $(n-1)$ times. This gives the upper-right entry of our matrix, $(n-1)$ times the Hopf map η.

0.4. **Relationship to the circle transfer.** The circle transfer for the trivial fibration $S^1 \to *$ is the map of spectra $\Sigma\Sigma^\infty_+(*) \to \Sigma^\infty_+ S^1$, or equivalently $\mathbb{S}^1 \to \mathbb{S}^0 \vee \mathbb{S}^1$, calculated by embedding S^1 into $\mathbb{R}^3$ and taking a Pontryagin-Thom collapse to get a map

$$S^3 \to S^2 \wedge S^1_+,$$

as in (1). *However,* when we identify the neighborhood of S^1 in $\mathbb{R}^3$ with $S^1 \times D^2$, we must add an *additional twist*, in other words we must compose with the homeomorphism of $S^1 \times D^2$ that rotates the disc D^2 around once as we go around S^1.

The reason for this modification is that the trivialization of the normal bundle of S^1 has to be compatible with the tangent bundle of S^1, which has a standard trivialization because S^1 is a Lie group. The compatibility is as follows: if we add both trivializations together, we need to get the identity of the trivial bundle $S^1 \times \mathbb{R}^3$, or at least an automorphism homotopic to the identity. And if we take the "spin around the z-axis" trivialization of the normal bundle, and add it to the tangent bundle, we instead get an automorphism of $S^1 \times \mathbb{R}^3$ with a single twist. (Visualize it! Use your hands!) So, we have to add a second twist in to cancel this out.

As a consistency check, let's verify that the circle transfer followed by the n-fold cover transfer agrees with the circle transfer:

(2)
$$\begin{tikzcd} & \mathbb{S}^1 \arrow[dl, "S^1 \text{ transfer}"'] \arrow[dr, "S^1 \text{ transfer}"] & \\ \mathbb{S}^0 \vee \mathbb{S}^1 \arrow[rr, "C_n \text{ transfer}"'] & & \mathbb{S}^0 \vee \mathbb{S}^1 \end{tikzcd}$$

In our perferred coordinates where we just spin around the z-axis, the circle transfer is given by the vector $\begin{pmatrix} \eta \\ 1 \end{pmatrix}$, the η representing the extra twist we had to add. So the compatibility becomes the matrix equation

$$\begin{pmatrix} n & (n-1)\eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ 1 \end{pmatrix} = \begin{pmatrix} \eta \\ 1 \end{pmatrix},$$

which holds because $2\eta = 0$.

0.5. **The answer in "circle transfer" coordinates.** If we change coordinates for $\mathbb{S}^0 \vee \mathbb{S}^1$ by the matrix

$$\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix},$$

the answer for the transfer for the n-fold cover becomes

$$\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n & (n-1)\eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}.$$

In other words, consider the equivalence

$$\mathbb{S}^0 \vee \mathbb{S}^1 \xrightarrow{\sim} \Sigma^\infty_+ S^1$$

given by the inclusion $S^0 \to S^1$ and the circle transfer. If we use *this* equivalence to relate $\Sigma^\infty_+ S^1$ to $\mathbb{S}^0 \vee \mathbb{S}^1$, and not the previous one, then the transfer for the n-fold cover has no off-diagonal term. This is consistent with the diagram (2), and in fact can be deduced from it, without any need to analyze Pontryagin-Thom collapse maps explicitly. (But where's the fun in that?)

0.6. **Concluding remarks.** I claim no originality here. This is just an explicit description of how to do the calculation, and a reference for people looking to get the answer right. The calculation appears in a few places in the literature, but sometimes incorrectly. The correct answer appears in [Hes96], proof of Lemma 1.5.1. It's also used in Lemma 3.15 of [CDD11].

References

[BG75] J. C. Becker and D. H. Gottlieb, *The transfer map and fiber bundles*, Topology **14** (1975), no. 1, 1–12.

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